ON THE THEORY OF ANISOTROPIC DIFFUSION OF ELECTRONS IN GASES

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Abstract

It is now recognized that when electrons move in a steady state of motion in a gas in an electric field the process of diffusion is in general anisotropic with a coefficient of diffusion D_{L} along or against the electric force $e\boldsymbol{E}$ that is not the same as the coefficient D for directions normal to $e\boldsymbol{E}$. A theoretical discussion of this phenomenon based upon the Maxwell–Boltzmann equation is given which also entails consideration of related matters such as the distribution function $f_0^*(c)$ for an isolated travelling group, the distribution of number density n, the equation of continuity and current density, and the relation of the theory of the travelling group to that of the steady stream.

I. Introduction

Systematic theoretical discussions (e.g. Parker 1963, 1965) of the motion of electrons in gases are based upon the Maxwell–Boltzmann equation, which is an equation to be satisfied by the product nf of the number density n and the velocity distribution function f. However, in order to derive formulae for the coefficients of transport, namely the coefficient of diffusion D and the drift velocity $\boldsymbol{W}$, it is supposed that f_0, the first term in the usual expansion for f in spherical harmonics (Section II), can be separated from the product nf_0. Similarly, the spatial distributions of the number density n in an isolated group of electrons drifting and diffusing through the gas in a uniform electric field or of electrons in a steady stream are found as solutions of a general equation of continuity satisfied by n independently of f, although the coefficients of transport D and W which appear as constant coefficients in the equation of continuity are integrals over all speeds of integrands that contain f_0 or $\partial f_0/\partial c$ as a factor. It was formerly supposed that diffusion continues to be isotropic in the presence of an electric field as it is when the field is absent, and it was the practice to represent its contribution in the equation of continuity by the single coefficient D. It is now recognized that a more faithful representation is in terms of an equivalent process of diffusion in the direction of, or against, the electric force $e\boldsymbol{E}$ with a coefficient D_{L} which is in general not equal to the coefficient D for directions at right angles to $\boldsymbol{E}$.

In the procedure that is adopted, cognizance is taken of the fact that in a uniform stream of electrons the distribution function f_0 is independent of position and is of the form f_0^* (see Section III). In the presence of spatial derivatives of n, on the other hand, such is no longer the case and a solution of the Maxwell–Boltzmann equation is attempted by postulating that f_0 depends on position through the spatial derivatives of n, with the postulated dependence of the form given in equation (70) and its special cases equations (31) and (42). In order to derive solutions applicable

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to an isolated travelling group, restraints are imposed on the coefficients of the partial spatial derivatives of n. These coefficients are functions of c. It proves possible to satisfy the Maxwell–Boltzmann equation provided $n(x, y, z, t)$ is a solution of an equation of continuity for n (equation (73)) which is the usual equation of continuity except that in the presence of an electric field apparent diffusion in the direction of and against the field takes place with a coefficient D_L different from D, the coefficient normal to the field, in agreement with observation. This theoretical approach also requires that a diffusing group that begins in a highly concentrated form spreads symmetrically about its centroid even when the latter is moving steadily in a uniform electric field. The usefulness of this theoretical procedure is to be judged by the degree of accord between its predictions and observation.

In what follows these matters are illustrated or touched upon in the course of a theoretical discussion of two cases of electron motion in gases, that of the isolated travelling group and that of a steady stream from an isolated source.

II. Symbols and Fundamental Equations

A point in configuration space is represented by its vector position $\boldsymbol{r}$ and an elementary volume that contains the point $\boldsymbol{r}$ by $\mathrm{d}\boldsymbol{r}$. Similarly, in velocity space the velocity $\boldsymbol{c}$ is represented by a velocity point which is the end point of the vector $\boldsymbol{c}$ and an element of velocity space that contains the point $\boldsymbol{c}$ is denoted by $\mathrm{d}\boldsymbol{c}$. We shall, however, extend the definition of $\mathrm{d}\boldsymbol{c}$ to include a spherical shell $(c, \mathrm{d}c)$ of velocity space with thickness $\mathrm{d}c$ and containing all velocity points with speeds in intervals c to $c+\mathrm{d}c$. The number of electrons whose positions at time t lie within $\mathrm{d}\boldsymbol{r}$ and whose velocity points lie within $\mathrm{d}\boldsymbol{c}$ is denoted by $nf\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}$, where $n = n(\boldsymbol{r}, t)$ and $f = f(\boldsymbol{c}, \boldsymbol{r}, t)$. The function f is the general velocity distribution function.

The function nf satisfies an equation of continuity involving both configuration space and velocity space. The formal expression for the rate of change of the membership of the class of electrons $nf\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}$ is $\{\mathrm{d}(nf)/\mathrm{d}t\}\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}$ and this rate is accounted for as follows when it is assumed that there are no processes that generate new electrons in space:

(1) The rate of net loss of members through transport of electrons because of their velocities $\boldsymbol{c}$ across the bounding surface of $\mathrm{d}\boldsymbol{r}$ is $\mathrm{div}_r(\boldsymbol{c}\,nf)\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}$, where div_r is the divergence operator in configuration space.

(2) The presence of an electric force $e\boldsymbol{E}$ on the electrons within $\mathrm{d}\boldsymbol{r}$ gives each electron an acceleration $e\boldsymbol{E}/m$ which causes the velocity points of those electrons to drift in the direction $e\boldsymbol{E}$ at the constant rate $e\boldsymbol{E}/m$, where $\boldsymbol{E}$ is independent of the time t. Consequently members are lost to the class $nf\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}$ by loss of velocity points from $\mathrm{d}\boldsymbol{c}$ brought about by the drift of points at the rate $e\boldsymbol{E}/m$ across its surface. This net loss is $\mathrm{div}_c(nf\,e\boldsymbol{E}/m)\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}$.

(3) In addition the quasi-discontinuous change in velocity that occurs when an electron encounters a molecule is represented by the abrupt displacement of its velocity point to another, usually distant, region of velocity space. Thus encounters remove electrons from or introduce electrons to the class $nf\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}$. The net rate of loss is denoted symbolically by $S\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}$.

When these terms are assembled into an equation and the common factor $\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}$ is removed the following equation is obtained

$$\mathrm{d}(nf)/\mathrm{d}t + \mathrm{div}_r(nf\,\boldsymbol{c}) + \mathrm{div}_c(nf\,e\boldsymbol{E}/m) + S = 0. \tag{1}$$

Equation (1) is the Maxwell–Boltzmann equation for nf. Since n and $e\boldsymbol{E}/m$ are independent of c, this equation is equivalent to (Chapman and Cowling 1952)

$$\mathrm{d}(nf)/\mathrm{d}t + \boldsymbol{c}\,.\,\mathrm{grad}_r(nf) + (e\boldsymbol{E}/m)\,.\,\mathrm{grad}_c(nf) + S = 0. \tag{2}$$

The chief motion of the electrons is the translatory motion between encounters represented by the velocities $\boldsymbol{c}$, but when spatial gradients of nf or an electric force $e\boldsymbol{E}$, or both, are present there is a net transport of electrons across an elementary surface $\mathrm{d}\boldsymbol{S}$ at a rate which we write as $n\,\boldsymbol{W}_{\mathrm{cv}}(c)\,\mathrm{d}\boldsymbol{S}$, where $\boldsymbol{W}_{\mathrm{cv}}(c)$ denotes the velocity of convective flow of the electrons of the shell $(c, \mathrm{d}c)$, in other words $\boldsymbol{W}_{\mathrm{cv}}(c)$ is the mean of the vectors $\boldsymbol{c}$ of the electrons of the shell. The velocity points of the shell cannot therefore be uniformly distributed within the shell $(c, \mathrm{d}c)$ since the resultant of the velocities $\boldsymbol{c}$ that these points represent is not zero. It is common practice to take account of this mean convective velocity by giving $f(\boldsymbol{c}, \boldsymbol{r}, t)$ the form (see, however, Appendix I)

$$f = f_0(c, \boldsymbol{r}, t) + f_1(c, \boldsymbol{r}, t)\cos\theta + \sum_{k=2}^{\infty} f_k(c, \boldsymbol{r}, t)\,\mathrm{P}_k(\cos\theta)$$

$$= f_0 + \sum_{k=1}^{\infty} f_k\,\mathrm{P}_k(\cos\theta),$$

where θ is the angle between the convective velocity $\boldsymbol{W}_{\mathrm{cv}}(c)$ of the electrons of the shell and a velocity $\boldsymbol{c}$. It follows that the point population of the shell is

$$n\,\mathrm{d}\boldsymbol{r}\,(4\pi c^2\,\mathrm{d}c)\int_0^{2\pi}\int_0^{\pi} f\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = (nf_0)(4\pi c^2\,\mathrm{d}c)\,\mathrm{d}\boldsymbol{r} \tag{3}$$

and that the convective speed of these electrons is

$$W_{\mathrm{cv}}(\boldsymbol{r}, c, t) = \frac{n\,\mathrm{d}\boldsymbol{c}\,\mathrm{d}\boldsymbol{r}}{nf_0\,\mathrm{d}\boldsymbol{c}\,\mathrm{d}\boldsymbol{r}}\int_0^{2\pi}\int_0^{\pi} c\cos\theta\,f\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = cf_1/3f_0,$$

where $\mathrm{d}\boldsymbol{c} = 4\pi c^2\,\mathrm{d}c$. For simplicity we write $W_{\mathrm{cv}}(c)$ for $W_{\mathrm{cv}}(\boldsymbol{r}, c, t)$. Since $\boldsymbol{W}_{\mathrm{cv}}(c)$ is a vector we may define a vector $\boldsymbol{f}_1(c, \boldsymbol{r}, t)$ through the relation

$$\boldsymbol{W}_{\mathrm{cv}}(c) = c\boldsymbol{f}_1/3f_0. \tag{4}$$

The next step is to find the form assumed by equation (1) when the element $\mathrm{d}\boldsymbol{c}$ is replaced by a shell $(c, \mathrm{d}c)$ with volume $4\pi c^2\,\mathrm{d}c$. The advantage of this procedure is that it is then unnecessary to consider the directions of the velocities $\boldsymbol{c}$ but only their speeds c.

The rate of change of the population of this class is $\{\mathrm{d}(nf_0)/\mathrm{d}t\}(4\pi c^2\,\mathrm{d}c)\mathrm{d}\boldsymbol{r}$, and consequently in equation (1) this expression replaces $\{\mathrm{d}(nf)/\mathrm{d}t\}\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}$. Since the mean convective velocity of this class of electrons is $\boldsymbol{W}_{\mathrm{cv}}(c)$, it follows that the term

$\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}\,\mathrm{div}_r(nf\,\boldsymbol{c})$ becomes $\mathrm{d}\boldsymbol{r}\,(4\pi c^2\,\mathrm{d}c)\,\mathrm{div}_r(n\,\boldsymbol{W}_{\mathrm{cv}}(c)\,f_0)$ which, from equation (4), is equivalent to

$$\mathrm{d}\boldsymbol{r}\,(4\pi c^2\,\mathrm{d}c)\,\tfrac{1}{3}c\,\mathrm{div}_r(n\boldsymbol{f}_1)\,.$$

We consider the remaining terms $\mathrm{div}_c(nf\,e\boldsymbol{E}/m)$ and S together since each represents a loss of points from the element $\mathrm{d}\boldsymbol{c}$ of velocity space. We seek the form of these terms when $\mathrm{d}\boldsymbol{c}$ is the shell $(c, \mathrm{d}c)$. Consider a spherical surface in velocity space with centre at the origin and radius c. The drift $e\boldsymbol{E}/m$ of points in velocity space causes them to cross the surface c both outwards and inwards. The density of points in velocity space is

$$\mathrm{d}\boldsymbol{r}\,(nf) = \mathrm{d}\boldsymbol{r}\,n\left(f_0 + \sum_{k=1}^{\infty} f_k\,\mathrm{P}_k(\cos\theta)\right)$$

in which $\theta = 0$ is the direction of $\boldsymbol{f}_1$. Take an element $c^2\,\mathrm{d}\omega$ of the spherical surface, where $\mathrm{d}\omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$, and let α be the angle between $e\boldsymbol{E}$ and the vector $\boldsymbol{c}$ which is the axis of $\mathrm{d}\omega$. If the angular coordinates of $e\boldsymbol{E}$ with respect to the direction of $\boldsymbol{f}_1$, that is, the polar axis $\theta = 0$, are (θ', ϕ') then

$$\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')\,.$$

The rate at which points cross $c^2\,\mathrm{d}\omega$ because of the drift $e\boldsymbol{E}/m$ is

$$(c^2\,\mathrm{d}\omega)(\mathrm{d}\boldsymbol{r}\,nf)(eE/m)\cos\alpha$$

so that the total outward flux of points over the whole sphere is at the rate

$$c^2\,\mathrm{d}\boldsymbol{r}\,n\left(\frac{eE}{m}\right)\int_0^{2\pi}\int_0^{\pi} f\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= \left(\frac{4\pi c^2}{3}\frac{eE}{m}f_1\cos\theta'\right)n\,\mathrm{d}\boldsymbol{r} = \left(\frac{4\pi c^2}{3}\frac{e\boldsymbol{E}}{m}\cdot n\boldsymbol{f}_1\right)\mathrm{d}\boldsymbol{r} = \sigma_E(c)\,\mathrm{d}\boldsymbol{r}\,,$$

where

$$\sigma_E(c) = \tfrac{4}{3}\pi c^2(e\boldsymbol{E}/m)\cdot n\boldsymbol{f}_1 \equiv \sigma_E\,. \tag{5}$$

Encounters between electrons and gas molecules cause velocity points to cross the surface of the sphere c in both senses but, except in the state of thermal equilibrium with $e\boldsymbol{E}$ absent, there is a net flux of points inwards. We denote the net rate of inward passage of velocity points from encounters by $\sigma_{\mathrm{coll}}(c)\,\mathrm{d}\boldsymbol{r}$. The total outward flux of points is therefore at the rate $\{\sigma_E(c) - \sigma_{\mathrm{coll}}(c)\}\mathrm{d}\boldsymbol{r}$. It follows that the rate of loss of velocity points from the shell $(c, \mathrm{d}c)$ is

$$\frac{\mathrm{d}}{\mathrm{d}c}\left(\sigma_E(c) - \sigma_{\mathrm{coll}}(c)\right)\mathrm{d}\boldsymbol{r}\,\mathrm{d}c$$

and this expression replaces the pair of terms $\{\mathrm{div}_c(nf\,e\boldsymbol{E}/m) + S\}\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}$ when $\mathrm{d}\boldsymbol{c}$ is the shell $(c, \mathrm{d}c)$. Thus when $\mathrm{d}\boldsymbol{c}$ is a shell $(c, \mathrm{d}c)$ equation (1) becomes

$$\frac{\mathrm{d}(nf_0)}{\mathrm{d}t} + \tfrac{1}{3}c\,\mathrm{div}_r(n\boldsymbol{f}_1) + \frac{1}{4\pi c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\sigma_E(c) - \sigma_{\mathrm{coll}}(c)\right) = 0\,, \tag{6}$$

in which $\sigma_E(c)$ is given by equation (5).

