COSMOLOGICAL MODELS WITH TWO FLUIDS

I. ROBERTSON–WALKER METRIC

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Abstract

Exact solutions in terms of elementary functions are given for flat, homogeneous and isotropic, relativistic cosmological models which contain two fluids, each with an equation of state of the form $p = (\nu_i - 1)\rho$ where p is the pressure, ρ is the density, ν_i $(i = 1, 2)$ is a constant, and $\nu_1/\nu_2 = (1+2n)/2(1+n)$ or $2(1+n)/(3+2n)$, $n = 0, 1, 2, \ldots$ For other forms of ν_1/ν_2, the relevant solution is given in terms of a hypergeometric function. The cases when one of the ν's is equal to $2/3$ or 2 are analogous to models with a Robertson–Walker metric with $k = \pm 1$ and to anisotropic models of the type discussed by Jacobs respectively. All solutions for the two-fluid models can be written in terms of elementary functions when $\nu_1 = 0$, which is analogous to a cosmological constant. The fact that all two-fluid solutions with $\nu_1 = 2/3$ which can be written in terms of elementary functions are given means that all such one-fluid solutions with $k = \pm 1$ are given.

I. Introduction

Many authors such as Harrison (1967) have published surveys of homogeneous and isotropic relativistic cosmological models for which the Robertson–Walker metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\{\mathrm{d}r^2/(1-kr^2) + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\} \tag{1}$$

holds and which contain one relativistic fluid with an equation of state

$$p = (\nu - 1)\rho\,, \qquad \nu \leqslant 2\,, \tag{2}$$

p and ρ being the pressure and density respectively of the fluid and ν a constant. Other authors such as Vajk (1969) and Hughston and Shepley (1970) have discussed models containing n noninteracting fluids with equations of state

$$p_i = (\nu_i - 1)\rho_i\,, \qquad i = 1, 2, \ldots n, \qquad 0 \leqslant \nu_i \leqslant 2\,, \tag{3}$$

where all the ν_i are constant. Some models with interacting fluids have also been discussed, for example, by McIntosh (1968*b*, 1970) and May and McVittie (1970). Models with the two fluids of dust ($\nu = 1$) and radiation ($\nu = 4/3$) have been closely examined by Chernin (1966), Jacobs (1967), McIntosh (1968*a*), and others.

In the present paper it is assumed that there are n fluids, each satisfying (3), and that the fluids do not interact. The conservation equations from Einstein's equations when the metric (1) holds then give

$$\rho_i R^{3\nu_i} = 3C_i/\kappa\,, \qquad i = 1, 2, \ldots n\,, \tag{4}$$

where the C_i are all constant and $\kappa = 8\pi G$ $(c = 1)$.

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Einstein's field equations with a nonzero cosmological constant λ give, in the case of the metric (1), the differential equations

$$3(\dot{R}^2+k)/R^2-\lambda = \kappa\rho = \kappa\sum_i \rho_i \tag{5a}$$

and

$$2\ddot{R}/R+(\dot{R}^2+k)/R^2-\lambda = -\kappa p = -\kappa\sum_i (\nu_i-1)\rho_i\,, \tag{5b}$$

where the dot denotes differentiation with respect to t and the summations are over $i = 1,2,\dots n$. It has been noted by many authors that the cosmological constant λ can be written on the middle and right-hand sides of (5a) and (5b) and counted as a fluid with

$$\kappa\rho_\lambda = \lambda = -\kappa p_\lambda\,, \qquad \nu_\lambda = 0\,. \tag{6}$$

Hughston and Shepley (1970) noted that the k/R^2 terms of (5) can also be transferred and counted as a fluid with

$$\kappa\rho_k = -3k/R^2 = -3\kappa p_k\,, \qquad \nu_k = 2/3\,. \tag{7}$$

The fluid with

$$\rho = p\,, \qquad \nu = 2\,, \tag{8}$$

can be termed a scalar fluid or an anisotropic fluid. Scalar fields in theories like those of Brans and Dicke (1961) or Hoyle and Narlikar (1963) act on the equations (5) like a fluid with this equation of state (see McIntosh 1970). Misner (1968) and Hughston and Shepley (1970) noted that so also do the anisotropic effects in the models discussed by Misner, Jacobs (1968), and Shikin (1968).

