AN EQUATION INCORPORATING CROSSING SYMMETRY, REGGE ASYMPTOTIC BEHAVIOUR, AND TWO–BODY UNITARITY

By J. A. Campbell*† and R. J. Yaes*

[*Manuscript received July* 17, 1969]

Summary

An explicitly crossing-symmetric generalization of Kadyshevsky's quasipotential equation is proposed. The equation has the two-body unitarity cuts in all three channels. Applied to a particular continuation of the Veneziano amplitude off the energy–momentum shell, it produces an amplitude with Regge asymptotic behaviour.

I. Introduction

Unitarity and crossing symmetry are two conditions which are expected on quite general grounds to apply to any amplitude for the scattering of hadrons or strongly interacting particles. A large amount of experimental information also supports the assumption of a specific asymptotic behaviour for hadronic scattering amplitudes at high energy and fixed momentum transfer. This relativistic behaviour is associated with the name of Regge, who first showed (1960) how it could arise in a nonrelativistic calculation based on the Schrödinger equation.

Attempts to derive amplitudes satisfying any more than one of the three conditions mentioned above have had limited success. Therefore the recent suggestion by Veneziano (1968) of an amplitude which has both Regge asymptotic behaviour and crossing symmetry has attracted much attention. Besides being symmetric, the Veneziano amplitude is real apart from isolated singularities. Thus it cannot take account of resonances of finite width. Although this means that it can only give an approximate description of the interactions of hadrons, it provides a loophole through which some form of unitarity can be imposed. As stated by Aaron, Amado, and Young (1968), the loophole is that an integral equation $T = V + VG_0T$ of the Lippmann–Schwinger (1950) type generates a T-matrix element T which obeys two-body unitarity, given an appropriate choice of the intermediate two-body Green's function G_0 and a real symmetric input V. Most of the applications of the equation have been nonrelativistic, having therefore no need of crossing symmetry, and V has been interpreted most often as a potential. We propose a crossing-symmetric generalization of the equation, in which the Veneziano amplitude is to be regarded as V. Our starting point is a relativistic quasipotential equation which has been examined by several Russian authors, particularly Kadyshevsky (1968), in considerable detail.

* Department of Physics and Center for Particle Theory, University of Texas at Austin, Austin, Texas 78712, U.S.A.

† Also a member of the Department of Computer Science, University of Texas at Austin.

After a review of the Veneziano amplitude in Section II, we state the equation and our auxiliary assumptions in Section III. A short discussion of the properties and uses of the equation follows in Section IV.

II. The Veneziano Amplitude

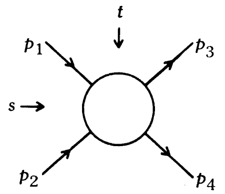

Fig. 1.—A process with two initial-state particles and two final-state particles which is illustrative of crossing symmetry. The s and t channels are indicated.

Consider the scattering of two spinless particles with four-momenta p_1 and p_2 to two particles with four-momenta p_3 and p_4 (i.e. $12 \to 34$), as shown in Figure 1. Three Lorentz-invariant variables for the process are

$$s = (p_1+p_2)^2 = (p_3+p_4)^2\,, \tag{1a}$$
$$t = (p_1-p_3)^2 = (p_4-p_2)^2\,, \tag{1b}$$
$$u = (p_1-p_4)^2 = (p_3-p_2)^2\,. \tag{1c}$$

In this spinless case, a single amplitude $A(s,t,u)$ describes the dynamics of the scattering event. The variable s is a measure of the total energy of the reaction and t is a measure of the four-momentum transfer.

Figure 1 also represents the reactions $1\bar{3} \to \bar{2}4$ and $1\bar{4} \to \bar{2}3$, where a bar denotes an antiparticle, whose four-momentum is the negative of the four-momentum for a corresponding particle. The statement of crossing symmetry is that the one amplitude describes all three reactions, so that

$$A(s,t,u) = A(t,s,u) = A(u,t,s)\,. \tag{2}$$

In Figure 1, the scattering in the s channel may proceed by the exchange of a particle P between particles 1 and 2. Similar exchanges can occur between 1 and $\bar{3}$ in the t channel and between 1 and $\bar{4}$ in the u channel. In many cases there is no unique P, because families of particles with different masses but the same discrete quantum numbers are well known. The generalization of the idea of exchange of a particle is the exchange of a Regge trajectory α, which carries information about all the particles P of a given family. A full account of the meaning of the trajectory functions is to be found in Collins and Squires (1968). A distinctive feature of the trajectories on which the observed hadrons lie is that they appear to be linear, that is, $\alpha(w) = a_0+aw$, where w is any one of the variables defined by (1). The values of a_0 differ between families, but for trajectories associated with known particles a is always in the neighbourhood of 1 GeV^{-2}.