When all encounters are elastic $\sigma_{\text{coll}}(c)$ can be shown to be

$$\sigma_{\text{coll}}(c) = 4\pi nc^2\nu\{(m/M)cf_0+\tfrac{1}{3}\langle C^2\rangle\,\partial f_0/\partial c\}, \tag{7}$$

where $\nu = Ncq_{\text{m}}(c)$, N being the molecular number density and $q_{\text{m}}(c)$ the collision cross section for momentum transfer, m/M is the mass ratio of the electron to the molecule and is assumed in the derivation of equation (7) to be very small in comparison with unity, and $\langle C^2\rangle$ is the mean square velocity of the molecules. When inelastic encounters are present the expression for $\sigma_{\text{coll}}(c)$ is supplemented by terms on the right-hand side of equation (7). These terms are not simple in general.

Equation (6), which is the six-dimensional equation of continuity applied to the volume element $\mathrm{d}\boldsymbol{r}$ and the shell $(c, \mathrm{d}c)$ involves two velocity distribution functions f_0 and f_1 and we require a second equation that involves these functions. Consideration of the gains and losses of momentum to the class $(nf)\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{c}$ when $\mathrm{d}\boldsymbol{c}$ is the shell $(c, \mathrm{d}c)$ leads to the equation

$$\frac{\mathrm{d}(n\boldsymbol{f}_1)}{\mathrm{d}t}+c\,\mathrm{grad}_r\left(nf_0+\frac{2nf_2}{5}\right)+\left(\frac{\partial f_0}{\partial c}+\frac{2}{5c^3}\frac{\partial(c^3f_2)}{\partial c}\right)\frac{n\,e\boldsymbol{E}}{m}+\nu n\boldsymbol{f}_1 = 0\,.$$

In practice the terms in f_2 are considered to be negligible and the working equation is taken to be

$$\frac{\mathrm{d}(n\boldsymbol{f}_1)}{\mathrm{d}t}+c\,\mathrm{grad}_r(nf_0)+\frac{e\boldsymbol{E}}{m}\frac{\partial(nf_0)}{\partial c}+\nu n\boldsymbol{f}_1 = 0\,. \tag{8}$$

Equations (6) and (8) respectively will be referred to as the scalar and vector equations for the shell.

When all encounters are elastic $\nu = Ncq_{\text{m}}(c)$ in equation (8), but when inelastic encounters are present the elastic momentum transfer cross section $q_{\text{m}}(c)$ is replaced by an equivalent cross section $q'_{\text{m}}(c)$ which, in practice, is little different from $q_{\text{m}}(c)$.

Equation (8) is equivalent to

$$-n\boldsymbol{f}_1 = \frac{c}{\nu}\,\mathrm{grad}_r(nf_0)+\boldsymbol{V}\frac{\partial(nf_0)}{\partial c}+\frac{1}{\nu}\frac{\mathrm{d}(n\boldsymbol{f}_1)}{\mathrm{d}t} \tag{9a}$$

in which

$$\boldsymbol{V} = e\boldsymbol{E}/m\nu\,. \tag{9b}$$

In laboratory experiments the collision frequency ν is large and the term $(1/\nu)\mathrm{d}(n\boldsymbol{f}_1)/\mathrm{d}t$ is negligibly small compared with the other terms on the right-hand side when $\boldsymbol{E}$ is a static field. This would not be so if $\boldsymbol{E}$ were to oscillate with an angular frequency comparable with ν, but since we are concerned here with cases where $\boldsymbol{E}$ is both static and uniform we shall neglect this term and write

$$n\boldsymbol{f}_1 = -(c/\nu)\mathrm{grad}_r(nf_0)-\boldsymbol{V}\,\partial(nf_0)/\partial c\,. \tag{10}$$

It follows from (4) that equation (10) is equivalent to

$$\boldsymbol{W}_{\text{cv}}(c)\,(nf_0) = -(\tfrac{1}{3}c^2/\nu)\mathrm{grad}_r(nf_0)-\tfrac{1}{3}c\boldsymbol{V}\,\partial(nf_0)/\partial c\,. \tag{11}$$

We now use equation (10) to eliminate f_1 from the scalar equation (6) which then becomes

$$\frac{\mathrm{d}(nf_0)}{\mathrm{d}t} - \mathrm{div}_r\left\{\mathrm{grad}_r\left(\frac{c^2}{3\nu}nf_0\right) + \tfrac{1}{3}c\boldsymbol{V}\frac{\partial(nf_0)}{\partial c}\right\}$$
$$-\frac{1}{c^2}\frac{\partial}{\partial c}\left(\tfrac{1}{3}c^3\boldsymbol{V}\cdot\mathrm{grad}_r(nf_0) + \tfrac{1}{3}c^2\nu V^2\frac{\partial(nf_0)}{\partial c} + \frac{\sigma_{\mathrm{coll}}(c)}{4\pi}\right) = 0, \qquad (12)$$

in which use has been made of equations (5) and (9b). When all encounters are elastic $\sigma_{\mathrm{coll}}(c)$ can be replaced by the expression (7) and equation (12) then becomes

$$\frac{\mathrm{d}(nf_0)}{\mathrm{d}t} - \mathrm{div}_r\left\{\mathrm{grad}_r\left(\frac{c^2}{3\nu}nf_0\right) + \tfrac{1}{3}c\boldsymbol{V}\frac{\partial(nf_0)}{\partial c}\right\}$$
$$-\frac{1}{3c^2}\frac{\partial}{\partial c}\left\{c^2\nu\left(\frac{c\boldsymbol{V}}{\nu}\cdot\mathrm{grad}_r(nf_0) + (V^2+\langle C^2\rangle)\frac{\partial(nf_0)}{\partial c} + \frac{3mc(nf_0)}{M}\right)\right\} = 0. \qquad (13)$$

The simplest application of equation (6) and its special forms (12) and (13) is to a steady uniform stream of electrons moving in a uniform field. In this example time and space differential coefficients vanish and equation (6) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}c}\left(\sigma_E(c) - \sigma_{\mathrm{coll}}(c)\right) = 0,$$

whence $\sigma_E - \sigma_{\mathrm{coll}} = \text{const.} = 0$ since both σ_E and σ_{coll} approach zero as $c \to \infty$. When all encounters are elastic this equation is equivalent to

$$(V^2+\langle C^2\rangle)\mathrm{d}f_0/\mathrm{d}c + (3m/M)cf_0 = 0,$$

whence

$$f_0 = A\exp\left(-\frac{3m}{M}\int_0^c \frac{c\,\mathrm{d}c}{V^2+\langle C^2\rangle}\right), \qquad (14)$$

where A is a constant. Equation (14) is Davydov's distribution function. In the steady state the net point flux $\sigma_E(c) - \sigma_{\mathrm{coll}}(c)$ outwards across all the spherical surfaces c is zero in this example.

As a step in the fuller discussion of equations (6) and (13) we first consider some properties of an isolated travelling group.

III. Isolated Travelling Group of Electrons

Consider a group of n_0 electrons that drift through a gas in a constant and uniform electric field $\boldsymbol{E}$. It is assumed that the group travels in an extensive region such that its total population

$$n_0 = \int_r n(\boldsymbol{r}, t)\,\mathrm{d}\boldsymbol{r}$$

resides within a large surface Σ on and near which n and its derivatives are zero.

The velocities of the electrons of the group are represented by n_0 velocity points in velocity space and the number of these points within an element $\mathrm{d}\boldsymbol{c}$ is given by $n_0 f^*(\boldsymbol{c},t)\,\mathrm{d}\boldsymbol{c}$. If $\mathrm{d}\boldsymbol{c}$ is the shell $(c, \mathrm{d}c)$ then because of the presence of a drift velocity the shell is not uniformly populated with velocity points. We therefore give the distribution function for the whole group $f^*(\boldsymbol{c},t)$ the form

$$f^*(\boldsymbol{c},t) = f_0^*(c,t) + \sum_{k=1}^{\infty} f_k^*(c,t)\,\mathrm{P}_k(\cos\theta),$$

where θ is the angle between a velocity vector $\boldsymbol{c}$ of the shell and the convective velocity $\boldsymbol{W}_{\mathrm{cv}}(c)$ of all the electrons of the shell. It follows that the population of the shell is

$$\int_\omega n_0 f^* c^2\,\mathrm{d}c\,\mathrm{d}\omega = n_0(c^2\,\mathrm{d}c)\int_0^\pi\int_0^{2\pi} f^*\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = (4\pi c^2\,\mathrm{d}c) n_0 f_0^*.$$

Since the total population of the group is

$$n_0 = 4\pi n_0 \int_0^\infty f_0^* c^2\,\mathrm{d}c,$$

it follows that

$$4\pi\int_0^\infty f_0^* c^2\,\mathrm{d}c = 1. \tag{15}$$

The mean value of the components of the velocities of the electrons of the shell along $\boldsymbol{f}_1$ is

$$W(c) = \frac{n_0}{4\pi n_0 f_0^* c^2\,\mathrm{d}c}\int_\omega (c\cos\theta) f c^2\,\mathrm{d}c\,\mathrm{d}\omega = c f_1^*/3 f_0^*,$$

whence

$$\boldsymbol{W}(c) = c\boldsymbol{f}_1^*/3 f_0^*. \tag{16}$$

The velocity $\boldsymbol{W}$ of the centroid of the whole group is the mean of $\boldsymbol{W}(c)$ taken over all shells. Thus

$$n_0 \boldsymbol{W} = 4\pi n_0 \int_0^\infty \boldsymbol{W}(c) f_0^* c^2\,\mathrm{d}c,$$

whence

$$\boldsymbol{W} = \tfrac{4}{3}\pi\int_0^\infty (c\boldsymbol{f}_1^*) c^2\,\mathrm{d}c. \tag{17}$$

The distribution function $f^*(\boldsymbol{c},t)$ for the group as a whole is related to the distribution function $f(\boldsymbol{c},\boldsymbol{r},t)$ for the electrons $n\,\mathrm{d}\boldsymbol{r}$ that constitute the group, since the total number of velocity points $n_0 f^*\,\mathrm{d}\boldsymbol{c}$ in the element $\mathrm{d}\boldsymbol{c}$ is the sum of the contributions $n\,\mathrm{d}\boldsymbol{r}\, f\,\mathrm{d}\boldsymbol{c}$, whence

$$n_0 f^* = \int_r n f\,\mathrm{d}\boldsymbol{r}, \tag{18}$$

so that f^* is the mean velocity distribution taken over all elementary volumes $\mathrm{d}\boldsymbol{r}$.

It follows from equation (18) that

$$n_0 f_0^* = \int_r nf_0 \, \mathrm{d}\boldsymbol{r} \qquad \text{and} \qquad n_0 f_1^* = \int_r nf_1 \, \mathrm{d}\boldsymbol{r}. \tag{19}$$

Consider equation (6). If each term is integrated over the whole of configuration space then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_r nf_0 \, \mathrm{d}\boldsymbol{r}\right) = \frac{\mathrm{d}(n_0 f_0^*)}{\mathrm{d}t} = n_0 \frac{\mathrm{d}f_0^*}{\mathrm{d}t};$$

$$\tfrac{1}{3}c \int_r \mathrm{div}_r(n\boldsymbol{f}_1) \, \mathrm{d}\boldsymbol{r} = \tfrac{1}{3}c \int_\Sigma n\boldsymbol{f}_1 \boldsymbol{.} \mathrm{d}\boldsymbol{S} = 0,$$

since n vanishes on the large surface Σ; and

$$\frac{1}{4\pi c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\int_r \{\sigma_E(c) - \sigma_{\mathrm{coll}}(c)\} \, \mathrm{d}\boldsymbol{r}\right) = \frac{1}{4\pi c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\overset{*}{\sigma}_E(c) - \overset{*}{\sigma}_{\mathrm{coll}}(c)\right),$$

where

$$\overset{*}{\sigma}_E(c) = \int_r \sigma_E(c) \, \mathrm{d}\boldsymbol{r} = \frac{4\pi c^2}{3}\frac{e\boldsymbol{E}}{m} \boldsymbol{.} \left(\int_r n\boldsymbol{f}_1 \, \mathrm{d}\boldsymbol{r}\right) = \frac{4\pi c^2}{3}\frac{e\boldsymbol{E}}{m} \boldsymbol{.} n_0 \boldsymbol{f}_1^*.$$

$\sigma_E^*(c)$ is the outward flux due to $e\boldsymbol{E}$ of points across the sphere c in velocity space, for the group as a whole. Similarly $\sigma_{\mathrm{coll}}^*(c)$ is found from $\sigma_{\mathrm{coll}}(c)$ by replacing nf_0 by $n_0 f_0^*$ and relates to the whole group.

Thus, for the whole group, equation (6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f_0^*(c,t)\right) = -\frac{1}{4\pi n_0 c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\overset{*}{\sigma}_E(c,t) - \overset{*}{\sigma}_{\mathrm{coll}}(c,t)\right). \tag{20}$$

When $e\boldsymbol{E}$, and therefore $\sigma_E^*(c)$, is zero the condition of equilibrium is one in which $\sigma_{\mathrm{coll}}^*(c)$ is zero. The distribution function is that of Maxwell, as is evident from equation (7) when all encounters are elastic.