Thus the main types of fluids encountered in cosmological models are:

$\nu = 2$,	scalar or anisotropic fluid	(9a)
$\nu = 4/3$,	radiation fluid	(9b)
$\nu = 1$,	dust fluid	(9c)
$\nu = 2/3$,	curvature fluid	(9d)
$\nu = 0$,	cosmological constant fluid	(9e)

In the following, the $k \neq 0$ and λ terms in equations (5) are included in the ρ and p terms so that the equations become

$$3\dot{R}^2/R^2 = \kappa\sum_i \rho_i\,, \tag{10a}$$

$$2\ddot{R}/R+\dot{R}^2/R^2 = \kappa\sum_i p_i = \kappa\sum_i (\nu_i-1)\rho_i\,. \tag{10b}$$

II. Behaviour of $\nu(t)$

Equations (2) and (3) give for n fluids

$$\nu-1 = p/\rho = \sum_i (\nu_i-1)\rho_i \,/\, \sum_i \rho_i \tag{11}$$

so that

$$\sum_i \rho_i(\nu_i-\nu) = 0 \qquad \text{or} \qquad \nu\rho = \sum_i \nu_i\rho_i\,. \tag{12}$$

Equations (4) and (12) combine to give

$$\nu = \sum_i C_i \nu_i R^{-3\nu_i} \,/\, \sum_i C_i R^{-3\nu_i}. \tag{13}$$

When this is differentiated with respect to t it follows that

$$-\tfrac{1}{3}\dot{\nu}\rho^2 H^{-1} = \sum_{i,j} \rho_i\rho_j(\nu_i-\nu_j)^2, \qquad i<j, \tag{14}$$

where H as usual denotes $\dot{R}/R$. If all the ρ_i are positive and R is a monotonic increasing function of t such that H is positive, ν is a monotonic decreasing function of t. Thus for an expanding n-fluid model with all the $\rho_i > 0$, ν changes from the highest value of ν_i through decreasing values to the lowest value as t increases. Thus, for example, if a model has a mixture of the five fluids in (9), it will start by being dominated by the scalar fluid, pass through a radiation-dominated stage, and so on.

If there are two fluids, equation (12) gives

$$\nu = (\nu_1\rho_1+\nu_2\rho_2)/(\rho_1+\rho_2)\,. \tag{15}$$

It follows that if

$$\nu_1 < \nu_2\,, \tag{16}$$

then

$$\rho_1 > 0,\quad \rho_2 > 0, \qquad \text{for} \qquad \nu_1 < \nu < \nu_2; \tag{17a}$$

$$\rho_1 > 0,\quad \rho_2 < 0, \qquad \nu < \nu_1; \tag{17b}$$

$$\rho_1 < 0,\quad \rho_2 > 0, \qquad \nu_2 < \nu\,. \tag{17c}$$

Thus if ρ_1 and ρ_2 have opposite signs, for H positive, ν is a monotonic increasing function. Since, from (10a), $\rho_1+\rho_2$ is positive, both ρ_1 and ρ_2 cannot be negative.

In all models with some of the ρ_i negative, $\rho = \sum \rho_i$ is positive and R reaches a maximum value when $H = 0$, $\rho = 0$. After this R decreases monotonically and H is negative. This happens, for example, in the oscillating models with $k = 0$ or -1 with $\rho_\lambda < 0$, or with $k = \pm 1$ and $\rho_\lambda < \rho_{\lambda,\mathrm{c}}$, some critical value of ρ_λ.

III. Two-fluid Model with Both ρ's Positive

When a model contains two fluids with both ρ's positive, equation (10a) gives

$$\dot{R}^2/R^2 = \tfrac{1}{3}\kappa(\rho_1+\rho_2) = C_1 R^{-3\nu_1} + C_2 R^{-3\nu_2} \tag{18}$$

such that

$$t+t_0 = \int R^{-1}(C_1 R^{-3\nu_1} + C_2 R^{-3\nu_2})^{-\frac{1}{2}}\,\mathrm{d}R\,, \tag{19}$$

where t_0 is a constant. This can be evaluated in terms of elementary functions for all ν_2 if $\nu_1 = 0$, that is, there is a cosmological constant fluid. The integral (19) can be rearranged to give an integral representation of a hypergeometric function with solution

$$3\nu_2(t+t_0) = 2C_2^{-\frac{1}{2}} R^{3\nu_2/2} F(\tfrac{1}{2}, b; b+1; -z), \tag{20}$$

where

$$b = \nu_2/2(\nu_2-\nu_1), \qquad z = (C_1/C_2)R^{3(\nu_2-\nu_1)}. \tag{21}$$

This hypergeometric function can be written in terms of elementary functions whenever b is a positive or negative integer or half-integer. This requirement is satisfied for two infinite sequences of ranges of ν_1/ν_2. Jacobs (1968) for the case of his "hard universes" gave solutions for the two infinite sequences of ν_1 when $\nu_2 = 2$. Solutions when ν_2 is not necessarily 2 can be written down in forms which are generalizations of Jacobs's solutions.

