RADIATION FROM ACCELERATED POINT DIPOLES IN CIRCULAR AND KEPLERIAN ORBITS

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[Manuscript received 20 September 1971]

Abstract

The frequency spectrum of electromagnetic radiation from accelerated magnetic dipoles is computed for circular and Keplerian motion. For circular motion three orientations of the dipole are considered: (1) parallel to the velocity vector, (2) parallel to the radius vector, and (3) perpendicular to the plane of the circle. For Keplerian motion the orientation is taken as perpendicular to the orbit. Expressions that are valid for velocities arbitrarily small and arbitrarily close to the velocity of light are derived. The mean power radiated is determined numerically.

I. INTRODUCTION

Radiation from accelerated point dipoles is relevant to the study of spinning white dwarf and neutron stars with shifted dipole fields, and to the motion of such stars in a Keplerian orbit. In recent years several authors (Ellis 1963, 1966; Ward 1964, 1965; Kolsrud and Leer 1967; Monaghan 1968) have established expressions for the radiation from an arbitrary point dipole moving in any manner. This work is extended in the present paper by calculating the spectrum to be expected from point dipoles in uniform circular motion or in a Keplerian orbit.

II. FIELDS OF THE DIPOLE

Consider a point dipole moving in an unspecified way. Let the dipole be a pure magnetic dipole in the reference frame at rest with the dipole. In any other reference frame the dipole will appear as a point magnetic dipole $\boldsymbol{\mu}$ and a point electric dipole $\boldsymbol{p}$ (Panofsky and Phillips 1962). Let the velocity of the dipole be $\boldsymbol{v}$ and let $\boldsymbol{R}$ be the vector from the point where the fields are measured to the position of the dipole. It may then be shown (Monaghan 1968) that the magnetic field due to the moving dipole is

$$\boldsymbol{B} = \left[\frac{3(\boldsymbol{\mu}\cdot\boldsymbol{n})\boldsymbol{n}-\boldsymbol{\mu}}{R^3} + \frac{R}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3(\boldsymbol{\mu}\cdot\boldsymbol{n})\boldsymbol{n}-\boldsymbol{\mu}}{R^3}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(\frac{3(\boldsymbol{\mu}\cdot\boldsymbol{n})\boldsymbol{n}-\boldsymbol{\mu}}{R^2c^2}\right)\frac{\mathrm{d}R}{\mathrm{d}t} - \frac{\boldsymbol{p}\times\boldsymbol{n}}{\kappa R^2 c}\right\}\right.$$
$$\left. + \frac{1}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\mu})+\boldsymbol{p}\times\boldsymbol{n}}{\kappa R}\right)\right]_{\mathrm{ret}}, \tag{1}$$

where c is the velocity of light,

$$\boldsymbol{n} = \boldsymbol{R}/R\,, \qquad \boldsymbol{p} = \boldsymbol{\beta}\times\boldsymbol{\mu}\,, \qquad \boldsymbol{\beta} = \boldsymbol{v}/c\,, \tag{2}$$

and

$$\kappa = 1-\boldsymbol{n}\cdot\boldsymbol{\beta}\,.$$

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The subscript ret indicates that the expression in square brackets is to be evaluated at the retarded time

$$t' = t - R(t')/c, \tag{3}$$

where the time dependence of R has been shown explicitly. The expression for the electric dipole moment in the second equation of (2) follows from the Lorentz transformation.

When R is very large only the last term on the right-hand side of (1) is important, and it is this term which gives the radiation. We therefore concentrate on $\boldsymbol{B}_{\mathrm{rad}}$ defined by

$$\boldsymbol{B}_{\mathrm{rad}} = \left[\frac{1}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\mu})+\boldsymbol{p}\times\boldsymbol{n}}{\kappa R}\right)\right]_{\mathrm{ret}}, \tag{4}$$

or, since derivatives of R may be neglected when only the radiation field is of interest,

$$\boldsymbol{B}_{\mathrm{rad}} = \left[\frac{1}{Rc^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\mu})+\boldsymbol{p}\times\boldsymbol{n}}{\kappa}\right)\right]_{\mathrm{ret}}. \tag{5}$$