If $e\boldsymbol{E}$ is applied at $t = 0$ then $\sigma_E^*(c,t)$ is established and the left-hand side of (20) is no longer zero. Both $\sigma_{\mathrm{coll}}^*(c,t)$ and $\sigma_E^*(c,t)$ change with time and a new equilibrium condition is approached with another distribution function that satisfies the condition $\sigma_E^*(c) = \sigma_{\mathrm{coll}}^*(c)$ for all speeds c. The net point flux over all spherical surfaces c is then zero. It follows that when all encounters are elastic the equilibrium distribution function for the group as a whole is Davydov's function as given by equation (14).

We consider next the form of $\boldsymbol{f}_1^*$ that appears in equation (17) for $\boldsymbol{W}$. Integrate each term of equation (9a) over the whole of configuration space and apply the second of equations (19). It then follows that

$$n_0 \boldsymbol{f}_1^* = -(c/\nu) \int_r \mathrm{grad}_r(nf_0) \, \mathrm{d}\boldsymbol{r} - n_0 \boldsymbol{V} \mathrm{d}f_0^*/\mathrm{d}c,$$

since $e\boldsymbol{E}$ is independent of position by hypothesis.

Consider

$$\int_r \operatorname{grad}_r(nf_0)\,\mathrm{d}\boldsymbol{r} = \iiint\limits_{-\infty}^{\infty}\left(\boldsymbol{i}_x\frac{\partial(nf_0)}{\partial x}+\boldsymbol{i}_y\frac{\partial(nf_0)}{\partial y}+\boldsymbol{i}_z\frac{\partial(nf_0)}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

But

$$\iint\limits_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{d}z\left(\int_{-\infty}^{\infty}\frac{\partial(nf_0)}{\partial x}\,\mathrm{d}x\right) = \iint\limits_{-\infty}^{\infty}[nf_0]_{-\infty}^{\infty}\,\mathrm{d}y\,\mathrm{d}z = 0\,,$$

since n vanishes at infinity. Likewise the remaining integrals vanish and it follows that

$$f_1^* = -V\,\mathrm{d}f_0^*/\mathrm{d}c\,, \tag{21}$$

where, from equation (9b), $V = e\boldsymbol{E}/m\nu$.

Equation (17) for the drift velocity $\boldsymbol{W}$, which as already remarked is the velocity of the centroid of the group, now becomes

$$\begin{aligned}\boldsymbol{W} &= -\frac{4\pi}{3}\int_0^{\infty} cV\frac{\mathrm{d}f_0^*}{\mathrm{d}c}c^2\,\mathrm{d}c = -\frac{4\pi}{3}\frac{e\boldsymbol{E}}{m}\int_0^{\infty}\frac{c^3}{\nu}\frac{\mathrm{d}f_0^*}{\mathrm{d}c}\,\mathrm{d}c \\ &= -\frac{4\pi}{3}\frac{e\boldsymbol{E}}{mN}\int_0^{\infty}\frac{c^2}{q_{\mathrm{m}}(c)}\frac{\mathrm{d}f_0^*}{\mathrm{d}c}\,\mathrm{d}c\,. \end{aligned}\tag{22}$$

An alternative formulation obtained by partial integration is

$$\boldsymbol{W} = \frac{4\pi}{3}\frac{e\boldsymbol{E}}{mN}\left\{-\frac{c^2}{q_{\mathrm{m}}(c)}f_0^*\Big|_0^{\infty} + \int_0^{\infty}\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{c^2}{q_{\mathrm{m}}(c)}\right)f_0^*\,c^2\,\mathrm{d}c\right\}.$$

If $c^2f_0^*/q_{\mathrm{m}}(c)$ is zero both when $c = 0$ and ∞ then

$$\boldsymbol{W} = \frac{4\pi}{3}\frac{e\boldsymbol{E}}{mN}\int_0^{\infty}\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{c^2}{q_{\mathrm{m}}(c)}\right)f_0^*\,c^2\,\mathrm{d}c = \frac{e\boldsymbol{E}}{3mN}\left\langle\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{c^2}{q_{\mathrm{m}}(c)}\right)\right\rangle. \tag{23}$$

The angle brackets in the final term on the right indicate the mean value of the enclosed quantity taken over all shells $(c, \mathrm{d}c)$.

When the group has attained its equilibrium distribution and if all encounters are elastic, f_0^* in equations (22) and (23) is independent of the time and is Davydov's function.

IV. Steady Stream from Spatially Limited Source: Calculation of Current Density

Since in Section IX the convective velocity $\boldsymbol{W}_{\mathrm{cv}} \equiv \boldsymbol{W}_{\mathrm{cv}}(\boldsymbol{r})$ is considered in relation to the current density $\boldsymbol{J}$, it is convenient to comment on this matter briefly at this point. We consider the case of a steady stream from a source which is limited in dimensions. The stream proceeds through the gas to infinity in a uniform electric field $\boldsymbol{E}$ and spreads by diffusion as it travels. Then n and its derivatives approach zero as $\boldsymbol{r} \to \infty$. Consider equation (6) in which we replace $\frac{1}{3}cf_1$ by $\boldsymbol{W}_{\mathrm{cv}}(c)\,f_0$ from equation (4). Since the stream is steady $\mathrm{d}(nf_0)/\mathrm{d}t$ is zero, so that in this application equation (6) becomes

$$\operatorname{div}_r\{nf_0\,\boldsymbol{W}_{\mathrm{cv}}(c)\}+\frac{1}{4\pi c^2}\frac{\mathrm{d}}{\mathrm{d}c}\Big(\sigma_E(c)-\sigma_{\mathrm{coll}}(c)\Big)=0\,.$$

Integrate each term of this equation over all shells of velocity space.† It then follows that

$$\operatorname{div}_r\left(n\,4\pi\int_0^\infty \boldsymbol{W}_{\mathrm{cv}}(c)\,f_0(c,\boldsymbol{r},t)\,c^2\,\mathrm{d}c\right)=0\,,$$

that is,

$$\operatorname{div}_r\{n\boldsymbol{W}_{\mathrm{cv}}(\boldsymbol{r})\}=0\,, \tag{24}$$

in which the total convective velocity $\boldsymbol{W}_{\mathrm{cv}}(\boldsymbol{r})$ at position $\boldsymbol{r}$ is

$$\boldsymbol{W}_{\mathrm{cv}}(\boldsymbol{r})=4\pi\int_0^\infty \boldsymbol{W}_{\mathrm{cv}}(c)\,f_0(c,\boldsymbol{r},t)\,c^2\,\mathrm{d}c \tag{25}$$

and $\boldsymbol{W}_{\mathrm{cv}}(c)$ is given by equation (11).

The current density in the stream is $\boldsymbol{J}=en\boldsymbol{W}_{\mathrm{cv}}$ and consequently equation (24) is equivalent to

$$\operatorname{div}_r\boldsymbol{J}=0\,. \tag{26}$$

If the source is surrounded by a closed surface σ, the total current i from the source is therefore the integral over σ of $\boldsymbol{J}\,.\,\mathrm{d}\boldsymbol{\sigma}$ and, because $\operatorname{div}_r\boldsymbol{J}=0$ throughout the space external to σ, the same current i flows across any closed surface surrounding σ. Thus the current across any elementary surface $\mathrm{d}\boldsymbol{S}$ or across any finite surface is to be calculated as

$$\int\boldsymbol{J}\,.\,\mathrm{d}\boldsymbol{S}=e\int n\boldsymbol{W}_{\mathrm{cv}}(\boldsymbol{r})\,.\,\mathrm{d}\boldsymbol{S}\,.$$

We can now resume the discussion of the full scalar equation in the case where all encounters are elastic.

V. Solution of Scalar Equation when All Encounters are Elastic

We seek a solution of equation (13) that describes an isolated travelling group of n_0 electrons whose centroid travels along the $+0z$ axis at speed W given by equations (22) and (23). It is also assumed that the velocity distribution function f_0^* for the group as a whole has attained its stable Davydov form as discussed in Section III.

Adopt a cartesian system of coordinates with $+0z$ parallel to $e\boldsymbol{E}$ and $\boldsymbol{V}$ so that $V=V_z$. In terms of these coordinates equation (13) becomes

$$-\frac{\mathrm{d}(nf_0)}{\mathrm{d}t}+\nabla^2\left(\frac{c^2}{3\nu}nf_0\right)+\tfrac{1}{3}cV\frac{\partial}{\partial z}\left(\frac{\partial(nf_0)}{\partial c}\right)$$
$$+\frac{1}{3c^2}\frac{\partial}{\partial c}\left\{c^2\nu\left(\frac{cV}{\nu}\frac{\partial(nf_0)}{\partial z}+(V^2+\langle C^2\rangle)\frac{\partial(nf_0)}{\partial c}+\frac{3mc(nf_0)}{M}\right)\right\}=0\,. \tag{27}$$

† That is to say, form the integral $4\pi\int_0^\infty(\;)c^2\,\mathrm{d}c$.

The constraints upon the solution that are required by its representation of a travelling group are:

(i) $n_0 f_0^* = \int_{\boldsymbol{r}} n f_0 \,\mathrm{d}\boldsymbol{r}$,

(ii) $1 = 4\pi \int_0^\infty f_0^* c^2 \,\mathrm{d}c = 4\pi \int_0^\infty f_0(c, \boldsymbol{r}, t)\, c^2 \,\mathrm{d}c$, and

(iii) (nf_0) and its spatial derivatives vanish as $\boldsymbol{r} \to \infty$.

We next consider a cartesian system with its origin travelling with the centroid of the group and with its axes parallel to the corresponding axes in the system at rest. If x', y', and z' are coordinates in the moving system, then $x' = x$, $y' = y$, $z' = z - Wt$, and $\partial^k/\partial z'^k = \partial^k/\partial z^k$. In order to distinguish time differentials in the two systems, that relating to a volume element $\mathrm{d}\boldsymbol{r}$ at rest is designated by $\mathrm{d}/\mathrm{d}t$ while that referring to an element $\mathrm{d}\boldsymbol{r}$ in the moving system by $\partial/\partial t$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = \frac{\partial}{\partial t} - W\frac{\partial}{\partial z} = \frac{\partial}{\partial t} - W\frac{\partial}{\partial z'},$$

and equation (27) becomes

$$-\frac{\partial(nf_0)}{\partial t} + \nabla^2\left(\frac{c^2}{3\nu} nf_0\right) + \tfrac{1}{3}cV\frac{\partial}{\partial z'}\left(\frac{\partial(nf_0)}{\partial c}\right) + W\frac{\partial(nf_0)}{\partial z'}$$
$$+ \frac{1}{3c^2}\frac{\partial}{\partial c}\left\{c^2\nu\left(\frac{cV}{\nu}\frac{\partial(nf_0)}{\partial z'} + (V^2 + \langle C^2\rangle)\frac{\partial(nf_0)}{\partial c} + \frac{3mc(nf_0)}{M}\right)\right\} = 0. \qquad (28)$$

Before attempting to solve the complete equation (27) it is convenient to illustrate by two simple but useful examples the procedure to be adopted. We therefore initially consider the special case of equation (27) in the absence of an electric force. We first note that in the complete equation (27) the pair of terms

$$(V^2 + \langle C^2\rangle)\partial(nf_0)/\partial c + 3mc(nf_0)/M$$

can be replaced by the single term

$$(V^2 + \langle C^2\rangle)\{\partial(f_0/f_0^*)/\partial c\}(nf_0^*)$$

in which f_0^* satisfies the equation

$$(V^2 + \langle C^2\rangle)\mathrm{d}f_0^*/\mathrm{d}c + 3mc f_0^*/M = 0$$

and is therefore Davydov's function. Equation (28) now becomes

$$-\frac{\partial(nf_0)}{\partial t} + \frac{c^2}{3\nu}\nabla^2(nf_0) + \tfrac{1}{3}cV\frac{\partial}{\partial z}\left(\frac{\partial(nf_0)}{\partial c}\right) + W\frac{\partial(nf_0)}{\partial z}$$
$$+ \frac{1}{3c^2}\frac{\partial}{\partial c}\left\{c^2\nu\left(\frac{cV}{\nu}\frac{\partial(nf_0)}{\partial z} + (V^2 + \langle C^2\rangle) f_0^*\frac{\partial(nf_0/f_0^*)}{\partial c}\right)\right\} = 0, \qquad (29)$$

in which for convenience $\partial/\partial z$ is written for $\partial/\partial z'$.

Motion with No Electric Field $\boldsymbol{E}$

In this case equations (28) and (29) become

$$-\frac{\mathrm{d}(nf_0)}{\mathrm{d}t}+\frac{c^2}{3\nu}\nabla^2(nf_0)+\frac{1}{3c^2}\frac{\partial}{\partial c}\left(c^2\nu\langle C^2\rangle f_0^*\frac{\partial(nf_0/f_0^*)}{\partial c}\right)=0, \tag{30}$$

in which f_0^* is now Maxwell's distribution function. This equation shows that f_0 in the presence of spatial derivatives of n is not the same as f_0^*, because if f_0 were equal to f_0^* the equation would reduce to

$$-\mathrm{d}n/\mathrm{d}t+(c^2/3\nu)\nabla^2 n=0,$$

which is absurd since n is not a function of c. It was noted that in a uniform steady stream f_0 becomes f_0^* and its form is given by equation (14). These facts imply that when temporal and spatial derivatives are present f_0 is a function of these derivatives such that the mean distribution function for the group is Maxwell's function f_0^*. Moreover, the form of equation (30) suggests the representation

$$f_0=f_0^*\left(1+\sum_{k=1}^{\infty}a_{2k}(c)\,n^{-1}\,(\nabla^2)^k n\right),$$

that is,

$$nf_0=nf_0^*+\sum_{k=1}^{\infty}f_0^*\,a_{2k}(c)\,(\nabla^2)^k n. \tag{31}$$

This representation, when constraint (iii) above is imposed, conforms to constraint (i) since

$$\int_r nf_0\,\mathrm{d}\boldsymbol{r}=f_0^*\int_r n\,\mathrm{d}\boldsymbol{r}=n_0 f_0^*.$$

In order to comply with constraint (ii) it is necessary to determine the $a_{2k}(c)$ so that for all $k>0$

$$4\pi\int_0^\infty a_{2k}(c)\,f_0^*\,c^2\,\mathrm{d}c\equiv\langle a_{2k}(c)\rangle=0. \tag{32}$$

Replace (nf_0) in equation (30) by its representation in equation (31) and regroup terms. Equation (30) then becomes

$$\begin{aligned}&\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(c^2\nu\langle C^2\rangle f_0^*\frac{\mathrm{d}(f_0^*/f_0^*)}{\mathrm{d}c}\right)n\\&+\left(-\frac{\mathrm{d}n}{\mathrm{d}t}+\frac{c^2}{3\nu}\nabla^2 n\right)f_0^*+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(c^2\nu\langle C^2\rangle f_0^*\frac{\mathrm{d}\{a_2(c)\}}{\mathrm{d}c}\right)\nabla^2 n\\&+\sum_{k=1}^{\infty}\left\{\nabla^2\left(-\frac{\mathrm{d}n}{\mathrm{d}t}+\frac{c^2}{3\nu}\nabla^2 n\right)a_{2k}(c)\,f_0^*+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(c^2\nu\langle C^2\rangle f_0^*\frac{\mathrm{d}\{a_{2k+2}(c)\}}{\mathrm{d}c}\right)(\nabla^2)^{k+1}n\right\}=0.\end{aligned} \tag{33}$$

In this equation the coefficient of n vanishes because $\mathrm{d}(f_0^*/f_0^*)/\mathrm{d}c\equiv\mathrm{d}(1)/\mathrm{d}c=0$.

