COSMOLOGICAL THEORY BASED ON LYRA'S GEOMETRY

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Abstract

Using a Robertson–Walker metric in a modified Riemannian manifold, a cosmological theory is obtained which gives rise to nonstatic perfect-fluid world models. The cosmological equations are formally identical to those of the normal relativistic theory with cosmological constant Λ. The advantage gained by the present theory is that the cosmological constant arises naturally from the geometry. Inspection of the field equations shows that Hoyle's C-field theory can be placed naturally within the framework of Lyra's geometry.

I. Introduction

Weyl (1918) proposed a modification of a Riemannian manifold in order to geometrize the whole of gravitation and electromagnetism. Although formally successful, Weyl's theory is generally considered to be physically unsatisfactory. Later Lyra (1951) proposed another modification of Riemannian geometry upon the basis of which Sen (1958) constructed a unified field theory analogous to that of Weyl's. Within the framework of Lyra's geometry, Sen (1957) also considered a static cosmological model which exhibited a redshift explainable in terms of a vector displacement field that arises naturally in the Lyra formalism.

In this paper we develop a theory, based upon Lyra's geometry, which admits nonstatic cosmological models and which turns out to be analogous to the normal relativistic cosmology (see e.g. McVittie 1965). However, before setting down our cosmological equations, we consider a brief outline of the main distinguishing features of the geometries mentioned above. The full details may be consulted elsewhere (Weyl 1918; Lyra 1951; Sen 1958; McVittie 1965).

II. Manifolds

In a Riemannian manifold the length of a vector ξ^{μ} is not changed when the vector undergoes parallel transfer and also the coefficients $\Gamma^{\alpha}_{\mu\nu}$ of the affine connection are symmetric in their lower indices. These characteristics represent that the affine connection of a Riemannian manifold is uniquely determined by the metric tensor $g_{\mu\nu}$ through the Christoffel symbols of the second kind:

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\} . \qquad (1)$$

Weyl modified the Riemannian geometry by dispensing with the conservation of length of a vector under infinitesimal parallel transfer. Assuming that the increment in length is proportional to the length of the vector itself and is a linear

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homogeneous function of the displacement vector $\mathrm{d}x^\mu$, a gauge vector $\phi_\mu(x)$ is introduced into the affine structure. A Weyl manifold is therefore characterized by the two independent quantities $g_{\mu\nu}$ and ϕ_μ, the coefficients $\Gamma^\alpha_{\mu\nu}$ of the affine connection being given by

$$\Gamma^\alpha_{\mu\nu} = \{^{\,\alpha}_{\mu\nu}\} + S^\alpha_{\mu\nu}, \tag{2}$$

where

$$S^\alpha_{\mu\nu} = \tfrac{1}{2}(\delta^\alpha_\mu \phi_\nu + \delta^\alpha_\nu \phi_\mu - g_{\nu\mu}\phi^\alpha) \tag{3}$$

and

$$\phi^\alpha = g^{\alpha\mu}\phi_\mu. \tag{4}$$

Lyra defined the displacement vector $\boldsymbol{PP'}$ between two neighbouring points $P(x^\mu)$ and $P'(x^\mu + \mathrm{d}x^\mu)$ by its components $x^0\,\mathrm{d}x^\mu$, where $x^0 = x^0(x^\mu)$ is a gauge function. The coordinate system x^μ and the gauge function x^0 together form a *reference system* (x^0, x^μ). The transformation to a new reference system $(\bar{x}^0, \bar{x}^\mu)$ is given by

$$\bar{x}^0 = \bar{x}^0(x^0, x^\mu), \qquad \bar{x}^\mu = \bar{x}^\mu(x^\alpha), \tag{5}$$

with

$$\partial\bar{x}^0/\partial x^0 \neq 0 \qquad \text{and Jacobian} \quad |\,\partial\bar{x}^\mu/\partial x^\nu\,| \neq 0.$$