Regge asymptotic behaviour for the amplitude A in the s channel when the momentum-transfer variable t is fixed and negative is

$$A(s,t,u) \sim s^{\alpha(t)-1}$$

for large s. This is equivalent to

$$A(s,t,u) \sim \{\alpha(s)\}^{\alpha(t)-1} \tag{3}$$

when the trajectories are linear. Similarly, in the t channel ($1\bar{3} \to \bar{2}4$),

$$A(t,s,u) \sim \{\alpha(t)\}^{\alpha(s)-1} \tag{4}$$

for large t and fixed negative s. Equation (2) implies the analogous u-channel behaviour.

From (1), the momentum-transfer variable is negative for any physical scattering process.

The basic Veneziano (1968) amplitude is given simply by

$$V(s,t,u) = \beta\left(\frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))}+\frac{\Gamma(1-\alpha(t))\,\Gamma(1-\alpha(u))}{\Gamma(2-\alpha(t)-\alpha(u))}+\frac{\Gamma(1-\alpha(u))\,\Gamma(1-\alpha(s))}{\Gamma(2-\alpha(u)-\alpha(s))}\right), \tag{5}$$

where the observed spacing of particles on trajectories is reproduced by the constraint $\alpha(s)+\alpha(t)+\alpha(u) = 2$, Γ is a gamma function, and β is a constant. The arguments of the gamma functions in other possible Veneziano amplitudes differ from those in (5) only by fixed integers. The crossing symmetry of (5) is explicit by comparison with (2). Regge asymptotic behaviour such as (3) and (4) follows from the large-z property of the gamma function, namely,

$$\Gamma(z+a)/\Gamma(z+b) \sim z^{a-b}\,. \tag{6}$$

The amplitude is real except for singularities whenever the argument of a gamma function in the numerator of (5) is equal to a negative integer $-n$ and the singularity is not cancelled by a zero elsewhere in (5). The singularities of the individual gamma functions are poles with residues $(-1)^n/n!$ when V is regarded as a function of complex variables.

III. Relativistic Quasipotential Equation

In nonrelativistic potential theory, the scattering amplitude off the energy shell (i.e. when energy is not necessarily conserved) is the solution of the Lippmann–Schwinger (1950) equation

$$T(p,q,E) = V(p,q)+\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\; V(p,k)\frac{1}{E+\mathrm{i}\epsilon-k^2}T(k,q,E)\,. \tag{7}$$

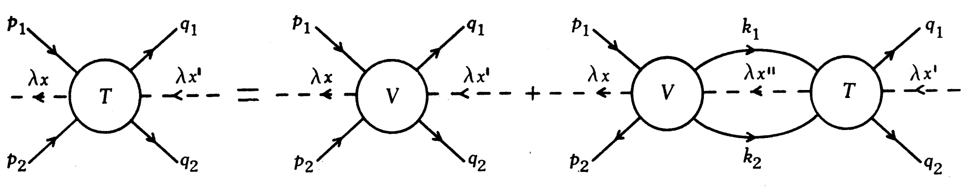

Fig. 2.—Diagram for the evolution of the process described by equations (7) and (9).

The significance of the various parts of (7) is illustrated in Figure 2, where the dashed lines are to be ignored for the moment. The quantity $V(p,q)$ is the Fourier transform of the potential. It is not difficult to show (Aaron, Amado, and Young 1968) that T satisfies the unitarity condition $\operatorname{Im} T = \pi T^* T$ if V is real and symmetric. On iteration, (7) gives the Born series

$$T = V+VG_0\,V+VG_0\,VG_0\,V+\ldots, \tag{8}$$

where unitarity is ensured by the inclusion of all terms of the Born series and by the substitution $G_0 = (E+\mathrm{i}\epsilon-k^2)^{-1}$. The physical scattering amplitude is recovered on-shell, when $p^2 = q^2 = E$.