If the second group of terms in $\nabla^2 n$ and $-\mathrm{d}n/\mathrm{d}t$ is equated to zero, the following equation is obtained:

$$\left(-\frac{\mathrm{d}n}{\mathrm{d}t}+\frac{c^2}{3\nu}\nabla^2 n\right)f_0^* + \frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(c^2\nu\langle C^2\rangle f_0^*\frac{\mathrm{d}\{a_2(c)\}}{\mathrm{d}c}\right)\nabla^2 n = 0. \tag{34}$$

Integrate each term of equation (34) over all shells of velocity space. Then, if $\mathrm{d}\{a_2(c)\}/\mathrm{d}c$ is finite at $c=0$ and $c^2\nu f_0^*(c)\,\mathrm{d}\{a_2(c)\}/\mathrm{d}c$ approaches zero as $c\to\infty$, it follows that

$$-\mathrm{d}n/\mathrm{d}t + D\,\nabla^2 n = 0, \tag{35}$$

where the isotropic coefficient of diffusion D is given by

$$D = 4\pi\int_0^\infty (c^2/3\nu)f_0^*\,c^2\,\mathrm{d}c. \tag{36}$$

Equation (35) is therefore the equation to be satisfied by the function $n(\boldsymbol{r},t)$ that gives the number density. Since, according to equation (35), $n(\boldsymbol{r},t)$ is such that $\mathrm{d}n(\boldsymbol{r},t)/\mathrm{d}t = D\,\nabla^2 n(\boldsymbol{r},t)$, $\mathrm{d}n/\mathrm{d}t$ can be eliminated from equation (33) which then takes the form

$$\left\{\left(\frac{c^2}{3\nu}-D\right)f_0^* + \frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(c^2\nu\langle C^2\rangle f_0^*\frac{\mathrm{d}\{a_2(c)\}}{\mathrm{d}c}\right)\right\}\nabla^2 n$$

$$+\sum_{k=1}^{\infty}\left\{\left(\frac{c^2}{3\nu}-D\right)a_{2k}(c)f_0^* + \frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(c^2\nu\langle C^2\rangle f_0^*\frac{\mathrm{d}\{a_{2k+2}(c)\}}{\mathrm{d}c}\right)\right\}(\nabla^2)^{k+1}n = 0. \tag{37}$$

In order to determine the coefficients $a_2(c)$, $a_4(c)$, . . ., we equate separately to zero the coefficients of the spatially dependent terms $\nabla^2 n$, $(\nabla^2)^2 n$, . . . $(\nabla^2)^k n$, ...

Thus $a_2(c)$ is to be found as a solution, subject to equation (32), of the equation

$$\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(c^2\nu\langle C^2\rangle f_0^*\frac{\mathrm{d}\{a_2(c)\}}{\mathrm{d}c}\right) = \left(D-\frac{c^2}{3\nu}\right)f_0^*, \tag{37a}$$

whence, since it is assumed that $c^2\nu\langle C^2\rangle f_0^*(c)\,\mathrm{d}\{a_2(c)\}/\mathrm{d}c$ is zero at $c=0$,

$$\frac{\mathrm{d}\{a_2(c)\}}{\mathrm{d}c} = \frac{3}{c^2\nu\langle C^2\rangle f_0^*(c)}\int_0^c\left(D-\frac{x^2}{3\nu(x)}\right)f_0^*(x)\,x^2\,\mathrm{d}x.$$

It follows that

$$a_2(c) = \phi_2(c) - \langle\phi_2(c)\rangle, \tag{38}$$

where

$$\phi_2(c) = \int_0^c \frac{3}{y^2\nu(y)\,f_0^*(y)\,\langle C^2\rangle}\left\{\int_0^y\left(D-\frac{x^2}{3\nu(x)}\right)f_0^*(x)\,x^2\,\mathrm{d}x\right\}\mathrm{d}y.$$

The constant of integration is given the form $\langle\phi_2(c)\rangle$ to make $a_2(c)$ satisfy (32).

Equation (37) is thus reduced to the summation. The coefficients of the terms $(\nabla^2)^{k+1}n$ are now separately equated to zero and provide equations from which any term $a_{2k+2}(c)$ can be seen to depend upon the preceding term $a_{2k}(c)$. Thus

$$a_{2k+2}(c) = \phi_{2k+2}(c) - \langle\phi_{2k+2}(c)\rangle\,, \tag{39}$$

where

$$\phi_{2k+2}(c) = \int_0^c \frac{3}{\langle C^2\rangle\,\nu(y)\,y^2 f_0^*(y)}\left\{\int_0^y \left(D - \frac{x^2}{3\nu}\right) a_{2k}(x)\, f_0^*(x)\, x^2\, \mathrm{d}x\right\} \mathrm{d}y\,.$$

Each function $a_{2k+2}(c)$ is therefore determined, in principle, when the preceding function $a_{2k}(c)$ is known. Since $a_2(c)$ can be found from equation (38) when $q_{\mathrm{m}}(c)$ is known, it follows that in principle as many of $a_4(c)$, $a_6(c)$, ... may be found as required.

Thus, in summary, the relation

$$nf_0 = nf_0^* + \sum_{k=1}^{\infty} f_0^*\, a_{2k}(c)\,(\nabla^2)^k n$$

satisfies equation (29) provided the sequence is convergent and the $a_{2k}(c)$ satisfy the relations (39) and $n(\boldsymbol{r},t)$ satisfies equation (35). It follows from equation (32) that, because $c^2 f_0^*(c)$ is always positive, $a_{2k}(c)$ cannot maintain the same sign throughout the range of integration and, from equation (39), that the sign of $a_{2k+2}(c)$ depends upon the algebraic magnitude of $\phi_{2k+2}(c)$ relative to $\langle\phi_{2k+2}(c)\rangle$. We recall that in any solution of equation (29) we have assumed that at $t = 0$ the distribution function of the isolated group as a whole has already reached its equilibrium form f_0^*.

We now consider the second relatively simple special case where the uniform electric force $e\boldsymbol{E}$ is present but only a single spatial coordinate (z) is relevant.

Motion in One Dimension

Let

$$Q(z,c,t) = \int\!\!\int_{-\infty}^{\infty} nf_0\, \mathrm{d}x\, \mathrm{d}y\,, \qquad q(z,t) = \int\!\!\int_{-\infty}^{\infty} n\, \mathrm{d}x\, \mathrm{d}y\,. \tag{40}$$

Form the integral $\iint_{-\infty}^{\infty}(\;)\,\mathrm{d}x\,\mathrm{d}y$ of each term of equation (29), which is thereby transformed to

$$-\frac{\partial Q}{\partial t} + \frac{c^2}{3\nu}\frac{\partial^2 Q}{\partial z^2} + \tfrac{1}{3}cV\frac{\partial^2 Q}{\partial z\,\partial c} + W\frac{\partial Q}{\partial z}$$
$$+\frac{1}{3c^2}\frac{\partial}{\partial c}\left\{c^2\nu\left(\frac{cV}{\nu}\frac{\partial Q}{\partial z} + (V^2 + \langle C^2\rangle)\, f_0^*\,\frac{\mathrm{d}(Q/f_0^*)}{\mathrm{d}c}\right)\right\} = 0 \tag{41}$$

because n and its derivatives vanish as $\boldsymbol{r} \to \infty$.

Replace Q in equation (41) by an assumed expansion

$$Q(z,c,t) = qf_0^* + \sum_{k=1}^{\infty} f_0^*(c)\, b_k(c)\frac{\partial^k}{\partial z^k} q(z,t)\,. \tag{42}$$

After regrouping and ordering of terms, equation (41) then becomes

$$\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(c^2\nu f_0^*\frac{\mathrm{d}(f_0^*/f_0^*)}{\mathrm{d}c}\right)q$$

$$+\left[\tfrac{1}{3}cV\frac{\mathrm{d}f_0^*}{\mathrm{d}c}+Wf_0^*+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}f_0^*+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_1(c)}{\mathrm{d}c}\right)\right\}\right]\frac{\partial q}{\partial z}$$

$$+\left(-\frac{\partial q}{\partial t}f_0^*+\left[\frac{c^2}{3\nu}f_0^*+\tfrac{1}{3}cV\frac{\mathrm{d}(b_1 f_0^*)}{\mathrm{d}c}+Wb_1 f_0^*\right.\right.$$

$$\left.\left.+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}b_1 f_0^*+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_2}{\mathrm{d}c}\right)\right\}\right]\frac{\partial^2 q}{\partial z^2}\right)$$

$$+\sum_{k=1}^{\infty}\frac{\partial^k}{\partial z^k}\left(-\frac{\partial q}{\partial t}f_0^* b_k+\left[\frac{c^2}{3\nu}b_k f_0^*+\tfrac{1}{3}cV\frac{\mathrm{d}(b_{k+1}f_0^*)}{\mathrm{d}c}+Wb_{k+1}f_0^*\right.\right.$$

$$\left.\left.+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}b_{k+1}f_0^*+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}\{b_{k+2}(c)\}}{\mathrm{d}c}\right)\right\}\right]\frac{\partial^2 q}{\partial z^2}\right)=0. \tag{43}$$

In this regrouping the term $\partial q/\partial t$ is not associated with the group in q or $\partial q/\partial z$ since the first vanishes identically and the second when $eE = 0$. It is necessary that equations (43) and (33) should be consistent.

As before the first term in equation (43) vanishes because $\mathrm{d}(f_0^*/f_0^*)/\mathrm{d}c = 0$. Next, equating to zero the coefficient of $\partial q/\partial z$ and deriving $b_1(c)$ from the resulting equation,

$$Wf_0^*+\tfrac{1}{3}cV\frac{\mathrm{d}f_0^*}{\mathrm{d}c}+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}f_0^*+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_1(c)}{\mathrm{d}c}\right)\right\}=0. \tag{44}$$

This gives

$$-c^2\nu(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_1}{\mathrm{d}c}=c^2\nu.\frac{cV}{\nu}f_0^*+3\int_0^c\left(\tfrac{1}{3}xV\frac{\mathrm{d}f_0^*}{\mathrm{d}x}+Wf_0^*\right)x^2\,\mathrm{d}x \tag{45}$$

on the assumption that $c^2\nu(V^2+\langle C^2\rangle)\,\mathrm{d}b_1/\mathrm{d}c = 0$ when $c = 0$. Moreover, as $c \to \infty$

$$3\int_0^c\left(\tfrac{1}{3}xV\frac{\mathrm{d}f_0^*}{\mathrm{d}c}+Wf_0^*\right)x^2\,\mathrm{d}x \to 0,$$

and consequently $f_0^*\,\mathrm{d}b_1/\mathrm{d}c \to 0$. Equation (45) gives

$$b_1(c) = -\int_0^c\frac{y\,V(y)\,\mathrm{d}y}{(V^2+\langle C^2\rangle)\nu(y)}$$

$$-\int_0^c\frac{3}{y^2\nu(y)\{V(y)^2+\langle C^2\rangle\}f_0^*(y)}\left\{\int_0^y\left(\tfrac{1}{3}xV\frac{\mathrm{d}f_0^*}{\mathrm{d}x}+Wf_0^*\right)x^2\,\mathrm{d}x\right\}\mathrm{d}y+\text{const.}$$

$$= -\psi_1(c)+\text{const.}, \tag{46}$$

where the constant is chosen to make $b_1(c)$ meet the general requirement of equation (32), namely

$$4\pi\int_0^\infty b_k(c)\,f_0^*(c)\,c^2\,\mathrm{d}c = \langle b_k(c)\rangle = 0. \tag{47}$$

It follows that

$$b_1(c) = \langle\psi_1(c)\rangle - \psi_1(c). \tag{48}$$

We next equate to zero the third group of terms in equation (43) and obtain the equation

$$-f_0^*\frac{\partial q}{\partial t}+\left[\frac{c^2}{3\nu}f_0^*+\tfrac{1}{3}cV\frac{\mathrm{d}(b_1 f_0^*)}{\mathrm{d}c}+Wb_1 f_0^*\right.$$
$$\left.+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}b_1 f_0^*+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_2}{\mathrm{d}c}\right)\right\}\right]\frac{\partial^2 q}{\partial z^2} = 0. \tag{49}$$

Integrate each term of equation (49) over all shells of velocity space. It then follows that

$$-\frac{\partial q}{\partial t}+D_\mathrm{L}\frac{\partial^2 q}{\partial z^2}+\left[c^2\nu\left(\frac{cV}{\nu}b_1 f_0^*+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_2}{\mathrm{d}c}\right)\right]_0^\infty\frac{\partial^2 q}{\partial z^2} = 0, \tag{50a}$$

where

$$D_\mathrm{L} = D+\frac{4\pi}{3}\int_0^\infty c^3 V\frac{\mathrm{d}(f_0^* b_1)}{\mathrm{d}c}\,\mathrm{d}c = D-\frac{4\pi}{3}\int_0^\infty b_1 f_0^*\frac{\mathrm{d}(c^3 V)}{\mathrm{d}c}\,\mathrm{d}c. \tag{50b}$$

The procedure for determining $b_2(c)$ is described below where it is shown that when $b_2(c)$ satisfies equation (57) the factor

$$[c^2\nu\{(cV/\nu)b_1 f_0^*+(V^2+\langle C^2\rangle)f_0^*\,\mathrm{d}b_2/\mathrm{d}c\}]_0^\infty$$

in equation (50a) is zero. Equation (50a) then becomes

$$-\partial q/\partial t+D_\mathrm{L}\,\partial^2 q/\partial z^2 = 0. \tag{51}$$

The function $q(z,t)$ is therefore restricted to the class of functions that satisfy (51).

