# COSMOLOGICAL MODELS WITH TWO FLUIDS

## II.* CONFORMAL AND CONFORMALLY FLAT METRICS

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*Abstract*

Similar solutions to those in Part I are given for two-fluid cosmological models when the Robertson–Walker metric in its usual form is replaced by "conformal" and "conformally flat" forms. In these cases the solutions can be written in terms of elementary functions when $(3\nu_1-2)/(3\nu_2-2) = 1-m^{-1}$, $m = 1, 2, 3, \ldots$ The relation of these solutions to one-fluid solutions with $k = \pm 1$ is also discussed.

## I. Introduction

It was shown in Part I (McIntosh 1972, present issue pp. 75–82) that for relativistic cosmological models with the Robertson–Walker metric in the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\{\mathrm{d}r^2/(1-kr^2) + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\}\,, \tag{1}$$

exact solutions for $R(t)$ can be found in terms of elementary functions for particular cases when $k = 0$ and there are two noninteracting fluids or when $k = \pm 1$ and there is one fluid. The fluids are assumed to have equations of state

$$p_i = (\nu_i - 1)\rho_i\,, \qquad 0 \leqslant \nu_i \leqslant 2\,, \quad i = 1, 2\,, \tag{2}$$

where the $\nu_i$ are constant and $p_i$ and $\rho_i$ are the pressure and density respectively of the $i$th fluid. For $k = 0$ and

$$\nu_1/\nu_2 = 1 - m^{-1}\,, \tag{3}$$

the solution is given by

$$3\nu_2(t+t_0) = 2C_2^{-\frac{1}{2}}\, R^{3\nu_2/2}\, F(\tfrac{1}{2}, \tfrac{1}{2}m; 1+\tfrac{1}{2}m; -z)\,, \tag{4}$$

where $C_2$ and $t_0$ are constants and

$$z = (C_1/C_2)R^{3(\nu_2-\nu_1)}\,. \tag{5}$$

Where $m$ is an integer, this hypergeometric function can be written in terms of elementary functions. The resultant series are given in Part I. Since there is no interaction between the fluids, the field equations together with the metric (1) yield the conservation equations

$$\rho_i R^{3\nu_i} = 3C_i/\kappa\,, \qquad i = 1, 2\,, \tag{6}$$

where the $C_i$ are constants and $\kappa = 8\pi G$ ($c = 1$).

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If $\nu_1 = 0$, $R(t)$ can be expressed in terms of elementary functions for all values of $\nu_2$. These solutions (equations (26) of Part I) are appropriate for the case of a single fluid and a nonzero cosmological constant, since the latter can be regarded as a fluid with $\nu = 0$.

A fluid with $\nu = 2/3$ has the same effect on the differential equations for $R(t)$ as does the curvature in the 3-space part of the metric (1). Thus the solutions for $R(t)$ with either $\nu_1$ or $\nu_2 = 2/3$ are those of a one-fluid model when (1) holds with $k = \pm 1$ as well as being those for a two-fluid model with $k = 0$ and one of the $\nu_i = 2/3$.

A two-fluid model with $\nu_2 = 2$ has the same solution for $R(t)$ as does a one-fluid anisotropic model or a one-fluid scalar tensor model. These solutions are also given by Jacobs (1968).

There are two other forms of the Robertson–Walker metric (1) which are frequently used in the literature. These are the "conformally flat" form discussed by Infeld and Schild (1945) and, in a cosmological context, by Tauber (1967), and the "conformal" form used, for example, by Vajk (1969), Wataghin (1969), Agnese (1970), and Gilman (1970).

Since the metric (1) is conformally flat, it can be written as

$$\mathrm{d}s^2 = \{f(v)/w^2\}\eta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \tag{7}$$

where

$$\eta_{ij} = \mathrm{diag}(1,-1,-1,-1), \qquad x^i = (\tau, x, y, z)\,, \tag{8}$$

$$v = \tau/w, \qquad w = 1-\tfrac{1}{4}ks^2 \quad (k = 0, \pm 1)\,, \tag{9}$$

and

$$s^2 = \tau^2-r^2\,, \qquad r^2 = x^2+y^2+z^2\,. \tag{10}$$

The conformal Minkowskian metric (7) is metrically, though not topologically, equivalent to the Robertson–Walker form (1) (Infeld and Schild 1945). The mapping between the two forms is obvious for $k = 0$, in which case

$$R(\tau)\,\mathrm{d}\tau = R(t)\,\mathrm{d}\tau = \mathrm{d}t\,, \tag{11}$$

and (1) becomes

$$\mathrm{d}s^2 = R^2(\tau)\{\mathrm{d}\tau^2-\mathrm{d}r^2-r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\}\,, \tag{12}$$

so that

$$w = 1, \qquad f(v) = f(\tau) = R^2(\tau)\,. \tag{13}$$

The conformal form occurs when the metric (1) is replaced by

$$\mathrm{d}s^2 = R^2(\tau)\left\{\mathrm{d}\tau^2-\left(\frac{\mathrm{d}r^2}{1-kr^2}+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\right)\right\} \tag{14}$$

under the mapping (11).