Where

$$b = 1, 2, 3, 4, \ldots, \tag{22a}$$

$$\nu_1/\nu_2 = (1+2n)/2(1+n) = \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{5}{6}, \tfrac{7}{8}, \ldots, \tag{22b}$$

$$n = (2\nu_1-\nu_2)/2(\nu_2-\nu_1) = 0, 1, 2, 3, \ldots, \tag{22c}$$

the solution (20) can be written as

$$t+t_0 = \frac{2(n+1)!}{3\nu_2(n+\frac{1}{2})!} \frac{C_2^{n+\frac{1}{2}}}{C_1^{n+1}} (1+z)^{\frac{1}{2}} \sum_{\lambda=0}^{n} (-1)^{n-\lambda} \frac{(\lambda-\frac{1}{2})!}{\lambda!} z^\lambda, \tag{23}$$

where z is given by (21).*

Where

$$b = \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{7}{2}, \tfrac{9}{2}, \ldots, \tag{24a}$$

$$\nu_1/\nu_2 = 2(1+m)/(3+2m) = \tfrac{2}{3}, \tfrac{4}{5}, \tfrac{6}{7}, \tfrac{8}{9}, \ldots, \tag{24b}$$

$$m = (3\nu_1-2\nu_2)/2(\nu_2-\nu_1) = 0, 1, 2, 3, \ldots, \tag{24c}$$

the solution (20) can be written as

$$t+t_0 = \frac{(-1)^m(2m+3)!}{3\nu_2\, 2^{2m+1}\{(m+1)!\}^2} \frac{C_2^{m+1}}{C_1^{m+3/2}}$$
$$\times \left(z^{\frac{1}{2}}(1+z)^{\frac{1}{2}} \sum_{\lambda=0}^{m} \frac{(-1)^\lambda 2^{2\lambda} (\lambda!)^2}{(2\lambda+1)!} z^\lambda - \ln\{z^{\frac{1}{2}} + (1+z)^{\frac{1}{2}}\} \right), \tag{25}$$

where z is again given by (21).

Negative values of b would arise if ν_1/ν_2 were taken to be greater than unity. The case $b = \frac{1}{2}$ occurs when

$$\nu_1 = 0, \qquad R^{3\nu_2} = (C_2/C_1)\sinh^2\{\tfrac{1}{2}C_1^{\frac{1}{2}}\, 3\nu_2(t+t_0)\} \tag{26}$$

(Harrison 1967). In all these cases, the requirement that $R = 0$ when $t = 0$ leads to $t_0 = 0$.

* It is to be noted that in equation (36a) of Jacobs (1968) $(n-\nu+\frac{1}{2})!$ should read $(n-\nu-\frac{1}{2})!$

Equation (22b) includes the following combinations of ν's:

$$(\nu_1, \nu_2) = (1,2), \quad (5/3,2), \quad (2/3,4/3), \quad (1,4/3), \tag{27}$$

and (24b) includes the combinations

$$(\nu_1, \nu_2) = (4/3,2), \quad (4/3,5/3), \quad (2/3,1). \tag{28}$$

Solutions in these cases are listed separately by Vajk (1969).* For example, for the flat space model with dust and radiation,

$$\nu_1 = 1, \quad \nu_2 = 4/3, \quad n = 1, \quad z = C_1 R/C_2 \tag{29}$$

and (23) gives

$$t+t_0 = \tfrac{2}{3} C_1^{-2}(C_1 R - 2C_2)(C_1 R + C_2)^{\frac{1}{2}}. \tag{30}$$

This solution has been given by many authors.