The quantity

$$q(z,t)\,\mathrm{d}z = \left(\iint_{-\infty}^{\infty} n(\boldsymbol{r},t)\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}z$$

is the number of electrons whose positions at time t lie between the planes $z =$ const. and $z+\mathrm{d}z =$ const. It follows that the total population of the isolated group is

$$n_0 = \int_{-\infty}^{\infty} q(z,t)\,\mathrm{d}z.$$

In a system of coordinates at rest equation (51) becomes

$$-\mathrm{d}q/\mathrm{d}t+D_\mathrm{L}\,\partial^2 q/\partial z^2-W\,\partial q/\partial z = 0 \tag{52}$$

and this is the equation satisfied by $q(z,t)$ in the system at rest.

To evaluate the remaining coefficients $b_k(c)$ with $k = 2, 3, 4, \ldots$, we replace $\partial q/\partial t$ in the groups that remain in equation (43) by $D_L\,\partial^2 q/\partial z^2$ from equation (51). Equation (43), since the coefficients of q and $\partial q/\partial z$ are zero, now becomes

$$\sum_{k=0}^{\infty}\left[\left(\frac{c^2}{3\nu}-D_L\right)b_k f_0^* + \tfrac{1}{3}cV\frac{\mathrm{d}(b_{k+1}f_0^*)}{\mathrm{d}c}+Wb_{k+1}f_0^* + \frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}b_{k+1}f_0^*+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_{k+2}}{\mathrm{d}c}\right)\right\}\right]\frac{\partial^{k+2}q}{\partial z^{k+2}} = 0 \tag{53}$$

with $b_0(c)\equiv 1$. The coefficients of the derivatives $\partial^{k+2}q/\partial z^{k+2}$ are then separately equated to zero to yield equations of the form

$$-\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}f_0^* b_{k+1}+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_{k+2}}{\mathrm{d}c}\right)\right\} = \left(\frac{c^2}{3\nu}-D_L\right)b_k f_0^* + \tfrac{1}{3}cV\frac{\mathrm{d}(b_{k+1}f_0^*)}{\mathrm{d}c}+Wb_{k+1}f_0^*,\quad k=0,1,2,\ldots. \tag{54}$$

In this way $b_{k+2}(c)$ is made to depend upon $b_{k+1}(c)$ and $b_k(c)$. In particular $b_2(c)$ is related directly to $b_1(c)$ and $b_0(c)\equiv 1$, both of which are known. Similarly $b_3(c)$ is determined from $b_2(c)$ and $b_1(c)$. As many as desired of the coefficients can, in principle, thus be found progressively.

The formal expression for the $b_{k+2}(c)$ derived from equation (53) with cognizance of equation (47) is

$$b_{k+2}(c) = \langle\psi_{k+2}(c)\rangle - \psi_{k+2}(c), \tag{55}$$

where

$$\psi_{k+2}(c) = \int_0^c \frac{3}{y^2\nu(y)\{V^2(y)+\langle C^2\rangle\}f_0^*(y)}\left[\int_0^y\left\{\left(\frac{x^2}{3\nu(x)}-D_L\right)b_k f_0^* + \tfrac{1}{3}xV\frac{\mathrm{d}(b_{k+1}f_0^*)}{\mathrm{d}x} + Wb_{k+1}f_0^*\right\}x^2\,\mathrm{d}x\right]\mathrm{d}y + \int_0^c\frac{y\,V(y)\,b_{k+1}\,\mathrm{d}y}{\nu(y)\{V^2(y)+\langle C^2\rangle\}}.$$

It remains to show that $b_2(c)$ can be so determined that equation (51) is valid. We replace $\partial q/\partial t$ in equation (49) by $D_L\,\partial^2 q/\partial z^2$ and obtain the following equation from which to determine $b_2(c)$,

$$-\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(c^2\nu(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_2(c)}{\mathrm{d}c}+c^3Vb_1(c)f_0^*\right) = \left(\frac{c^2}{3\nu}-D_L\right)f_0^* + Wb_1f_0^* + \tfrac{1}{3}cV\frac{\mathrm{d}(b_1f_0^*)}{\mathrm{d}c}. \tag{56}$$

Adopt as a first integral the equation

$$c^2\nu(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_2(c)}{\mathrm{d}c}+c^3Vb_1(c)f_0^* = 3\left\{\int_0^c\left(\frac{x^2}{3\nu(x)}-D_L\right)f_0^*x^2\,\mathrm{d}x + \int_0^c\left(Wb_1f_0^*+\tfrac{1}{3}xV\frac{\mathrm{d}(b_1f_0^*)}{\mathrm{d}x}\right)x^2\,\mathrm{d}x\right\}. \tag{57}$$

The right-hand side vanishes as $c \to 0$ (and it also vanishes when $c \to \infty$ since it then becomes $3\{(D-D_{\mathrm{L}})+(D_{\mathrm{L}}-D)\}/4\pi = 0$). It follows therefore that

$$[c^2\nu\{(V^2+\langle C^2\rangle)f_0^* \,\mathrm{d}b_2/\mathrm{d}c + (cV/\nu)b_1(c)\,f_0^*\}]_0^\infty$$

vanishes at both limits as required.

Equation (43) is now reduced to the final summation term and is, with $\partial q/\partial t$ replaced by $D_{\mathrm{L}}\,\partial^2 q/\partial z^2$,

$$\sum_{k=1}^{\infty}\left[\left(\frac{c^2}{3\nu}-D_{\mathrm{L}}\right)b_k f_0^* + Wb_{k+1}f_0^* + \tfrac{1}{3}cV\frac{\mathrm{d}(b_{k+1}f_0^*)}{\mathrm{d}c} + \frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}b_{k+1}f_0^* + (V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_{k+2}}{\mathrm{d}c}\right)\right\}\frac{\partial^{k+2}q}{\partial z^{k+2}}\right] = 0. \tag{58}$$

In order to evaluate the coefficients $b_k(c)$ in which $k > 1$, we equate separately to zero the coefficients of the $\partial^{k+2}q/\partial z^{k+2}$ as described above. Of the coefficients $b_k(c)$ the most important is $b_1(c)$ since it appears in the formula for D_{L} (equation (50b)). We therefore consider $b_1(c)$ and D_{L} in the section that follows. The coefficient D_{L} is the *longitudinal coefficient of diffusion* and D the *lateral* or, alternatively, the *isotropic coefficient of diffusion.*

The fact that diffusion is apparently anisotropic when electrons move in a gas in the presence of an electric field remained long unremarked, although Wannier (1952) commented on this phenomenon in relation to the motion of ions in strong electric fields. The first reference in the literature to such a possibility in the motion of electrons was made by Wagner, Davis, and Hurst (1967), who suggested anisotropic diffusion to explain apparent anomalies between coefficients of diffusion measured by time of flight methods and those measured by the Townsend–Huxley method of the spreading stream. The suggestion was shown to be correct by Parker and Lowke (1969) who developed a theory of the phenomenon by use of the technique of Fourier transformations to obtain a solution of the scalar equation from which D_{L} was derived. The topic has also been discussed by Skullerud (1969). These authors have computed curves and tables of D_{L}/D for various gases, monatomic and diatomic, as functions of E/N.

The formula for D_{L} given in equation (50b) was derived by Parker and Lowke (1969) but in another notation and by a procedure different from that followed here.

VI. Formulae for $b_1(c)$ and D_{L} when q_{m} is Constant and $V^2 \gg \langle C^2\rangle$

Equation (45) is equivalent to

$$-f_0^*\frac{\mathrm{d}b_1}{\mathrm{d}c} = \frac{cV}{\nu(V^2+\langle C^2\rangle)}f_0^* + \frac{3}{c^2\nu(V^2+\langle C^2\rangle)}\int_0^c\left(\tfrac{1}{3}xV\frac{\mathrm{d}f_0^*}{\mathrm{d}x} + Wf_0^*\right)x^2\,\mathrm{d}x. \tag{59}$$

When $V^2 \gg \langle C^2\rangle$ over the effective range of integration the factor $cV/\nu(V^2+\langle C^2\rangle)$ is in effect $c/\nu V = (m/eE)c$ and the factor $c^2\nu(V^2+\langle C^2\rangle)$ becomes $\nu(cV)^2 = \nu(eE/mNq_{\mathrm{m}})^2$. The factor eE/m is related to the drift velocity W through equation (23) which gives

$$W = (eE/mN)\tfrac{1}{3}\langle c^{-2}\,\mathrm{d}(c^2/q_{\mathrm{m}})/\mathrm{d}c\rangle. \tag{60}$$

Two special cases are of interest:

(i) $\nu = Ncq_{\mathrm{m}}$ *is independent of* c, that is to say, $q_{\mathrm{m}}(c) \propto c^{-1}$. Equation (60) shows in this case that the drift velocity is

$$W = eE/m\nu = V. \tag{61}$$

It then follows from equations (50b) and (47) that $D_{\mathrm{L}} = D$. In general, however, where $q_{\mathrm{m}}(c)$ is not proportional to c^{-1}, D_{L} is not equal to D except when $e\boldsymbol{E}$ is zero.

(ii) q_{m} *is independent of* c. This case is useful in practice because in circumstances where q_{m} varies slowly with c it can be treated as independent of c.

It is shown in Appendix II that when q_{m} is independent of c and $V^2 \gg \langle C^2 \rangle$, the distribution function is

$$f_0^*(c) = \{\pi \Gamma(\tfrac{3}{4})\, \alpha^3\}^{-1} \exp\{-(c/\alpha)^4\}, \tag{62}$$

where

$$\alpha^4 = (4M/3m)(eE/mNq_{\mathrm{m}})^2 \tag{63}$$

and

$$b_1(c) = \frac{\pi^{\frac{1}{2}}}{2\lambda} \frac{\langle c^2 \rangle - c^2}{\alpha^2} + \frac{\pi}{2\lambda} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\frac{7}{4})} \frac{\langle c^{4n+3} \rangle - c^{4n+3}}{(4n+3)\alpha^{4n+3}}$$

$$- \frac{\pi}{2\lambda} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\frac{5}{2})} \frac{\langle c^{4n+6} \rangle - c^{4n+6}}{(4n+6)\alpha^{4n+6}}, \tag{64}$$

in which the mean value of an xth power of c is written

$$\langle c^x \rangle = 4\pi \int_0^\infty f_0^*\, c^x c^2\, \mathrm{d}c = \alpha^x\, \Gamma(\tfrac{3}{4} + \tfrac{1}{4}x)/\Gamma(\tfrac{3}{4}) \tag{65}$$

and

$$2\lambda = W/D. \tag{66}$$

With this definition of λ and the replacement of V by $eE/mNcq_{\mathrm{m}}$ and q_{m} constant, the formula for D_{L}/D (equation (50b)) can be transformed to read

$$\frac{D_{\mathrm{L}}}{D} = 1 - \frac{4\pi}{3D} \int_0^\infty b_1(c)\, f_0^* \frac{\mathrm{d}(c^3 V)}{\mathrm{d}c}\, \mathrm{d}c$$

$$= 1 - \frac{2\Gamma(\frac{3}{4})\alpha\lambda}{\pi^{\frac{1}{2}}} \left(4\pi \int_0^\infty b_1(c)\, f_0^*(c)\, c\, \mathrm{d}c \right). \tag{67}$$

It can then be shown that when $b_1(c)$ in equation (67) is replaced by its representative series as given by equation (64) (or equation (A18) of Appendix II) the following formula is found for D_{L}/D:

$$\frac{D_{\mathrm{L}}}{D} = 1 - \Gamma(\tfrac{3}{4}) \frac{\langle c^2 \rangle \langle c^{-1} \rangle - \langle c \rangle}{\alpha} - \pi^{\frac{1}{2}}\, \Gamma(\tfrac{3}{4}) \left(\sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\frac{7}{4})} \frac{\langle c^{4n+3} \rangle \langle c^{-1} \rangle - \langle c^{4n+2} \rangle}{(4n+3)\alpha^{4n+2}} \right.$$

$$\left. - \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\frac{5}{2})} \frac{\langle c^{4n+6} \rangle \langle c^{-1} \rangle - \langle c^{4n+5} \rangle}{(4n+6)\alpha^{4n+5}} \right). \tag{68}$$

By means of equation (65) (or (A7) of Appendix II) this formula is transformed to

$$\frac{D_{\mathrm{L}}}{D} = 1-\left(\frac{\Gamma(\frac{5}{4})\pi^{\frac{1}{2}}}{\Gamma(\frac{3}{4})}-1\right)-\pi^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+\frac{7}{4})}\frac{\Gamma(n+\frac{3}{2})\pi^{\frac{1}{2}}/\Gamma(\frac{3}{4})-\Gamma(n+\frac{5}{4})}{4n+3}\right.$$
$$\left.-\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+\frac{5}{2})}\frac{\Gamma(n+\frac{9}{4})\pi^{\frac{1}{2}}/\Gamma(\frac{3}{4})-\Gamma(n+2)}{4n+6}\right). \tag{69}$$

This expression gives for the special case under discussion, that is, $q_{\mathrm{m}} = \text{const.}$ and $\langle C^2\rangle/V^2 \ll 1$ (high field limit), the value $D_{\mathrm{L}}/D = 0\cdot495$, which is the value found by quadrature (Lowke and Parker 1969) when all encounters are elastic.