The two forms are obviously identical when $k = 0$. Solutions in the cases when there is one fluid with $k = 0, \pm 1$ have been given basically by Tauber (1967) for the metric (7) and explicitly by Gilman (1970) for the metric (14). In the present paper, two-fluid solutions are discussed for both metrics when $k = 0$ and they are related to the one-fluid solutions when $k = \pm 1$.

## II. Basic Equations for Metric (7)

Tauber (1967) has discussed the $k = +1$ case in terms of the variables defined in equations (7)–(10), and has discussed the $k = -1$ case with $f(v)/w^2$ replaced by $\exp(\Gamma(s))$. This latter step is not possible with $k = +1$, and all cases ($k = 0, \pm 1$) will be discussed here in terms of the variables of equations (7)–(10). The following equations are then a generalization of Tauber's $k = +1$ equations.

The partial derivatives of $w$ and $v$ are

$$w_i = -\tfrac{1}{2}kss_i = -\tfrac{1}{2}k\eta_{ij}x^j, \quad w^i = -\tfrac{1}{2}kx^i, \tag{15}$$

$$v_i = w^{-1}[\eta_{i0} - vw_i], \qquad v^i = w^{-1}[\eta^{i0} - vw^i], \tag{16}$$

such that also*

$$w_i w^i = \tfrac{1}{4}s^2k^2 \doteq k(1-w), \tag{17a}$$

$$w_i v^i = -(vk/2w)(2-w), \tag{17b}$$

$$v_i v^i = (1+kv^2)/w^2. \tag{17c}$$

The Einstein tensor is then given by

$$G_{ij} = v_i v_j(\psi'' - \tfrac{1}{2}\psi'^2) - (\eta_{ij}/w^2)\{(\psi'' + \tfrac{1}{4}\psi'^2)(1+kv^2) + 3k(\psi'v+1)\}, \tag{18}$$

where

$$\psi = \ln f \tag{19}$$

and the primes denote differentiation with respect to $v$.

For a perfect fluid the energy–momentum tensor has the form

$$T_{ij} = f(v)[(p+\rho)\{v_i v_j/(1+kv^2)\} - (p/w^2)\eta_{ij}], \tag{20}$$

so that, upon equating terms proportional to $v_i v_j$ and $\eta_{ij}$, the field equations

$$G_{ij} = -\kappa T_{ij} \tag{21}$$

yield the differential equations

$$(1+kv^2)(\psi'' - \tfrac{1}{2}\psi'^2) = -\kappa\, e^{\psi}(p+\rho) \tag{22a}$$

and

$$(1+kv^2)(\psi'' + \tfrac{1}{4}\psi'^2) + 3k(\psi'v+1) = -\kappa\, e^{\psi}p. \tag{22b}$$

The conservation laws yield

$$\rho' + \tfrac{3}{2}(p+\rho)\{\psi' + 2kv/(1+kv^2)\} = 0. \tag{23}$$

* There is a factor of $\frac{1}{2}$ missing in the equation equivalent to (17b) in Tauber's (1967) paper. It should also be noted that in his equation (2.11a) the $a$ in the term $(12a/A^2)$ should be deleted.

Equations (22a) and (22b) can then be replaced by (23) and

$$\tfrac{3}{4}(1+kv^2)\psi'^2+3k(\psi'v+1) = \kappa e^{\psi}\rho. \tag{24}$$

Where there are $n$ fluids, each having an equation of state of the form (2), and where there is no interaction between the fluids, the relation

$$\kappa\rho_i = 3A_i V^{-3\nu_i/2}, \qquad i = 1, 2, \ldots n, \tag{25}$$

follows from (23), where the $A_i$ are constants and

$$V = e^{\psi}(1+kv^2). \tag{26}$$

Equation (24) can then be written as

$$(1+kv^2)V'/2V = \left(\sum_{i=1}^{n} A_i V^{(2-3\nu_i)/2} - k\right)^{\frac{1}{2}}. \tag{27}$$

A fluid with $\nu_i = 2/3$ and $A_i = -k$ gives the same contribution on the right-hand side of (27) as does the $-k$ term. But as $k$ also appears on the left-hand side of (27), a $\nu_i = 2/3$ fluid does not give the same form of $V$ as does the curvature term.