It is shown by Lyra (1951) and Sen (1957, 1958) that in any general reference system the coefficients of the affine connection are determined by the independent quantities $\Gamma^\alpha_{\mu\nu}$ and ϕ_μ, where the vector field quantities ϕ_μ appear as a natural consequence of the introduction of the gauge function x^0 into the structureless manifold, and the $\Gamma^\alpha_{\mu\nu}$ are related to the $g_{\mu\nu}$ and x^0 through

$$\Gamma^\alpha_{\mu\nu} = (x^0)^{-1}\{^{\,\alpha}_{\mu\nu}\} + S^\alpha_{\mu\nu}, \tag{6}$$

where the $S^\alpha_{\mu\nu}$ are defined as in (3). The infinitesimal parallel transfer of a vector ξ^α is given by

$$\delta\xi^\alpha = -\tilde{\Gamma}^\alpha_{\mu\nu}\,\xi^\mu x^0\,\mathrm{d}x^\nu, \tag{7}$$

where

$$\tilde{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \tfrac{1}{2}\delta^\alpha_\mu \phi_\nu, \tag{8}$$

the $\tilde{\Gamma}^\alpha_{\mu\nu}$ being unsymmetric, but $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$.

The metric in Lyra's geometry is invariant under both coordinate and gauge transformations and is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\,x^0\,\mathrm{d}x^\mu\,x^0\,\mathrm{d}x^\nu, \tag{9}$$

where $g_{\mu\nu}$ is a second rank symmetric tensor. The parallel transfer of a vector is integrable, that is, the length of a vector is conserved upon parallel transport, as in Riemannian geometry.

A geodesic in the Lyra manifold is defined to be a curve $x^\mu = x^\mu(s)$ whose tangent vector $\xi^\mu = (\mathrm{d}x^\mu/\mathrm{d}s)x^0$ is transferred parallel to itself. Therefore a geodesic will not, in general, be a curve of extremal length given by $\delta\int \mathrm{d}s = 0$, in contrast to the Riemannian case. However, Sen (1960) has proved that a sufficient condition for coincidence of these two classes of curves is that $\phi_\mu = \Phi_\mu$, where Φ_μ is defined in equation (12) below.

The curvature tensor $K^{\alpha}_{\lambda\mu\nu}$ of Lyra's geometry is defined in the same manner as the curvature tensor $R^{\alpha}_{\lambda\mu\nu}$ of Riemannian geometry, and is given by

$$K^{\alpha}_{\lambda\mu\nu} = \frac{1}{(x^0)^2}\left\{\frac{\partial}{\partial x^{\mu}}\left(x^0\,\tilde{\Gamma}^{\alpha}_{\lambda\nu}\right) - \frac{\partial}{\partial x^{\nu}}\left(x^0\,\tilde{\Gamma}^{\alpha}_{\lambda\mu}\right) + x^0\,\tilde{\Gamma}^{\alpha}_{\beta\mu}\,x^0\,\tilde{\Gamma}^{\beta}_{\lambda\nu} - x^0\,\tilde{\Gamma}^{\alpha}_{\beta\nu}\,x^0\,\tilde{\Gamma}^{\beta}_{\lambda\mu}\right\}, \tag{10}$$

where the $\tilde{\Gamma}^{\alpha}_{\mu\nu}$ are defined in (8). The curvature scalar of Lyra's geometry is

$$K = R(x^0)^{-2} + 3(x^0)^{-1}\phi^{\mu}_{;\mu} + \tfrac{3}{2}\phi^{\mu}\phi_{\mu} + 2\Phi_{\mu}\phi^{\mu}, \tag{11}$$

where R is the Riemann curvature scalar, the semicolon denotes covariant differentiation with respect to the Christoffel symbols of the second kind in the Riemannian sense, and Φ_{μ} is defined by

$$\Phi_{\mu} = (x^0)^{-1}\,\partial\{\log(x^0)^2\}/\partial x^{\mu}. \tag{12}$$

The volume integral I is given by

$$I = \int L(-g)^{\frac{1}{2}}\,x^0\,\mathrm{d}x^1\,x^0\,\mathrm{d}x^2 \ldots x^0\,\mathrm{d}x^n, \tag{13}$$

where L is a scalar, and is an absolute invariant in this geometry.