VII. Motion in Three Dimensions in a Uniform Field

It is necessary to allow for the presence of mixed derivatives of the form $\partial^l\{(\partial^2/\partial x^2)^m\}/\partial z^l$ and we therefore adopt the following representation of nf_0:

$$nf_0 = f_0^*\left(n+\sum_{k=1}^{\infty}a_{2k}(\nabla_{x,y}^2)^k+\sum_{k=1}^{\infty}b_k\frac{\partial^k n}{\partial z^k}+\sum_{l,m}^{\infty}c_{lm}\frac{\partial^l}{\partial z^l}(\nabla_{x,y}^2)^m n\right), \tag{70}$$

in which $\nabla_{x,y}^2 \equiv \partial^2/\partial x^2+\partial^2/\partial y^2$ and a_{2k}, b_k, and c_{lm} imply $a_{2k}(c)$, $b_k(c)$, and $c_{lm}(c)$ respectively. It suffices to show how the coefficients of lowest order are determined and to establish the equation that corresponds to equations (35), (51), and (52).

When nf_0 is replaced in equation (29) by the right-hand side of (70) and the terms are grouped in ascending order of the spatial derivatives, it is found that

$$\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{c^2}{\nu}(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}(f_0^*/f_0^*)}{\mathrm{d}c}\right)n$$
$$+\left[\tfrac{1}{3}cV\frac{\mathrm{d}f_0^*}{\mathrm{d}c}+Wf_0^*+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{\frac{c^2}{\nu}\left(\frac{cV}{\nu}f_0^*+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_1}{\mathrm{d}c}\right)\right\}\right]\frac{\partial n}{\partial z}$$
$$+\left(-\frac{\partial n}{\partial t}f_0^*+\frac{c^2}{3\nu}f_0^*\left(\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}\right)+\left(\frac{c^2}{3\nu}f_0^*+\tfrac{1}{3}cV\frac{\mathrm{d}(b_1 f_0^*)}{\mathrm{d}c}+Wb_1 f_0^*\right)\frac{\partial^2 n}{\partial z^2}\right.$$
$$\left.+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left[\frac{c^2}{\nu}\left\{\left(\frac{cV}{\nu}b_1+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_2}{\mathrm{d}c}\right)\frac{\partial^2 n}{\partial z^2}+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}a_2}{\mathrm{d}c}\left(\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}\right)\right\}\right]\right)$$
$$+\left(-b_1 f_0^*\frac{\partial}{\partial z}\left(\frac{\partial n}{\partial t}\right)+\left(\frac{c^2}{3\nu}b_1 f_0^*+\tfrac{1}{3}cV\frac{\mathrm{d}(a_2 f_0^*)}{\mathrm{d}c}+Wa_2 f_0^*\right)\left(\frac{\partial^3 n}{\partial z\,\partial x^2}+\frac{\partial^3 n}{\partial z\,\partial y^2}\right)\right.$$
$$+\left(\frac{c^2}{3\nu}b_1 f_0^*+\tfrac{1}{3}cV\frac{\mathrm{d}(b_2 f_0^*)}{\mathrm{d}c}+Wb_2 f_0^*\right)\frac{\partial^3 n}{\partial z^3}$$
$$+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left[\frac{c^2}{\nu}\left\{\left(\frac{cV}{\nu}b_2+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_3}{\mathrm{d}c}\right)\frac{\partial^3 n}{\partial z^3}\right.\right.$$
$$\left.\left.\left.+\left(\frac{cV}{\nu}a_2+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}c_{11}}{\mathrm{d}c}\right)\left(\frac{\partial^3 n}{\partial z\,\partial x^2}+\frac{\partial^3 n}{\partial z\,\partial y^2}\right)\right\}\right]\right)+S = 0, \tag{71}$$

the final term S being a sum of mixed and pure spatial derivatives with coefficients that are functions of c plus terms with $\partial n/\partial t$ as a factor. The coefficient of n in equation (71) vanishes identically. To determine b_1 we equate to zero the coefficient of $\partial n/\partial z$. It is then seen that b_1 satisfies equation (44) and is identical with the b_1 of the one-dimensional case. Next, equating to zero the group of terms with $\partial n/\partial t$ in association with second-order derivatives and integrating each term of this group over all shells of velocity space, we arrive at the following equation to be satisfied by $n(\boldsymbol{r},t)$, with the requirement that the coefficients $a_{2k}(c)$ etc. conform to equation (47),

$$-\frac{\partial n}{\partial t}+D\left(\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}\right)+D_{\mathrm{L}}\frac{\partial^2 n}{\partial z^2}=0, \tag{72}$$

in which D and D_{L} are the coefficients defined by equation (50b).

In the stationary system of coordinates equation (72) is replaced by

$$-\frac{\mathrm{d}n}{\mathrm{d}t}+D\left(\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}\right)+D_{\mathrm{L}}\frac{\partial^2 n}{\partial z^2}-W\frac{\partial n}{\partial z}=0. \tag{73}$$

We are now able to replace $\partial n/\partial t$ in equation (71) by $D\,\nabla^2_{x,y}n+D_{\mathrm{L}}\,\partial^2 n/\partial z^2$ from equation (72). When this is done equation (71) becomes (since the terms in n and $\partial n/\partial z$ have vanished)

$$\begin{aligned}
&\left\{\left(\frac{c^2}{3\nu}-D\right)f_0^*+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(c^2\nu(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}a_2}{\mathrm{d}c}\right)\right\}\left(\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}\right)\\
&\quad+\left[\frac{c^2}{3\nu}f_0^*+\tfrac{1}{3}cV\frac{\mathrm{d}(b_1 f_0^*)}{\mathrm{d}c}+Wb_1 f_0^*-D_{\mathrm{L}}f_0^*\right.\\
&\qquad\left.+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}b_1+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_2}{\mathrm{d}c}\right)\right\}\right]\frac{\partial^2 n}{\partial z^2}\\
&\quad+\left[\frac{c^2}{3\nu}b_1 f_0^*+\tfrac{1}{3}cV\frac{\mathrm{d}(a_2 f_0^*)}{\mathrm{d}c}+Wa_2 f_0^*-Db_1 f_0^*\right.\\
&\qquad\left.+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}a_2+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}c_{11}}{\mathrm{d}c}\right)\right\}\right]\left(\frac{\partial^3 n}{\partial z\,\partial x^2}+\frac{\partial^3 n}{\partial z\,\partial y^2}\right)\\
&+\left[\frac{c^3}{3\nu}b_1 f_0^*+\tfrac{1}{3}cV\frac{\mathrm{d}(b_2 f_0^*)}{\mathrm{d}c}+Wb_2 f_0^*+\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}b_2+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_3}{\mathrm{d}c}\right)\right\}\right]\frac{\partial^3 n}{\partial z^3}\\
&\qquad\qquad\qquad\qquad+S'=0,
\end{aligned} \tag{74}$$

where S' indicates the sum S with $\partial n/\partial t$ replaced by $D\,\nabla^2 n+(D_{\mathrm{L}}-D)\partial^2 n/\partial z^2$.

The coefficients of the spatial derivatives are equated to zero to provide equations from which the coefficients $a_{2k}(c)$ etc. can be found. For instance, $a_2(c)$ is required to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}c}\left(c^2\nu(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}a_2}{\mathrm{d}c}\right)+3c^2\left(\frac{c^2}{3\nu}-D\right)f_0^*=0. \tag{75}$$

Similarly, $b_2(c)$ and $c_{11}(c)$ are solutions respectively of

$$\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}b_1+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}b_2}{\mathrm{d}c}\right)\right\}+3c^2\left(\frac{c^2}{3\nu}f_0^*+\tfrac{1}{3}cV\frac{\mathrm{d}(b_1f_0^*)}{\mathrm{d}c}+Wb_1f_0^*-D_{\mathrm{L}}f_0^*\right)=0 \tag{76}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}c}\left\{c^2\nu\left(\frac{cV}{\nu}a_2+(V^2+\langle C^2\rangle)f_0^*\frac{\mathrm{d}c_{11}}{\mathrm{d}c}\right)\right\}$$

$$+3c^2\left(\frac{c^2}{3\nu}b_1f_0^*+\tfrac{1}{3}cV\frac{\mathrm{d}(a_2f_0^*)}{\mathrm{d}c}+Wa_2f_0^*-Db_1f_0^*\right)=0,$$

and so on. Thus $b_2(c)$ can be found in principle since b_1 is known, and c_{11} because a_2 and b_1 are known. Similarly b_3 is found in terms of b_2 and b_1. The same procedure is adopted to find the coefficients included in S' by equating the coefficients of the spatial derivatives individually to zero. In practice no use is made of any coefficients except $b_1(c)$ which determines the form of D_{L}.

We note that when $eE = 0$ equations (75) and (76) reduce to the same form as that satisfied by a_2 in equation (37a); moreover c_{11} vanishes. In general the b_{2k} become a_{2k} and the b_{2k+1} and $c_{l,m}$ vanish. Thus the expansion in equation (70) reduces to that in equation (31) as it should.

VIII. Extension of the Theory to a Steady Stream

The discussion has hitherto concerned the properties of an isolated travelling group but it follows from the fact that a steady stream of electrons in a gas can be regarded as a sequence of overlapping travelling groups that it is legitimate to apply equation (73) with $\mathrm{d}n/\mathrm{d}t = 0$ to the calculation of the distribution of number density $n(\boldsymbol{r},t)$ in a steady stream. It is useful to illustrate this general comment by a specific example, that of a steady stream from an isolated point source.

We consider in the first instance a steady stream of electrons from a pole source at the origin. Let the electric field be uniform and the electric force $e\boldsymbol{E}$ be directed along $+0z$. Suppose the stream to carry current i; this current may be regarded as a continuous succession of elementary groups of electrons with populations $(i\,\mathrm{d}t)/e$ liberated by the source with a spatial δ-function concentration. The time-independent number density in the stream may be regarded as the sum of the contributions from those elementary groups whose times of travel range continuously from zero to infinity. Thus the number density in the steady stream can be represented in the form

$$n(x,y,z)=\sum_{m=1}^{\infty}\Delta_m n,$$

where the $\Delta_m n$ are the number densities contributed by the elementary groups. It follows that each Δn satisfies equation (31) which refers to the moving system of coordinates and therefore that $n = \sum \Delta_m n$ does so also. We therefore replace $\sum \Delta_m n$ by n and integrate over all shells $(c, \mathrm{d}c)$ and find that n for the steady stream satisfies the same equation in the moving system as n for a travelling group, namely

$$\frac{\partial n}{\partial t} = D\left(\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}\right)+D_{\mathrm{L}}\frac{\partial^2 n}{\partial z'^2}.$$

However, when this equation is referred to the stationary system it takes the form

$$D\left(\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}\right)+D_{\mathrm{L}}\frac{\partial^2 n}{\partial z^2}-W\frac{\partial n}{\partial z} = 0, \tag{77}$$

with $\mathrm{d}n/\mathrm{d}t = 0$ since $n = \sum \Delta_m n$ is independent of time. We note the important fact that D and D_{L} are the same for the steady stream as for the travelling group. Similarly the formulae for all the coefficients in equation (70) remain unchanged.

In order to illustrate more precisely the representation of a steady stream as a succession of overlapping elementary travelling groups we note that the centroid of an elementary group $(i\,\mathrm{d}t)/e$ that has travelled for time t lies at the point $(0, 0, Wt)$ and consequently that the number density contributed by the group at the point (x, y, z) is

$$\Delta n = \frac{i\,\mathrm{d}t}{e(4\pi Dt)(4\pi D_{\mathrm{L}}t)^{\frac{1}{2}}}\exp\left(-\frac{\rho^2}{4Dt}\right)\exp\left(-\frac{(z-Wt)^2}{4D_{\mathrm{L}}t}\right).$$

The total number density at (x, y, z) is therefore

$$n(x,y,z) = \frac{i}{e(4\pi D)(4\pi D_{\mathrm{L}})^{\frac{1}{2}}}\int_0^\infty t^{-3/2}\exp\left(-\frac{\rho^2}{4Dt}-\frac{(z-Wt)^2}{4D_{\mathrm{L}}t}\right)\mathrm{d}t.$$

Putting $2\lambda_{\mathrm{L}} = W/D_{\mathrm{L}}$ and $\tau = tW^2/4D_{\mathrm{L}} = \frac{1}{2}\lambda_{\mathrm{L}}\,Wt$ gives

$$n(x,y,z) = \frac{i\,D_{\mathrm{L}}^{\frac{1}{2}}\,\lambda_{\mathrm{L}}\exp(\lambda_{\mathrm{L}}z)}{e(4\pi D)(4\pi D_{\mathrm{L}})^{\frac{1}{2}}}\int_0^\infty \tau^{-3/2}\exp\left(-\tau-\lambda_{\mathrm{L}}^2\frac{z^2+(D_{\mathrm{L}}/D)\rho^2}{4\tau}\right)\mathrm{d}\tau.$$

But (Watson 1944, Section 6.22, equation (15))

$$\int_0^\infty \tau^{-(\nu+1)}\exp(-\tau-s^2/4\tau)\,\mathrm{d}\tau = 2(2/s)^\nu\,\mathrm{K}_\nu(s),$$

where $\mathrm{K}_\nu(s)$ is a modified Bessel function of the second kind and order ν. Consequently,

$$\begin{aligned} n(x,y,z) &= \frac{i\lambda_{\mathrm{L}}\,2^{3/2}}{e(4\pi D)(4\pi)^{\frac{1}{2}}}\lambda_{\mathrm{L}}^{-\frac{1}{2}}\{z^2+(D_{\mathrm{L}}/D)\rho^2\}^{-\frac{1}{4}}\exp(\lambda_{\mathrm{L}}z)\,\mathrm{K}_{\frac{1}{2}}[\lambda_{\mathrm{L}}\{z^2+(D_{\mathrm{L}}/D)\rho^2\}^{\frac{1}{2}}] \\ &= \frac{i}{e(4\pi D)}\frac{\exp\{-\lambda_{\mathrm{L}}(r'-z)\}}{r'}, \end{aligned} \tag{78}$$

where

$$r' = \{z^2+(D_{\mathrm{L}}/D)\rho^2\}^{\frac{1}{2}}.$$

This expression for $n(x, y, z)$ in the steady stream does in fact satisfy equation (77).