With $k = 0$ and a single fluid, equation (27) gives

$$f(v) = e^{\psi} = (Bv+C)^{4/(3\nu-2)}, \tag{28}$$

where $B$ and $C$ are constants. Then, from (13),

$$R(\tau) = (B\tau+C)^{2/(3\nu-2)}. \tag{29}$$

This could have been immediately derived from (11) with

$$R(t) = (Bt+C)^{2/3\nu}, \tag{30}$$

the equivalent Robertson–Walker solution.

When $k = -1$ and with a single fluid, equation (27) gives

$$\exp\{\tfrac{1}{4}(3\nu-2)\psi\} = B(1+v)^{(2-3\nu)/2}+C(1-v)^{(2-3\nu)/2}, \tag{31}$$

where

$$A_1 = -4BC \tag{32}$$

and $B$ and $C$ are constants.

When $k = +1$,* equation (27) gives

$$\exp\{\tfrac{1}{4}(3\nu-2)\} = (1+v^2)^{(2-3\nu)/4} B\sin(\theta+\theta_0) \tag{33}$$

or

$$V^{(3\nu-2)/4} = B\sin(\theta+\theta_0), \tag{34}$$

where

$$2(\theta+\theta_0) = (3\nu-2)\arctan(v+v_0), \tag{35}$$

$B$, $\theta_0$, and $v_0$ are constants, and

$$A_1 = B^2. \tag{36}$$

* For $k = +1$ Tauber (1967) gives the solution as $F(3\nu/4, (3\nu-2)/4; 3\nu/2; 1+kv^2)$. (This solution also holds for $k = -1$.) He states that the solution becomes degenerate for particular values of $\nu$. In fact it is degenerate for all $\nu$.

If $V = 0$ when $v = 0$, then $\theta_0 = v_0 = 0$ in the $k = +1$ case and $B+C = 0$ in the $k = -1$ case.

## III. Basic Equations for Metric (14)

The metric (14) and Einstein's field equations give the differential equations (cf. Agnese 1970)

$$3R_\tau^2/R^4 + 3k/R^2 = \kappa\rho \tag{37a}$$

and

$$2R_{\tau\tau}/R^3 - R_\tau^2/R^4 + k/R^2 = -\kappa p\,, \tag{37b}$$

where

$$R_\tau = \mathrm{d}R/\mathrm{d}\tau\,. \tag{38}$$

Under the transformation (11) these take the form of the usual Friedmann equations of the Robertson–Walker metric. Thus where there are $n$ noninteracting fluids with equations of state (2),

$$\kappa\rho_i(\tau)\,R^{3\nu_i}(\tau) = 3A_i \tag{39a}$$

and

$$R_\tau^2 = R^2\left(\sum_{i=1}^{n} A_i R^{2-3\nu_i} - k\right). \tag{39b}$$

Gilman (1970) gives the one-fluid solutions as

$$\kappa\rho(\tau)\,R^{3\nu}(t) = 3A\,, \tag{40a}$$

$$R^{3\nu-2}(\tau) = AS^2\{\tfrac{1}{2}(3\nu-2)(\tau+\tau_0)\}\,, \tag{40b}$$

where $A$ is a constant and

$$\left.\begin{aligned} S(\chi) &= \sin\chi \qquad \text{for} \qquad & k &= +1\,,\\ &= \chi & &= \phantom{+}0\,,\\ &= \sinh\chi & &= -1\,. \end{aligned}\right\} \tag{41}$$

## IV. Two-fluid Solutions for Both Metrics with $k = 0$

With $k = 0$, the metrics (7) and (14) are the same with

$$v = \tau, \qquad V(v) = f(v) = R^2(\tau)\,. \tag{42}$$

For two fluids, equations (27) and (39b) both give

$$\tau+\tau_0 = \int R^{-1}(A_1 R^{2-3\nu_1} + A_2 R^{2-3\nu_2})^{-\frac{1}{2}}\,\mathrm{d}R\,. \tag{43}$$