Logunov and Tavkhelidze (1963) have shown that an equation similar to (7) holds for a relativistic scattering amplitude. We shall use a form of that equation due to Kadyshevsky (1968) and to Kadyshevsky and Mateev (1968):

$$
\begin{aligned}
T(p_1,p_2,\lambda x;q_1,q_2,\lambda x') &= V(p_1,p_2,\lambda x;q_1,q_2,\lambda x') \\
&+\frac{1}{(2\pi)^3}\int \mathrm{d}^4k_1\,\mathrm{d}^4k_2\, V(p_1,p_2,\lambda x;k_1,k_2,\lambda x'')\,\theta(k_1^0)\,\delta(k_1^2-m^2)\,\theta(k_2^0)\,\delta(k_2^2-m^2)\,\frac{\mathrm{d}x''}{x''-\mathrm{i}\epsilon} \\
&\times T(k_1,k_2,\lambda x'';q_1,q_2,\lambda x')\,\delta^{(4)}(k_1+k_2-\lambda x''-q_1-q_2+\lambda x'),
\end{aligned}
\tag{9}
$$

where δ and $\delta^{(4)}$ are one-dimensional and four-dimensional Dirac delta functions and k_i^0 is the energy of the particle with four-momentum k_i and mass m. The step function θ is zero if its argument is negative and unity if its argument is positive. Equation (9) has direct reference to Figure 2, because of the novel off-shell continuations for T and V. All particles represented in Figure 2 by solid lines are assumed to be on their mass shells, but the deviation from the on-shell case overall is expressed by the use of extra "particles", indicated here by dashed lines. These quasiparticles are nonphysical objects whose sole purpose is to carry the four-momentum that measures the departure from the on-shell situation. In Figure 2, λ is a four-momentum which can be chosen parallel to any combination of the four-momenta of the ordinary external particles, and x, x', and x'' are scale factors.

We suggest that the natural crossing-symmetric generalization of (9) is

$$
\begin{aligned}
T(p_1,p_2,\lambda x;q_1,q_2,\lambda x') &= V(p_1,p_2,\lambda x;q_1,q_2,\lambda x') \\
&+\frac{1}{(2\pi)^3}\int \mathrm{d}^4k_1\,\mathrm{d}^4k_2\,\theta(k_1^0)\,\delta(k_1^2-m^2)\,\theta(k_2^0)\,\delta(k_2^2-m^2)\,\frac{\mathrm{d}x''}{x''-\mathrm{i}\epsilon} \\
&\times\{\delta^{(4)}(k_1+k_2-\lambda x''-q_1-q_2+\lambda x')\,V(p_1,p_2,\lambda x;k_1,k_2,\lambda x'')\,T(k_1,k_2,\lambda x'';q_1,q_2,\lambda x') \\
&+\delta^{(4)}(k_1+k_2-\lambda x''+p_2-q_2+\lambda x')\,V(p_1,-q_1,\lambda x;k_1,k_2,\lambda x'')\,T(k_1,k_2,\lambda x'';-p_2,q_2,\lambda x') \\
&+\delta^{(4)}(k_1+k_2-\lambda x''+p_2-q_1+\lambda x')\,V(p_1,-q_2,\lambda x;k_1,k_2,\lambda x'')\,T(k_1,k_2,\lambda x'';-p_2,q_1,\lambda x')\},
\end{aligned}
\tag{10}
$$

which contains two-body unitarity in all channels. The proof of unitarity is long but not conceptually difficult. The diagram which corresponds to (10) is Figure 3. From the definitions (1) and (2), with p_3 and p_4 replaced by q_1 and q_2, it is easy to see that (10) is crossing-symmetric by construction, provided that V is symmetric and that we set the four-vector $\lambda = (1,0,0,0)$.

In passing, we remark that two-body unitarity is exact for equation (9), even on an iteration of the type (8), because any vertical section through Figure 2 intersects only the lines for two intermediate particles. The quasiparticle is not counted. Similar sections through Figure 3, when the iteration (8) has been performed, cut intermediate lines for multiparticle states. All the topologically possible two-particle states for each channel are included in the set, but the iterated calculation contains an unspecified amount of three-body unitarity, four-body unitarity, and so on. It is a price to be paid for the inclusion of crossing symmetry that two-body unitarity cannot be isolated exactly. However, the assumption (13) made below about the off-shell behaviour of V in order to cause the integrals in (10) to converge and have the right behaviour also seems to guarantee rapid convergence of any iteration, thus implying that the principal unitarity correction which the integral term in (10) makes to V comes from two-body unitarity.