If we now put $x^0 = 1$, the so-called normal gauge, and also put $L = K$, the curvature scalar, then (11), (12), and (13) become respectively, for the four-dimensional Lyra manifold,

$$K = R + 3\phi^{\mu}_{;\mu} + \tfrac{3}{2}\phi^{\mu}\phi_{\mu}, \tag{14}$$

$$\Phi_{\mu} = 0, \tag{15}$$

$$I = \int K(-g)^{\frac{1}{2}}\,\mathrm{d}^4x, \tag{16}$$

where d^4x is the element of volume in the four-space.

We now proceed to give a cosmological interpretation to the vector displacement field ϕ_{μ}.

III. Cosmological Theory

The model to be constructed may be viewed as an ideal fluid whose fundamental particles are to be identified with galaxies. We assume the Robertson–Walker metric

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \frac{S^2(t)}{(1+\frac{1}{4}kr^2)^2}\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right), \tag{17}$$

where r, θ, ϕ are comoving coordinates, t is the proper time as measured by a comoving observer, k is a constant, and $S(t)$ is the scale factor of expansion (the so-called radius of the universe).

The fluid energy–momentum tensor is

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu} + pc^{-2}(u^{\mu} u^{\nu} - g^{\mu\nu}), \tag{18}$$

where $u^{\mu} \equiv \mathrm{d}x^{\mu}/\mathrm{d}s$, $\rho = \rho(t)$ is the average density of matter, $p = p(t)$ is the average internal pressure, and

$$T_1^1 = T_2^2 = T_3^3 = -pc^{-2}, \qquad T_4^4 = \rho, \qquad T_\nu^\mu = 0 \quad (\mu \neq \nu). \tag{19}$$

The Greek indices run from 1 to 4 and we have made the identifications $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, $x^4 = ct$.

The field equations may be obtained from the variational principle

$$\delta(I+J) = 0, \tag{20}$$

where I is the integral invariant (16) and

$$J = \int \mathscr{L}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x, \tag{21}$$

where $\mathscr{L}$ is the Lagrangian density of matter. The well-known method (see e.g. Landau and Lifshitz 1962) gives the field equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \tfrac{3}{2}\phi_\mu\phi_\nu - \tfrac{3}{4}g_{\mu\nu}\phi_\alpha\phi^\alpha = -\kappa T_{\mu\nu}, \tag{22}$$

where $\kappa = 8\pi G/c^2$, G being the gravitational constant; R is the Riemann curvature scalar; and the energy–momentum tensor $T_{\mu\nu}$ has arisen from the variation of J. Using the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R, \tag{23}$$

we may rewrite (22) in the mixed form

$$G_\nu^\mu + \tfrac{3}{2}\phi^\mu\phi_\nu - \tfrac{3}{4}\delta_\nu^\mu\phi^\alpha\phi_\alpha = -\kappa T_\nu^\mu, \tag{24}$$

where we have made use of (4). For the metric (17) we have

$$-G_1^1 = -G_2^2 = -G_3^3 = \frac{k}{S^2} + \frac{\dot{S}^2}{c^2S^2} + \frac{2\ddot{S}}{c^2S} \tag{25}$$

and

$$-G_4^4 = \frac{3k}{S^2} + \frac{3\dot{S}^2}{c^2S^2}, \tag{26}$$

where dots denote differentiation with respect to t.