IX. Current Density and Current

In order to simplify the discussion we first consider the specific case of diffusion in the absence of an electric field. It follows directly from equation (12) that when $V = 0$

$$-\frac{\mathrm{d}(nf_0)}{\mathrm{d}t} + \frac{c^2}{3\nu}\nabla^2(nf_0) + \frac{1}{4\pi c^2}\frac{\partial(\sigma_{\mathrm{coll}})}{\partial c} = 0. \tag{79}$$

We note that equation (30) is the special case of equation (79) when all encounters are elastic. In (79) we replace nf_0 by its series representation postulated in equation (31) and then integrate each term over all shells of velocity space. The third term of (79) thus becomes $\sigma_{\mathrm{coll}}|_0^\infty$ and vanishes at both limits. The following equation is therefore obtained

$$-\frac{\mathrm{d}n}{\mathrm{d}t} + D\nabla^2 n + \sum_{k=1}^{\infty} D_k(\nabla^2)^k n = 0, \tag{80}$$

where

$$D_k = (4\pi/3)\int_0^\infty (c^2/\nu) f_0^* a_{2k}(c)\, c^2\,\mathrm{d}c. \tag{81}$$

However, from equation (35), which is the equation of continuity for n, it follows that

$$\sum_{k=1}^{\infty} D_k(\nabla^2)^k n = 0.$$

Thus the total contribution to the particle flux from spatial derivatives of n of order higher than the first is zero and the flux is given simply by

$$-D\,\mathrm{grad}_r\, n\,.\,\mathrm{d}\boldsymbol{S}.$$

We consider next the general case in which an electric field is present. We refer again to equation (12) with V no longer equal to zero. In this equation replace nf_0 by its series representation in equation (70) and again integrate over all shells $(c, \mathrm{d}c)$ of velocity space. It will be found that equation (12) then yields the following equation

$$-\frac{\mathrm{d}n}{\mathrm{d}t} + D\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}\right) + D_{\mathrm{L}}\frac{\partial^2 n}{\partial z^2} - W\frac{\partial n}{\partial z} + R = 0, \tag{82}$$

where R is the sum of terms dependent upon higher order spatial derivatives of n analogous to the third term of equation (80). We have, however, postulated that n shall be a solution of equation (73) and consequently $R = 0$. It follows from equation (73) that the particle flux across a vector element of surface $\mathrm{d}\boldsymbol{S}$ is

$$n\,\boldsymbol{W}_{\mathrm{cv}}\,.\,\mathrm{d}\boldsymbol{S},$$

where

$$n(W_{\mathrm{cv}})_x = -D\,\partial n/\partial x, \tag{83}$$

$$n(W_{\mathrm{cv}})_y = -D\,\partial n/\partial y\,, \tag{84}$$

$$n(W_{\mathrm{cv}})_z = -D_{\mathrm{L}}\,\partial n/\partial z + nW\,. \tag{85}$$

Equations (83) and (84) are the equations normally used to calculate fluxes of particles across surfaces. Equation (85) is different from the standard form in that in the presence of an electric field D is replaced by D_{L}.

It is to be noted that the mutual cancellation of the contributions of the higher order spatial derivatives in equations (80) and (82) is a consequence of the form postulated in equation (31) for the dependence of f_0 on position and of the constraints (i), (ii), and (iii) that follow equation (27).

An important practical application of equations (77) and (85) is the Townsend–Huxley method for investigating the properties of the motions of electrons in gases and collision cross sections. The quantity measured in these experiments is the ratio R of the current to a central disk to the total current when a steady stream of electrons is received by a plane metal electrode.

In previous analyses that have related the ratio R to the geometry of the apparatus and to the transport coefficients, two assumptions were made, namely that diffusion is isotropic and that the hole in the cathode acts as a pole source of electrons. The first assumption appeared reasonable prior to the experimental and theoretical work that revealed the extent of the anisotropicity in certain circumstances (Wagner, Davis, and Hurst 1967; Parker and Lowke 1969). The second assumption, although violating the normally accepted boundary condition $n = 0$ over the cathode except at the source hole, appeared to be justified by the success which the formula based on it enjoyed in predicting consistent values of the ratio D/W from a large body of experimental data (Huxley and Crompton 1955; Crompton and Jory 1962; Crompton, Elford, and Gascoigne 1965). A detailed analysis by Hurst and Liley (1965), while retaining the assumption of isotropic diffusion in regions of the apparatus removed from the boundaries, showed that the experimental results could be accounted for on the assumption of artificial reflection coefficients at the electrodes. In a recent paper Lowke (1971) has analysed the problem assuming anisotropic diffusion and the boundary condition $n = 0$ at both anode and cathode. His analysis shows that the earlier semi-empirical formula for R is applicable with little error for a wide range of experimental conditions provided the momentum transfer cross section is constant. The anomaly of many years standing has been thereby explained.

In order to relate the theory of the experiment based on equation (77) to the earlier theory, we assume first that the source of current is an isolated pole source. Then according to the discussion given in Section VIII the appropriate solution of equation (77) for the number density in the stream is

$$n = \frac{i}{4\pi De}\frac{\exp\{-\lambda_{\mathrm{L}}(r'-z)\}}{r'}\,, \tag{86}$$

where $r'^2 = z^2+\rho'^2$ with $\rho'^2 = (D_{\mathrm{L}}/D)\rho^2$ and $\rho^2 = x^2+y^2$.

Equation (86) refers to a stream that proceeds to infinity. When the stream falls upon an infinite plane electrode in the plane $z = h$ over which $n = 0$, it is necessary to add another term which is a separate solution of equation (77). The appropriate solution that gives $n = 0$ when $z = h$ is

$$n = \frac{i\exp(\lambda_L z)}{4\pi De}\left(\frac{\exp(-\lambda_L r')}{r'} - \frac{\exp(-\lambda_L r_1')}{r_1'}\right), \tag{87}$$

in which

$$2\lambda_L = W/D_L, \qquad (r_1')^2 = (2h-z)^2+\rho'^2.$$

The current received by a central circular disk with radius b and centre on the axis $0z$ is

$$i_{0b} = -2\pi e D_L \int_0^b \left(\frac{\partial n}{\partial z}\right)_{z=h} \rho\, \mathrm{d}\rho.$$

When n is replaced by the right-hand side of equation (87) it is found after reduction that

$$R_{\text{pole}} = i_{0b}/i = 1-(h/d')\exp\{-\lambda_L(d'-h)\}, \tag{88}$$

in which $d'^2 = h^2+(D_L/D)b^2 = h^2+(\lambda/\lambda_L)b^2$ with $2\lambda = W/D$.

If the source is now assumed to be a dipole instead of a pole, thus satisfying the normally accepted boundary condition at the cathode, the distribution of number density in the presence of a plane electrode in the plane $z = h$ is

$$n \propto -\exp(\lambda_L z)\frac{\partial}{\partial z}\left(\frac{\exp(-\lambda_L r')}{r'} + \frac{\exp(-\lambda_L r_1')}{r_1'}\right) \tag{89}$$

and the current ratio becomes

$$R_{\text{dipole}} = \frac{i_{0b}}{i} = 1-\left(\frac{h}{d'} - \frac{1-h^2/d'^2}{\lambda_L h}\right)\frac{h}{d'}\exp\{-\lambda_L(d'-h)\}. \tag{90}$$

This formula assumes a simpler and more useful form if the design of the apparatus is such that $(b/h)^2 \ll 1$. When such is the case,

$$\begin{aligned}\lambda_L(d'-h) &= \lambda_L[h\{1+(\lambda/\lambda_L)b^2/h^2\}^{\frac{1}{2}}-h] \\ &= \lambda b^2/2h - \lambda^2 b^4/8\lambda_L h^3 + \dots \\ &\simeq \lambda b^2/2h.\end{aligned}$$

However, when $(b/h)^2 \ll 1$

$$\lambda(d-h) \simeq \lambda b^2/2h$$

and consequently

$$\exp\{-\lambda_L(d'-h)\} \simeq \exp\{-\lambda(d-h)\}.$$

Moreover, when $\lambda_L h \gg 1$ the factor $h/d' - (1-h^2/d'^2)/\lambda_L h \simeq h/d$. Thus in practice in an apparatus in which $b = 0\cdot5$ cm and $h = 10$ cm, formula (90) is insignificantly different from

$$R_{\text{dipole}} = 1-(h/d)\exp\{-\lambda(d-h)\}. \tag{91}$$

This formula was originally derived on the assumption of a pole source and isotropic diffusion. That equation (91) results from these assumptions can be seen by substituting $\lambda_{\mathrm{L}} = \lambda$ (that is, $D_{\mathrm{L}} = D$) into equation (88). As has already been stated, the formula was found to agree with the experimental measurements with remarkable accuracy over a wide range of experimental parameters; in fact good agreement was found for a wider range of the parameters than is to be expected from a comparison of equations (90) and (91) using the values of b and h appropriate to the experiments of, for example, Crompton and Jory (1962). This now surprising agreement has recently been shown (Lowke 1971) to be a consequence of the fact that, for the gases for which the most extensive investigations of the applicability of equation (91) have been made, the momentum transfer cross section is approximately constant in the range of investigation and, as a consequence, $D_{\mathrm{L}}/D \simeq 0\cdot 5$. However, at the time of the experimental investigations of the validity of the ratio formula, there remained the theoretical inconsistency that the assumption of a pole source did not give $n = 0$ over the metal surface of the plane $z = 0$ of the cathode. The present theory resolves this difficulty. We note that when $(b/h)^2$ is kept very much less than unity the quantity measured by the Townsend–Huxley method is $2\lambda = W/D$ and not $2\lambda_{\mathrm{L}} = W/D_{\mathrm{L}}$ as might have been expected, although erroneously, from the form of equations (88) and (90).

X. Inelastic Collisions

The analysis presented in this paper leads to results that are in close agreement with the measured properties of the motions of electrons in both monatomic and other gases, but the detailed analysis has been based on the supposition that all encounters are elastic and is therefore not strictly applicable to gases other than monatomic gases. We note, however, that the vector equation (8), which is an expression of the law of conservation of momentum, is not concerned essentially with the nature of the encounters whether elastic or inelastic and that even in the scalar equation (6) it is only the term $\sigma_{\mathrm{coll}}(c)$ that is directly affected by the nature of the encounters. Although it is possible to derive an expression for $\sigma_{\mathrm{coll}}(c)$ when the influence of inelastic encounters is important, a result that leads to a simple formula for f_0^*, such as Davydov's formula, is not obtained. One possible course to follow, other than one of immediate numerical analysis, is to assume some form for $f_0^*(c)$, as for instance Druyvesteyn's distribution formula $f_0^* = \text{const.}\times\exp(c/\alpha)^4$ in high field conditions where $V^2 \gg \langle C^2\rangle$. In this case α is no longer given by equation (63) but is directly related to the mean square velocity of the electrons. It is found, however, that when inelastic encounters are important the factor $(3m/M)c$ in equation (28) is replaced by some more general function of c, say $h(c)$, and that in consequence $f_0^*(c)$ becomes

$$f_0^*(c) = \text{const.}\times\exp\left(-\int_0^c \frac{h(c)}{V^2+\langle C^2\rangle}\,\mathrm{d}c\right),$$

which includes Davydov's function as a special case. The formal analysis can then proceed as for the case of elastic encounters. This analysis again leads to the basic equation (73) while the conclusions of the previous section remain unchanged.

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Appendix I

When $\mathrm{grad}_r\, nf$ and $e\boldsymbol{E}$ are not parallel it is impossible to choose a direction for the polar axis such that $f(c)$ is independent of the azimuth ϕ as supposed in the representation

$$f = \sum_{k=0}^{\infty} f_k\, \mathrm{P}_k(\cos\theta)\,.$$

Take an arbitrary direction for the polar axis and assume that when both $\mathrm{grad}_r\, nf$ and $e\boldsymbol{E}$ are present each "polarizes" the distribution of velocity point density in the shell $(c, \mathrm{d}c)$ with an axis of symmetry about its own direction. Let the direction of $\mathrm{grad}_r\, nf$ be $(\theta_{\mathrm{G}}, \phi_{\mathrm{G}})$ and of $e\boldsymbol{E}$ be (θ_E, ϕ_E) in the reference system. Also let γ_{G} and γ_E be the angles between an arbitrary direction (θ, ϕ) and the directions $(\theta_{\mathrm{G}}, \phi_{\mathrm{G}})$ and (θ_E, ϕ_E) respectively. Now suppose that

$$f = f_0 + \sum_{k=1}^{\infty} f_{k\mathrm{G}}\, \mathrm{P}_k(\cos\gamma_{\mathrm{G}}) + \sum_{k=1}^{\infty} f_{kE}\, \mathrm{P}_k(\cos\gamma_E)\,, \tag{A1}$$

where

$$\cos\gamma_{\mathrm{G}} = \cos\theta\cos\theta_{\mathrm{G}} + \sin\theta\sin\theta_{\mathrm{G}}\cos(\phi-\phi_{\mathrm{G}})$$

and

$$\cos\gamma_E = \cos\theta\cos\theta_E + \sin\theta\sin\theta_E\cos(\phi-\phi_E)\,.$$

But, from the addition theorem for spherical harmonics,

$$\mathrm{P}_k(\cos\gamma_{\mathrm{G}}) = \mathrm{P}_k(\cos\theta)\,\mathrm{P}_k(\cos\theta_{\mathrm{G}}) + 2\sum_{m=1}^{k} \frac{(k-m)!}{(k+m)!} \cos\{m(\phi-\phi_{\mathrm{G}})\}\, \mathrm{P}_k^m(\cos\theta)\, \mathrm{P}_k^m(\cos\theta_{\mathrm{G}})$$

and

$$\mathrm{P}_k(\cos\gamma_E) = \mathrm{P}_k(\cos\theta)\,\mathrm{P}_k(\cos\theta_E) + 2\sum_{m=1}^{k}\frac{(k-m)!}{(k+m)!}\cos\{m(\phi-\phi_E)\}\,\mathrm{P}_k^m(\cos\theta)\,\mathrm{P}_k^m(\cos\theta_E),$$

where

$$\mathrm{P}_k^m(\mu) = (1-\mu^2)^{\frac{1}{2}m}\,\mathrm{d}^m\{\mathrm{P}_k(\mu)\}/\mathrm{d}\mu^m.$$