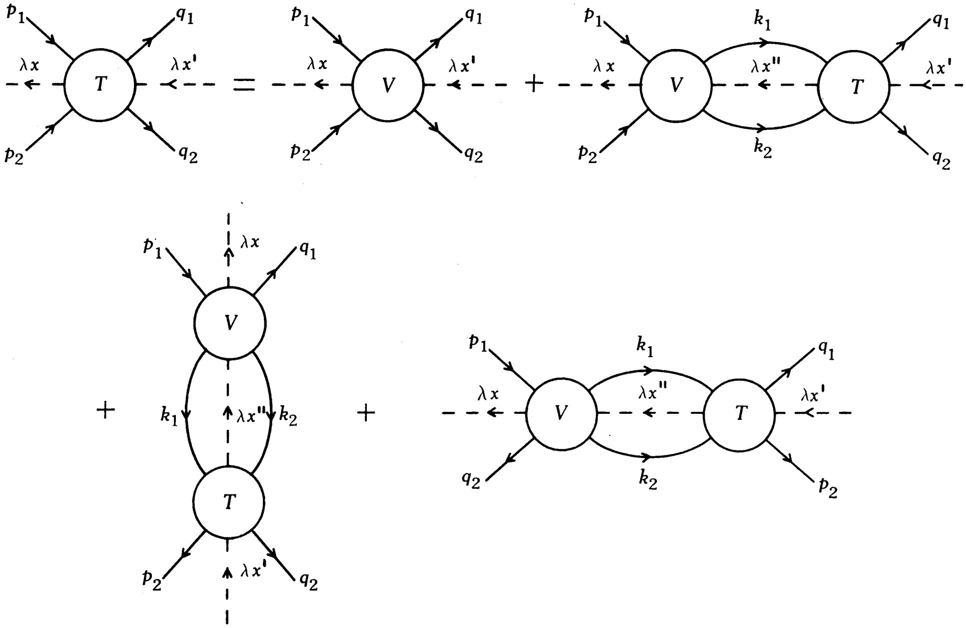

Fig. 3.—Diagram for the evolution of the process described by equation (10).

If (5) is to be used together with (10), an off-shell continuation of the Veneziano amplitude is required. Chan (1969) and Chan and Tsun (1969) have already given a prescription for a Veneziano amplitude for a reaction with six external particles, which allows us to include the quasiparticles, but the prescription is hard to use in calculations and proofs. We therefore note that the method of off-shell continuation is arbitrary, subject to the restriction that the original amplitude should be recovered in the on-shell limit ($x \to 0$ and $x' \to 0$ simultaneously), and propose a continuation which simplifies the task of making calculations with (10). We begin by substituting into (5) the off-shell arguments

$$s = (p_1 + p_2 - \lambda x)^2\,, \tag{11a}$$

$$t = (p_1 - \tfrac{1}{2}\lambda x - k_1 + \tfrac{1}{2}\lambda x'')^2 = t_{pk}\,, \tag{11b}$$

$$u = (p_1 - \tfrac{1}{2}\lambda x - k_2 + \tfrac{1}{2}\lambda x'')^2 = u_{pk}\,. \tag{11c}$$

The relations (11) are not enough to complete our continuation, because the integrals in (10) may still diverge. This is most apparent if we consider the part of V that is dependent only on s and t, assume that T has the same Regge asymptotic behaviour (3) as V in the s channel, and substitute that form back into the appropriate integral term of (10) as a test of consistency. Because the form behaves like

$$\Gamma(1-\alpha(t))\{-\alpha(s)\}^{\alpha(t)-1} \tag{12}$$

according to (6), factors of $\Gamma(1-\alpha(t_{pk}))$ and $\Gamma(1-\alpha(t_{kq}))$, where

$$t_{kq} = (k_1 - \tfrac{1}{2}\lambda x'' - q_1 + \tfrac{1}{2}\lambda x')^2,$$

will occur in the integral. Although $\Gamma(1-\alpha(t))$ is constant for fixed t, the factors involving t_{pk} and t_{kq} are not, and the gamma functions under the integral sign can therefore have large arguments in part of the range of integration. Stirling's approximation indicates that division by factors that behave like $\exp(x'' \log x'')$ as $x'' \to \infty$ is needed to settle all questions of convergence. We have found that the simplest way to insert such factors, and at the same time to maintain Regge asymptotic behaviour for T, is to consider β in (5) as being no longer a constant, but a function of kinematical variables which is also symmetric. We use the functional form