We now restrict the vector displacement field ϕ_μ to be constant and assume it to be the time-like vector

$$\phi_\mu = (0,0,0,\beta), \tag{27}$$

where β is a constant. Then using equations (25) and (26) we get the field equations (24) in the explicit form

$$3\left(\frac{k}{S^2} + \frac{\dot{S}^2}{c^2S^2}\right) - \tfrac{3}{4}\beta^2 = \kappa\rho, \tag{28}$$

and

$$\frac{k}{S^2} + \frac{\dot{S}^2}{c^2S^2} + \frac{2\ddot{S}}{c^2S} + \tfrac{3}{4}\beta^2 = -\frac{\kappa p}{c^2}. \tag{29}$$

We compare these two equations with those of the normal relativistic cosmology based on Riemannian geometry:

$$3\left(\frac{k}{S^2}+\frac{\dot{S}^2}{c^2S^2}\right)-\Lambda = \kappa\rho, \tag{30}$$

$$\frac{k}{S^2}+\frac{\dot{S}^2}{c^2S^2}+\frac{2\ddot{S}}{c^2S}-\Lambda = -\frac{\kappa p}{c^2}. \tag{31}$$

We observe that, apart from a difference of sign between the last terms on the left-hand sides of (29) and (31), the equations are identical; the number β^2, and therefore ϕ_μ, plays the role of the cosmological constant Λ.

Differentiating (28) with respect to t, we get

$$(3\dot{S}/c^2S^3)(2S\ddot{S}-2\dot{S}^2-2kc^2) = \kappa\dot{\rho}. \tag{32}$$

Using (29), this gives

$$3\dot{S}\{(\kappa p+\tfrac{3}{4}\beta^2c^2)S^2+3kc^2+3\dot{S}^2\}+\kappa c^2S^3\dot{\rho} = 0. \tag{33}$$

Employing (28) again, we obtain from (33) the equation

$$3\dot{S}\{(\kappa p+\tfrac{3}{4}\beta^2c^2)S^2+(\kappa c^2\rho+\tfrac{3}{4}\beta^2c^2)S^2\}+\kappa c^2S^3\dot{\rho} = 0 \tag{34}$$

or

$$\mathrm{d}(c^2\rho S^3)/\mathrm{d}t+p\,\mathrm{d}(S^3)/\mathrm{d}t = -\mathrm{d}(hS^3)/\mathrm{d}t, \tag{35}$$

where $h = 3\beta^2c^2/2\kappa$ is a constant. Considering a comoving element of volume V in the three-space $t =$ constant, we may write $V = S^3$ and $M = \rho V$, where M is the mass of the volume. The total energy E in V is then $E = Mc^2 = c^2\rho S^3$, and so (35) can be written in the form

$$\mathrm{d}E+p\,\mathrm{d}V = -h\,\mathrm{d}V, \tag{36}$$

showing that the mass–energy conservation law does not hold in this cosmology (cf. $\mathrm{d}E+p\,\mathrm{d}V = 0$ in the Riemannian-based cosmology).

In the case of a *pressure-free model,* equation (35) reduces to

$$(\rho+3\beta^2/2\kappa)S^3 = B, \tag{37}$$

B being a constant. Addition of (28) and (29) gives

$$\frac{4k}{S^2}+\frac{4\dot{S}^2}{c^2S^2}+\frac{2\ddot{S}}{c^2S} = \kappa\left(\rho-\frac{p}{c^2}\right), \tag{38}$$

while subtracting three times (29) from (28) yields

$$\frac{6\ddot{S}}{c^2S}+3\beta^2 = -\kappa\left(\rho+\frac{3p}{c^2}\right), \tag{39}$$

which upon setting $p = 0$ reduces to

$$6\ddot{S}/c^2S+3\beta^2 = -\kappa\rho. \tag{40}$$

Eliminating ρ between (28) and (40) leads to

$$2S\ddot{S}+\dot{S}^2+kc^2+\tfrac{3}{4}\beta^2c^2S^2 = 0\,, \tag{41}$$

which is equivalent to

$$\mathrm{d}(S\dot{S}^2)/\mathrm{d}t+(kc^2+\tfrac{3}{4}\beta^2c^2S^2)\dot{S} = 0\,. \tag{42}$$

This possesses the first integral

$$S\dot{S}^2+kc^2S+\tfrac{1}{4}\beta^2c^2S^3 = A\,, \tag{43}$$

A being a constant. Substituting in (41), we get

$$\ddot{S} = -A/2S^2-\tfrac{1}{4}\beta^2c^2S\,. \tag{44}$$

From (40) and (44), for negligible p, we find

$$6A = (3\beta^2c^2+2kc^2\rho)S^3\,. \tag{45}$$

Substituting for A in (43), we have

$$\dot{S}^2-(\tfrac{1}{4}\beta^2c^2+\tfrac{1}{3}\kappa c^2\rho)S^2+kc^2 = 0\,. \tag{46}$$