When, for example, $\nu_2 = 1$, equations (22b) and (24b) yield

$$\nu_2 = 1, \qquad \nu_1 = \begin{cases} \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \ldots \\ \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots \end{cases} \tag{31}$$

When $\nu_1 = 1$, they yield

$$\nu_1 = 1, \qquad \nu_2 = \begin{cases} 2, \frac{4}{3}, \frac{6}{5}, \ldots \\ \frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \ldots \end{cases} \tag{32}$$

Thus $t = t(R)$ can be written in terms of elementary functions in the two-fluid case with dust and another fluid having a value of ν given by (31) or (32).

IV. Two-fluid Model with One ρ Negative

When one of ρ_1 and ρ_2 is negative, say ρ_1, the integral (19) is replaced by

$$t+t_0 = \int R^{-1}(B_2 R^{-3\nu_2} - B_1 R^{-3\nu_1})^{-\frac{1}{2}}\, \mathrm{d}R, \tag{33}$$

where

$$B_2 = C_2, \qquad B_1 = -C_1 > 0. \tag{34}$$

This has as solution

$$3\nu_2(t+t_0) = 2B_2^{-\frac{1}{2}} R^{3\nu_2/2} F(\tfrac{1}{2}, b; b+1; z), \tag{35}$$

where b and z are given by (21).

With ν_1/ν_2 given by (22b) this becomes

$$t+t_0 = \frac{2(n+1)!}{3\nu_2(n+\frac{1}{2})!}\frac{B_2^{n+\frac{1}{2}}}{B_1^{n+1}}\left(1-(1-z)^{\frac{1}{2}} \sum_{\lambda=0}^{n} \frac{(\lambda-\frac{1}{2})!}{\lambda!} z^{\lambda}\right), \tag{36}$$

* Vajk's solution (B9a) for $\nu_1 = 5/3$, $\nu_2 = 2$ has $(A(\tau)-b)^{\frac{1}{2}}$ on the left-hand side. This should read $(A(\tau)+b)^{\frac{1}{2}}$.

and with ν_1/ν_2 given by (24b) it becomes

$$t+t_0 = \frac{(2m+3)!}{3\nu_2\, 2^{2m+1}\{(m+1)!\}^2} \frac{B_2^{m+1}}{B_1^{m+3/2}} \times \left(\arcsin z^{\frac{1}{2}} - z^{\frac{1}{2}}(1-z)^{\frac{1}{2}} \sum_{\lambda=0}^{m} \frac{2^{2\lambda}(\lambda!)^2}{(2\lambda+1)!} z^{\lambda}\right). \tag{37}$$

When $b = \frac{1}{2}$,

$$\nu_1 = 0, \qquad R^{3\nu_2} = (B_2/B_1)\sin^2\{\tfrac{1}{2}B_1^{\frac{1}{2}}\, 3\nu_2(t+t_0)\}. \tag{38}$$

In all these cases, $t_0 = 0$ if $R = 0$ when $t = 0$.

V. One-fluid Model with $k = \pm 1$

As mentioned in Section I, the k/R^2 terms in equations (5) act in those equations in the same way as does a fluid with $\nu = 2/3$. In this case the C_1 of equation (4) is equal to unity. For a two-fluid model with one of the ν's, say ν_1, equal to 2/3, with both ρ_1 and ρ_2 positive, and with the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\{\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\}, \tag{39}$$

$t = t(R)$ can be obtained from either (23) or (25) if ν_1/ν_2 fits into the sequences (22b) or (24b) respectively, or else it can be expressed as a hypergeometric function by equation (20). Then ρ_2 is known since

$$\rho_2 R^{3\nu_2} = C_2. \tag{40}$$

Exactly the same expression $t(R)$ gives the one-fluid solution when $k = -1$ in the Robertson–Walker metric (1). In this case $\rho(t)$ has the same form as $\rho_2(t)$ in equation (40).

Similarly, if $\nu_1 = 2/3$ and ρ_1 is negative, $t = t(R)$ from (35), (36), or (37) is the same R as that in the one-fluid case with $k = +1$ in metric (1). In both cases ρ_2 is given by (40) with $C_2 = B_2$.

Similar results can be obtained with ρ_1 positive and ρ_2 negative.

Thus equations (23), (25), (36), and (37) give all the one-fluid solutions which can be written in terms of elementary functions when $k = \pm 1$.