The radiated power is given by the Poynting vector. Since the electrical field is equal in magnitude and perpendicular to $\boldsymbol{B}_{\mathrm{rad}}$ in the radiation zone, the radiated power per unit solid angle is then

$$\mathrm{d}P/\mathrm{d}\Omega = (c/4\pi)\,|\,R\boldsymbol{B}_{\mathrm{rad}}\,|^2. \tag{6}$$

III. Power Spectra

Set

$$A(t) = (c/4\pi)^{\frac{1}{2}}[R\boldsymbol{B}_{\mathrm{rad}}]_{\mathrm{ret}}, \tag{7}$$

so that

$$\mathrm{d}P/\mathrm{d}\Omega = |\,A(t)\,|^2. \tag{8}$$

If the motion is periodic, with period T, it is convenient to consider the mean power radiated per unit solid angle. This quantity is given by

$$\mathrm{d}w/\mathrm{d}\Omega = T^{-1}\int_0^T |\,A(t)\,|^2\,\mathrm{d}t. \tag{9}$$

Since $A(t)$ is periodic by assumption, it can be written in the form

$$A(t) = \sum_{-\infty}^{\infty} \boldsymbol{a}_s \exp(\mathrm{i}s\omega t), \tag{10}$$

where $\omega = 2\pi/T$. From Parseval's theorem and the fact that $A(t)$ is real, we find

$$\mathrm{d}w/\mathrm{d}\Omega = 2\sum_{s=0}^{\infty}{}'\,|\,\boldsymbol{a}_s\,|^2, \tag{11}$$

where the prime indicates that half the first term is taken. The power radiated per unit solid angle in the sth component is then

$$\mathrm{d}P_s/\mathrm{d}\Omega = 2\,|\boldsymbol{a}_s|^2. \tag{12}$$

From the usual expression for Fourier coefficients,

$$\boldsymbol{a}_s = T^{-1}\int_0^T A(t)\exp(-\mathrm{i}s\omega t)\,\mathrm{d}t, \tag{13}$$

so that, with the help of (5) and (7), we find

$$\boldsymbol{a}_s = \frac{1}{(4\pi c^3)^{\frac{1}{2}}T}\int_0^T \left[\frac{\mathrm{d}^2\boldsymbol{G}}{\mathrm{d}t^2}\right]_{\mathrm{ret}} \exp(-\mathrm{i}s\omega t)\,\mathrm{d}t, \tag{14}$$

where

$$\boldsymbol{G} = \{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\mu}) + (\boldsymbol{\beta}\times\boldsymbol{\mu})\times\boldsymbol{n}\}/\kappa. \tag{15}$$

It is convenient to work with the retarded time t', noting from the third equation of (2) and from (3) that

$$\mathrm{d}t = \kappa\,\mathrm{d}t'.$$

Accordingly, (14) becomes

$$\boldsymbol{a}_s = \frac{1}{(4\pi c^3)^{\frac{1}{2}}T}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{1}{\kappa}\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}t'}\right)\exp\left\{-\mathrm{i}s\omega\left(t'+\frac{R}{c}\right)\right\}\mathrm{d}t'. \tag{16}$$

Recalling that $\boldsymbol{R}$ is the vector from the point where the fields are measured to the position of the dipole, and letting $\boldsymbol{x}$ be the vector from the origin to the observation point and $\boldsymbol{r}$ the vector from the origin to the dipole, we have

$$\boldsymbol{x} = \boldsymbol{r}-\boldsymbol{R}$$

so that, when $|\boldsymbol{x}| \gg |\boldsymbol{r}|$,

$$R \sim x - \boldsymbol{n}.\boldsymbol{r}. \tag{17}$$

Consequently (16) becomes, apart from an overall phase factor $\exp(-\mathrm{i}s\omega x/c)$ which we henceforth omit,

$$\boldsymbol{a}_s = \frac{1}{(4\pi c^3)^{\frac{1}{2}}T}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{1}{\kappa}\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}t'}\right)\exp\left\{-\mathrm{i}s\omega\left(t'-\frac{\boldsymbol{n}.\boldsymbol{r}}{c}\right)\right\}\mathrm{d}t'. \tag{18}$$