From (37) we can write

$$\kappa c^2\rho S^2 = 3Cc^2/S-\tfrac{3}{2}\beta^2c^2S^2\,, \tag{47}$$

where $3C = \kappa B$ is a constant, and then (46) becomes

$$\dot{S}^2/c^2 = C/S-k-\tfrac{1}{4}\beta^2S^2\,, \tag{48}$$

which should be compared with the normal relativistic result in a Riemannian manifold, namely

$$\dot{S}^2/c^2 = C/S-k+\tfrac{1}{3}\Lambda S^2\,. \tag{49}$$

If β^2 is set equal to $-4\Lambda/3$, it can be seen that (48) and (49) are identical. We have achieved almost complete equivalence between the present cosmological theory and the usual relativistic theory, the only disturbing factor being the inconsistency in sign between equations (29) and (31) as mentioned above. However, if one considers the particular class of models for which $\beta = 0$, that is, ϕ_μ vanishes, the correspondence between the two theories is complete. This is to be expected, for in the case $\beta = 0$ the equations (22) reduce to Einstein's field equations.

The various models which arise through assigning specific values to k and β^2 in equation (48) may be discussed in a manner similar to that for the models arising from equation (49) (see e.g. Bondi 1960).

We note here that Sen (1957) considered a *static* model based on Lyra's geometry by taking ϕ_μ to be a constant *imaginary* time-like vector, i.e. $\beta^2 < 0$. This choice was necessary for a redshift to be shown in the static model. Now by adopting the nonstatic Robertson–Walker metric (17) we already have a redshift without the necessity of restricting $\phi_\mu = (0,0,0,\beta)$ to be imaginary. Indeed, one may even take the point of view that it is preferable to have a real ϕ_μ and then $\beta^2 \geqslant 0$ narrows down the choice of possible model universes arising from (48) to those for which $\Lambda \leqslant 0$ in the corresponding normal relativistic cosmology.

IV. Conclusions

The object of this investigation has been to develop a cosmological theory, within the framework of Lyra's geometry, which would admit nonstatic world models. By choosing a special gauge $x^0 = 1$ we have set up field equations which can be solved with the use of the Robertson–Walker metric. The resulting cosmological equations are then identified with those derived within a Riemannian framework, the vector displacement field ϕ_μ in Lyra's geometry playing the role of the cosmological constant Λ in the normal relativistic treatment. The essential difference between the two theories lies in the fact that ϕ_μ arises naturally from the concept of gauge in Lyra's geometry, whereas Λ is introduced in an *ad hoc* fashion in the usual treatment.

We have made the special assumptions that ϕ_μ is constant and that $x^0 = 1$ in the present paper. The next step, currently under investigation, is to remove these restrictions on ϕ_μ and x^0.

It is interesting to note that Hoyle (1948) introduced a vector field similar to the present (27) in order to obtain continuous creation of matter in a steady-state model. Now we can write the field equations (22) in the form

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R = -\kappa\{T^{\mu\nu}-f(\phi^\mu\phi^\nu-\tfrac{1}{2}g^{\mu\nu}\phi_\alpha\phi^\alpha)\}, \tag{50}$$

where $f = -3c^2/16\pi G$ is a constant, and this equation is identical to Hoyle's C-field equation when we identify our vector field ϕ_μ with Hoyle's creation field C_μ. We noted in the previous section (equation (36)) that the usual energy conservation law does not hold in the present cosmology, so we may expect from that some energy replenishment (or depletion). On this view, we may take ϕ_μ to be a source (or sink) vector field. Further, Hoyle's C_μ vector was *introduced* in order to account for the non-conservation of energy. In the present theory the ϕ_μ arises *naturally* in the geometry and we later give it the present interpretation.

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