VI. Anisotropic Models

Jacobs (1968), Shikin (1968), and others have studied cosmological models with the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \{A^2(t)\,\mathrm{d}x^2 + B^2(t)\,\mathrm{d}y^2 + W^2(t)\,\mathrm{d}z^2\}. \tag{41}$$

Where R is the geometric mean of A, B, and W, that is,

$$R^3 = ABW, \tag{42}$$

this can be written as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\{P^2(t)\,\mathrm{d}x^2 + Q^2(t)\,\mathrm{d}y^2 + \mathrm{d}z^2/P^2(t)\,Q^2(t)\}, \tag{43}$$

with

$$A = RP, \qquad B = RQ, \qquad W = R/PQ. \tag{44}$$

Then equations (5a) and (5b) are replaced by

$$3\dot{R}^2/R^2 - \dot{P}^2/P^2 - \dot{Q}^2/Q^2 - \dot{P}\dot{Q}/PQ = \kappa\rho \tag{45a}$$

and

$$2\ddot{R}/R + \dot{R}^2/R^2 + \dot{P}^2/P^2 + \dot{Q}^2/Q^2 + \dot{P}\dot{Q}/PQ = -\kappa p\,, \tag{45b}$$

or

$$3\dot{R}^2/R^2 = \kappa(\rho + \rho_a) \tag{46a}$$

and

$$2\ddot{R}/R + \dot{R}^2/R^2 = -\kappa(p + p_a)\,, \tag{46b}$$

where

$$\kappa\rho_a = \kappa p_a = \dot{P}^2/P^2 + \dot{Q}^2/Q^2 + \dot{P}\dot{Q}/PQ, \qquad \nu_a = 2\,. \tag{47}$$

Thus, as mentioned in Section I, the effect of the anisotropic functions P and Q on the differential equations (5a) and (5b) is the same as that of a fluid with $\nu = 2$. Since

$$\rho_a R^6 = C_a\,, \tag{48}$$

ρ_a must be a monotonic decreasing function of t in an expanding universe. This does not, however, give information about the individual behaviour of P and Q with respect to t. Two of the functions P, Q, and $1/PQ$ may either increase or decrease monotonically. There is a pancake (cigar) singularity in the limit as t approaches zero if two increase (decrease). The discussion on types of singularities is then a discussion on relative behaviour of P and Q.

When one of the three functions A, B, and W is equal to the geometric mean R, say,

$$A = R, \qquad P = 1\,, \tag{49}$$

then

$$\kappa\rho_a = \dot{Q}^2/Q^2\,. \tag{50}$$

The same contribution is also obtained from the scalar Q where

$$T_{Q\,ij} = (2/\kappa Q)(Q_i\,Q_j - \tfrac{1}{2}g_{ij}\,Q_k\,Q^k)\,, \qquad Q_i = \partial Q/\partial x^i\,, \tag{51}$$

in the field equations

$$G_{ij} = -\kappa(T_{ij} + T_{Q\,ij})\,, \tag{52}$$

as in the scalar tensor theories of Brans and Dicke (1961) and Hoyle and Narlikar (1963).

Thus a one-fluid anisotropic model, a one-fluid scalar tensor model, and a two-fluid model with $\nu_2 = 2$ all give the same form

$$\rho_1(t) = C_1 R^{-3\nu_1}\,, \tag{53}$$

where R is given by (20) or (35), i.e. by (23), (25), (36), or (37) if it can be written in terms of elementary functions.

It was shown in Section II that for an expanding universe, ν is a monotonic decreasing function. Thus, for a two-fluid model with $\nu_2 = 2$ (the highest possible value of ν), the model acts like one filled with the $\nu_2 = 2$ fluid for early t and like one filled with the other fluid for later t. It is then obvious that anisotropic models with the metric (41) will tend towards isotropic ones as t increases, as was noted by Jacobs (1968). He also noted that his "Zel'dovich universe", i.e. an anisotropic model with a $\nu = 2$ fluid, remains anisotropic. This is obvious since it acts like a one-fluid model with $\nu = 2$. The solution is thus $R \propto t^{\frac{1}{3}}$.

VII. Conclusions

Although all the solutions for models with two noninteracting fluids where the fluids have the most common values of ν as given by (9) are in the literature, the two general solutions (23) and (25) where both ρ_1 and ρ_2 are positive include all these solutions and many others as well. It is not expected that many of the combinations of ν's where the solutions are new will be needed very often, but the general solutions will be useful in obtaining general properties as well as in simplifying the present solutions. Similarly it is not expected that the new solutions with one of ρ_1 and ρ_2 negative will be often used except for cases with $\nu_1 = 0$ $(\lambda < 0)$ and $\nu_1 = 2/3$ $(k = +1)$.

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