$$\beta = \beta(x, x'') = \beta_0 / \Gamma(1 - cstux^2 - cstux''^2), \tag{13}$$

where β_0 and c are constants. The crossing-symmetric version of (5), with the extension (13), is

$$V = \beta(x, x'')\left(\frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))} + \frac{\Gamma(1-\alpha(t))\,\Gamma(1-\alpha(u))}{\Gamma(2-\alpha(t)-\alpha(u))} + \frac{\Gamma(1-\alpha(u))\,\Gamma(1-\alpha(s))}{\Gamma(2-\alpha(u)-\alpha(s))}\right). \tag{14}$$

It is straightforward to observe that there are still problems with convergence and asymptotic behaviour if $c = 0$, and also that the Regge asymptotic behaviour of T in (10) is exactly that of the inhomogeneous term V in the case that c is extremely large, because then the integral contributes almost nothing to T. The calculation of a minimum permissible c is somewhat involved, but one result (not necessarily the best lower bound) is that $c > \mu^{-8}$, where μ is the mass of the heaviest external particle in Figure 3. This completes our off-shell prescription for V.

The general equations which we propose, therefore, to produce an amplitude of the Veneziano type with corrections from two-body unitarity are (14), (11), and (10), with λ equal to the constant four-vector $(1, 0, 0, 0)$.

IV. Discussion

The demonstration of two-body unitarity in each channel in (10) has already been mentioned in Section III. The method of proof follows that of Kadyshevsky (1968), but it does not depend on the choice of any particular coordinate system.

With the off-shell function (13), the Regge asymptotic behaviour of T is exactly the same as that of V. This is so because, for the s channel where the asymptotic behaviour is given by (3), equation (13) introduces into the integral term a factor of $\alpha(s)$ raised to a negative power of s. Thus, for large s, T and V are effectively identical, which is in any case what one expects physically. When the kinematical variables are permuted according to (2), the same argument holds asymptotically for the other channels.

We have so far considered only general and asymptotic properties of the scattering amplitude. The intense present interest in Veneziano's formula exists because the hypothesis of duality between channels predicts that one can calculate low energy parameters like scattering lengths with the help of high energy input to (5). Harari (1969) and Rosner (1969) give a simple picture of some implications of duality. It is equally possible to carry out such a low energy calculation with the T given by (10), which contains some of the refinements of unitarity that are missing from V. The presence of the step functions θ in (10), amongst other things, makes it hard to apply conventional techniques for handling integral equations (e.g. reduction to matrix equations) to our expression for T. Numerical methods and iteration are the only practical tools available. We are now determining numerically the properties of elastic scattering of positive by negative pions according to (10) and (14). This is probably the simplest test, because duality allows only the first term of (14) to survive. Moreover, for comparison, there are already estimates of these properties based on V alone; a particularly elegant example is due to Lovelace (1968).

Low energy calculations depend strongly on the nature and position of poles of the amplitude. Singularities will have to be detected by our numerical methods, as there is no reason to believe that the poles of T and V are in identical positions. The work of Amado (1963) on the Lippmann–Schwinger equation suggests that no extra families of poles not described by V will appear in T, but that the positions of the input poles will be shifted in the output. This is equivalent to a change in the initial equation $\alpha(w) = a_0+aw$ for a Regge trajectory that is implicit whenever (5) is used by itself. Thus unitarity may be said to modify the trajectories.

The constants β_0 and c are arbitrary here, subject to the inequality on c given in Section III, but in an actual calculation based on T other constraints from assumptions not mentioned above restrict them quite strongly. For example, in any scheme which attempts to link the Regge trajectories used in V with theories of current algebras, it should be possible to overdetermine β_0 and c and therefore investigate the consistency of our entire present technique.

V. Conclusions

We have presented an equation for an amplitude that has the same desirable properties (i.e. crossing symmetry and Regge asymptotic behaviour) for the interactions of hadrons as the Veneziano amplitude, but which also takes into account two-body unitarity. The equation is in a form suitable for numerical computation of low energy parameters according to the hypothesis of duality.

VI. Acknowledgments

We wish to thank Dr. Masao Yamada and Professor I. T. Todorov for helpful discussions. This work was supported in part by the National Science Foundation.

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