This has the same form as the integral (19) in Part I, and hence the solution can be written in terms of elementary functions whenever

$$\mu_1/\mu_2 = 1 - m^{-1}\,, \tag{44}$$

where $m$ is an integer and

$$3\mu_i = 3\nu_i - 2\,. \tag{45}$$

In such cases the solutions may be obtained from Section III of Part I by replacing $C_i$ by $A_i$ and $\nu_i$ by $\mu_i$. Thus the general solution when $A_1$ and $A_2$ are positive is

$$3\mu_2(\tau+\tau_0) = 2A_2^{-\frac{1}{2}} R^{3\mu_2/2} F(\tfrac{1}{2}, \tfrac{1}{2}m; \tfrac{1}{2}m+1; -z)\,, \tag{46}$$

where

$$m = \mu_2/(\mu_2-\mu_1) = (3\nu_2-2)/3(\nu_2-\nu_1)\,, \tag{47}$$

$$z = (A_1/A_2)R^{3(\mu_2-\mu_1)} = (A_1/A_2)R^{3(\nu_2-\nu_1)}\,. \tag{48}$$

Thus when $m$ is even, for example, and when $A_1$ and $A_2$ are positive

$$\tau+\tau_0 = \frac{2(n+1)!}{(3\nu_2-2)(n+\frac{1}{2})!}\frac{A_2^{n+\frac{1}{2}}}{A_1^{n+1}}(1+z)^{\frac{1}{2}} \sum_{\lambda=0}^{n} (-1)^{n-\lambda}\frac{(\lambda-\frac{1}{2})!}{\lambda!}z^\lambda\,, \tag{49}$$

with $m = 2(n+1)$, $n = 0, 1, \ldots$ The other three solutions (including the two cases where one of the $A_i$ is positive and the other negative) may be deduced from Part I.

When $\mu_1 = 0$ ($\nu_1 = 2/3$) and both $A_1$ and $A_2$ are positive

$$R^{3\mu_2} = R^{3\nu_2-2} = |(A_2/A_1)|\sinh^2\{\tfrac{1}{2}|A_1|^{\frac{1}{2}}(3\nu_2-2)(\tau+\tau_0)\}\,. \tag{50}$$

With $A_1$ negative, the sinh is replaced by sin.

## V. One-fluid Solutions for $k = \pm 1$

It was shown in the previous section that for $k = 0$ it is possible to find a solution when $\nu_1 = 2/3$ for any $\nu_2$. In terms of the metric (14) the solutions for $k = \pm 1$ have already been given. Solution (50) with $A_1 = 1$ is the same as solution (40b) with $k = -1$. Similarly (50) with $A_1 = -1$ is the same as (40b) with $k = +1$.

For the conformally flat case the situation is slightly more complicated. The left-hand side of equation (27) is $(1+kv^2)V'/2V$ and thus it is no longer true that $\tau = v$. The solutions equivalent to (50) in this case ($A_1 = 1$) are

$$\arctan(v+v_0) = 2/(3\nu-2)\arcsin(A_2^{-\frac{1}{2}}\, V^{(3\nu-2)/4}), \quad k = +1\,, \tag{51a}$$

$$\operatorname{artanh}(v+v_0) = 2/(3\nu-2)\operatorname{arsinh}(A_2^{-\frac{1}{2}}\, V^{(3\nu-2)/4}), \quad k = -1\,. \tag{51b}$$

These are the same as solutions (33) and (31).

## VI. Conclusions

As was the case in Part I, the solutions for the most common values of $\nu_1$ and $\nu_2$ are already in the literature and are included here in the more general solutions. The solutions in terms of elementary functions for the metrics (7) and (14) generally occur for different cases of $(\nu_1, \nu_2)$ from those for the Robertson–Walker metric (1). Such solutions exist for the same values of $(\nu_1, \nu_2)$ when and only when $R(\tau)\,\mathrm{d}\tau = \mathrm{d}t$ (equation (11)) can be integrated in terms of elementary functions. For example, such special combinations as

$$(\nu_1, \nu_2) = (1, 4/3), \quad (4/3, 2), \quad (5/3, 2), \quad (4/3, 5/3) \tag{52}$$

allow solutions in any of the three metrics (e.g. Vajk 1969, where his $\psi$ is equivalent to the present $\tau$). But the case with one fluid with $\mu_1 = -2/3$ ($\nu_1 = 0$) cannot be integrated in terms of elementary functions for any value of $\mu_2$ for $0 < \mu_2 \leqslant 4/3$. Thus no solution of this type can be found for the metrics (7) and (14) when there is a cosmological constant, unlike the Robertson–Walker case, for which solutions can be found for all $\nu_2$ when $\nu_1 = 0$.

Solutions for models with three noninteracting fluids are limited to a small range of $(\nu_1, \nu_2, \nu_3)$. The only solutions using common values of $\nu$ are

$$(\nu_1, \nu_2, \nu_3) = (2/3, 1, 4/3), \quad (4/3, 5/3, 2)$$

(Vajk 1969) and

$$(\nu_1, \nu_2, \nu_3) = (0, 1, 2)\,.$$

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