Let the polar axis $\theta = 0$ lie in the plane containing the directions $(\theta_\mathrm{G}, \phi_\mathrm{G})$ and (θ_E, ϕ_E), that is to say, the polar points on the sphere lie on the same line of longitude which we may take as $\phi = 0$. It then follows that

$$f = f_0 + \sum_{k=1}^{\infty}\Big(\{f_{k\mathrm{G}}\,\mathrm{P}_k(\cos\theta_\mathrm{G}) + f_{kE}\,\mathrm{P}_k(\cos\theta_E)\}\mathrm{P}_k(\cos\theta) + 2\sum_{m=1}^{k}\{f_{k\mathrm{G}}\,\mathrm{P}_k^m(\cos\theta_\mathrm{G}) + f_{kE}\,\mathrm{P}_k^m(\cos\theta_E)\}\frac{(k-m)!}{(k+m)!}\mathrm{P}_k^m(\cos\theta)\cos m\phi\Big). \tag{A2}$$

Since $\mathrm{P}_k^m(1) = 0$, the value of f when $\theta = 0$ (at the pole of reference) is

$$f_{\theta=0} = f_0 + \sum_{k=1}^{\infty}\{f_{k\mathrm{G}}\,\mathrm{P}_k(\cos\theta_\mathrm{G}) + f_{kE}\,\mathrm{P}_k(\cos\theta_E)\},$$

in which the lowest order term dependent upon θ is $(k = 1)$

$$f_{1\mathrm{G}}\cos\theta_\mathrm{G} + f_{1E}\cos\theta_E. \tag{A3}$$

Let the direction of the axis $\theta = 0$ lie in the plane containing the directions $(\theta_\mathrm{G}, 0)$ and $(\theta_E, 0)$ and be chosen to give $f_{1\mathrm{G}}\cos\theta_\mathrm{G} + f_{1E}\cos\theta_E$ its maximum value. Since $\theta_\mathrm{G} - \theta_E$ is constant it follows that $\mathrm{d}\theta_\mathrm{G}/\mathrm{d}\theta = \mathrm{d}\theta_E/\mathrm{d}\theta$ and that the condition for a maximum is therefore

$$f_{1\mathrm{G}}\sin\theta_\mathrm{G} + f_{1E}\sin\theta_E = 0. \tag{A4}$$

If we define vectors $\boldsymbol{f}_{1\mathrm{G}}$ and $\boldsymbol{f}_{1E}$ which have the magnitudes and directions $\{f_{1\mathrm{G}}, (\theta_\mathrm{G}, 0)\}$ and $\{f_{1E}, (\theta_E, 0)\}$ respectively, then the vector $\boldsymbol{f}_1 = \boldsymbol{f}_{1\mathrm{G}} + \boldsymbol{f}_{1E}$ has the direction of the axis $\theta = 0$, since the projection of $\boldsymbol{f}_{1\mathrm{G}} + \boldsymbol{f}_{1E}$ on the plane normal to $\boldsymbol{f}_1$ is

$$f_{1\mathrm{G}}\sin\theta_\mathrm{G} + f_{1E}\sin\theta_E$$

and is zero. It is this choice of axis that is in fact adopted in the derivation of the basic equations in Section II, that is to say, the direction of $\boldsymbol{f}_1$ is adopted as the polar axis of reference. Equation (10), which is based on considerations of conservation of momentum, exhibits $\boldsymbol{f}_1$ as the vector sum of

$$f_{1\mathrm{G}} = -(c/\nu)\mathrm{grad}_r(nf_0) \qquad \text{and} \qquad f_{1E} = -V\,\partial(nf_0)/\partial c.$$

Returning to equation (A1), in which the polar axis $\theta = 0$ is arbitrarily directed with respect to the axes of polarization, this equation is equivalent to

$$f = f_0 + \sum_{k=1}^{\infty} f_k\,\mathrm{P}_k(\cos\theta) + \sum_{k=1}^{\infty}\left\{2\sum_{m=1}^{k} A_{km}\frac{(k-m)!}{(k+m)!}\Big(\cos\{m(\phi-\phi_\mathrm{G})\} + \cos\{m(\phi-\phi_E)\}\Big)\mathrm{P}_k^m(\cos\theta)\right\}, \tag{A5}$$

where

$$f_k = f_{k\mathrm{G}}\,\mathrm{P}_k(\cos\theta_{\mathrm{G}}) + f_{kE}\,\mathrm{P}_k(\cos\theta_E)$$

and

$$A_{km} = f_{k\mathrm{G}}\,\mathrm{P}_k^m(\cos\theta_{\mathrm{G}}) + f_{kE}\,\mathrm{P}_k^m(\cos\theta_E)\,.$$

In particular

$$f_1 = f_{1\mathrm{G}}\cos\theta_{\mathrm{G}} + f_{1E}\cos\theta_E$$

and, because $f_{1\mathrm{G}}$ and f_{1E} are the magnitudes of $\boldsymbol{f}_{1\mathrm{G}}$ and $\boldsymbol{f}_{1E}$, the coefficient f_1 when the reference axis and the axes of polarization are arbitrarily directed becomes the sum of projections of $\boldsymbol{f}_{1\mathrm{G}}$ and $\boldsymbol{f}_{1E}$ on the reference axis $\theta = 0$. The mean value of f around a parallel of latitude $\theta =$ const. is

$$(2\pi)^{-1}\int_0^{2\pi} f\,\mathrm{d}\phi$$

and in the process of integration terms containing $\cos\{m(\phi-\phi_{\mathrm{G}})\}$ and $\cos\{m(\phi-\phi_E)\}$ are eliminated.

The right-hand side of equation (A5) is thereby reduced to

$$f_0 + \sum_{k=1}^{\infty} f_k\,\mathrm{P}_k(\cos\theta)\,,$$

which is the form adopted for f in Section II. There, however, in addition the direction of the axis $\theta = 0$ is taken as that of $\boldsymbol{f}_1$. Thus the representation of f in a form independent of ϕ is in fact the replacement of f by its mean value around a parallel of latitude. However, had ϕ been retained at this stage it would have been eliminated in the course of integration over the whole sphere. It is therefore convenient to eliminate the dependence upon ϕ at the outset.

Appendix II

Derivation of Formula for $b_1(c)$

The coefficient $b_1(c)$ is to be found from equation (59), Section VI. The discussion is restricted to the special case of $q_{\mathrm{m}} =$ const. and $V^2 \gg \langle C^2\rangle$. In this case Davydov's distribution function gives

$$f_0^*(c) = \text{const.} \times \exp\left(-\frac{3m}{M}\int_0^c \frac{c\,\mathrm{d}c}{V^2+\langle C^2\rangle}\right) \simeq \text{const.} \times \exp\{-(c/\alpha)^4\}\,,$$

where

$$\alpha^4 = (4M/3m)(eE/mNq_{\mathrm{m}})^2\,,$$

since $V = eE/mNcq_{\mathrm{m}}(c)$. Because

$$4\pi\int_0^{\infty} f_0^*\,c^2\,\mathrm{d}c = 1\,,$$

it follows that

$$f_0^* = \{\alpha^3\pi\Gamma(\tfrac{3}{4})\}^{-1}\exp\{-(c/\alpha)^4\}\,. \tag{A6}$$

The mean value of the xth power of c is

$$\langle c^x\rangle = \{\alpha^3\pi\,\Gamma(\tfrac{3}{4})\}^{-1}\int_0^\infty \exp\{-(c/\alpha)^4\}\,c^{\,x+2}\,\mathrm{d}c = \alpha^x\Gamma(\tfrac{3}{4}+\tfrac{1}{4}x)/\Gamma(\tfrac{3}{4}). \tag{A7}$$

Also, when q_{m} is independent of c, it follows from equations (36) and (23) that

$$D = \frac{\langle c\rangle}{3Nq_{\mathrm{m}}}, \qquad W = \frac{2eE}{3m}\frac{\langle c^{-1}\rangle}{Nq_{\mathrm{m}}} = \frac{2eE}{m}\frac{\langle c^{-1}\rangle}{\langle c\rangle}D,$$

that is to say,

$$2\lambda = W/D = (2eE/m)\langle c^{-1}\rangle/\langle c\rangle,$$

or

$$eE/m = \lambda\langle c\rangle/\langle c^{-1}\rangle. \tag{A8}$$

When $V^2 \gg \langle C^2\rangle$ equation (59) reduces to the form

$$-f_0^*\frac{\mathrm{d}b_1}{\mathrm{d}c} \simeq \frac{mc}{eE}f_0^* + \frac{W}{D}\frac{\langle c^{-1}\rangle^2}{\lambda^2\langle c\rangle}\int_0^c\left(f_0^* + \frac{1}{2\langle c^{-1}\rangle}\frac{\mathrm{d}f_0^*}{\mathrm{d}x}\right)x^2\,\mathrm{d}x$$

which, because of equations (A7) and (A8), is equivalent to

$$-f_0^*\frac{\mathrm{d}\{b_1(c)\}}{\mathrm{d}c} \simeq \frac{\pi^{\frac{1}{2}}}{\alpha^2\lambda}cf_0^* + \frac{2\pi}{\Gamma(\frac{3}{4})\alpha^3\lambda c}\int_0^c\left\{1-\frac{3\Gamma(\frac{3}{4})}{\pi^{\frac{1}{2}}}\left(\frac{x}{\alpha}\right)^3\right\}f_0^*x^2\,\mathrm{d}x. \tag{A9}$$

If $s = (x/\alpha)^4$ then equation (A9) can be written, using (A6), in the form

$$-\exp\{-(c/\alpha)^4\}\frac{\mathrm{d}\{b_1(c)\}}{\mathrm{d}c} \simeq \frac{\pi^{\frac{1}{2}}c}{\alpha^2\lambda}\exp\{-(c/\alpha)^4\}$$
$$+\frac{\pi}{2\Gamma(\frac{3}{4})\lambda c}\int_0^{(c/\alpha)^4}\{s^{-\frac{1}{4}}-2\pi^{-\frac{1}{2}}\Gamma(\tfrac{3}{4})s^{\frac{1}{2}}\}\exp(-s)\,\mathrm{d}s. \tag{A10}$$

The definition of the incomplete gamma function $\gamma(a,x)$ is

$$\gamma(a,x) = \int_0^x \exp(-t)\,t^{a-1}\,\mathrm{d}t, \tag{A11}$$

whence it follows that

$$\int_0^{(c/\alpha)^4}\{s^{-\frac{1}{4}}-2\pi^{-\frac{1}{2}}\Gamma(\tfrac{3}{4})s^{\frac{1}{2}}\}\exp(-s)\,\mathrm{d}s$$
$$= \gamma(\tfrac{3}{4},(c/\alpha)^4) - 2\pi^{-\frac{1}{2}}\Gamma(\tfrac{3}{4})\gamma(\tfrac{3}{2},(c/\alpha)^4). \tag{A12}$$

It is known, however, and may be established by successive integration by parts, that

$$\gamma(a,x) = a^{-1}x^a\exp(-x)\,M(1,1+a,x), \tag{A13}$$

where

$$M(a,b,x) \equiv {}_1\mathrm{F}_1(a,b,x) = \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{x^n}{n!} \tag{A14}$$

is a confluent hypergeometric function. In equation (A14) the symbols $(a)_n$ and $(b)_n$ carry their usual meanings:

$$(a)_n = a(a+1)(a+2)\ldots(a+n-1) = \Gamma(a+n)/\Gamma(a),$$

$$(b)_n = \Gamma(b+n)/\Gamma(b), \qquad (a)_0 = (b)_0 = 1.$$

Equation (A10) can therefore be given the form

$$-\frac{\mathrm{d}\{b_1(c)\}}{\mathrm{d}c} = \frac{\pi^{\frac{1}{2}}c}{\lambda\alpha^2} + \frac{2\pi}{3\Gamma(\frac{3}{4})\lambda c}\Big((c/\alpha)^3 M(1, \tfrac{7}{4}, (c/\alpha)^4)\Big)$$

$$-\frac{2\pi^{\frac{1}{2}}}{3\lambda c}\Big((c/\alpha)^6 M(1, \tfrac{5}{2}, (c/\alpha)^4)\Big). \qquad \text{(A15)}$$

When $a = 1$ then $(a)_n = n!$ and consequently equation (A14) gives

$$M(1, b, x) = \sum_{n=0}^{\infty} \{\Gamma(b)/\Gamma(n+b)\}x^n. \qquad \text{(A16)}$$

Equation (A15) now becomes

$$-\frac{\mathrm{d}\{b_1(c)\}}{\mathrm{d}c} = \frac{\pi^{\frac{1}{2}}c}{\lambda\alpha^2} + \frac{2\pi}{3\Gamma(\frac{3}{4})\lambda c}\left((c/\alpha)^3 \sum_{n=0}^{\infty} \frac{\Gamma(\frac{7}{4})}{\Gamma(n+\frac{7}{4})}(c/\alpha)^{4n}\right)$$

$$-\frac{2\pi^{\frac{1}{2}}}{3\lambda c}\left((c/\alpha)^6 \sum_{n=0}^{\infty} \frac{\Gamma(\frac{5}{2})}{\Gamma(n+\frac{5}{2})}(c/\alpha)^{4n}\right). \qquad \text{(A17)}$$

When the constant of integration is chosen to conform with equation (47) we find that $b_1(c)$ is represented by the series

$$b_1(c) = \frac{\pi^{\frac{1}{2}}}{2\lambda}\frac{\langle c^2\rangle - c^2}{\alpha^2}$$

$$+\frac{\pi}{2\lambda}\sum_{n=0}^{\infty}\left(\frac{1}{\Gamma(n+\frac{7}{4})}\frac{\langle c^{4n+3}\rangle - c^{4n+3}}{(4n+3)\alpha^{4n+3}} - \frac{1}{\Gamma(n+\frac{5}{2})}\frac{\langle c^{4n+6}\rangle - c^{4n+6}}{(4n+6)\alpha^{4n+6}}\right), \qquad \text{(A18)}$$

where $V^2 \gg \langle C^2\rangle$ and q_{m} is independent of c.