Integrating by parts twice, and noting that in the radiation zone

$$\mathrm{d}(t'-\boldsymbol{n}.\boldsymbol{r}/c)/\mathrm{d}t = \kappa,$$

we find

$$\boldsymbol{a}_s = \frac{s^2\omega^2}{(4\pi c^3)^{\frac{1}{2}}T}\int_0^T \{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\mu}) + (\boldsymbol{\beta}\times\boldsymbol{\mu})\times\boldsymbol{n}\}\exp\left\{-\mathrm{i}s\omega\left(t'-\frac{\boldsymbol{n}.\boldsymbol{r}}{c}\right)\right\}\mathrm{d}t'. \tag{19}$$

IV. Uniform Motion in a Circle

The case of uniform circular motion is relevant to a spinning neutron star with a shifted dipole field of the type recently proposed for magnetic stars (Landstreet 1970; Preston 1970). Once the point dipole is shifted from the centre, it will be carried on a circular trajectory at uniform velocity (provided the slowing down due to radiation is neglected). We consider three orientations of the dipole: (*a*) parallel to $\boldsymbol{\beta}$; (*b*) parallel to the radius vector, and (*c*) perpendicular to the plane of the circle. The case of general alignment may be treated, but it leads to complicated expressions which are not very informative.

(*a*) $\boldsymbol{\mu}$ *Parallel to* $\boldsymbol{\beta}$

For this orientation equation (19) becomes

$$\boldsymbol{a}_s = \frac{s^2\omega^2}{(4\pi c^3)^{\frac{1}{2}}T}\boldsymbol{n}\times\left[\boldsymbol{n}\times\int_0^T \boldsymbol{\mu}\exp\left\{-\mathrm{i}s\omega\left(t'-\frac{\boldsymbol{n}\cdot\boldsymbol{r}}{c}\right)\right\}\mathrm{d}t'\right]. \tag{20}$$

We use a spherical polar coordinate system with origin at the centre of the circle and polar axis z perpendicular to the circle. The azimuthal angle of the dipole is denoted by ϕ, the radius of the circle by a, the constant angular velocity by ω, and the polar angle of the line of sight by θ. Without loss of generality, ϕ can be measured from the plane containing $\boldsymbol{n}$ and the polar axis. With the above coordinate system and the use of the instantaneous Lorentz transform, the components and magnitude of $\boldsymbol{\mu}$ are given by

$$\boldsymbol{\mu} = (\mu_0/\gamma)(-\sin\phi, \cos\phi, 0), \tag{21}$$

where the vector components correspond to the x, y, z cartesian axes, γ is the relativistic factor

$$\gamma = (1-\beta^2)^{-\frac{1}{2}}, \tag{22}$$

and μ_0 is the value of μ in the rest frame of the dipole. Substituting (21) into (20) we find $\boldsymbol{a}_s$ depends on integrals of the form

$$\int_0^{2\pi}\exp[-\mathrm{i}s(\phi-\omega a(\sin\theta\cos\phi)/c)]\begin{Bmatrix}1\\ \sin\phi\\ \cos\phi\end{Bmatrix}\mathrm{d}\phi,$$

where one of the terms in the braces may occur in the integrand. Such integrals can be expressed in terms of Bessel functions (Watson 1958) with the result that

$$|\boldsymbol{a}_s|^2 = \frac{s^4\omega^4\mu_0^2}{4\pi c^3\gamma^2}\left(\{\mathrm{J}_s'(z)\}^2+\frac{\cot^2\theta}{\beta^2}\mathrm{J}_s^2(z)\right), \tag{23}$$

where

$$z = s\omega a(\sin\theta)/c$$

and the standard notation for Bessel functions has been used. The prime denotes a derivative of the Bessel function with respect to its argument. The expression (23) is the dipole analogue of the formula derived by Schott (1912) for the power radiated by a charged particle in uniform motion. It may be noted, in the interest of historical accuracy, that the prediction of the harmonics in the fields of a charged particle in uniform circular motion and the calculation of the first few terms are due to Heaviside (1904).

When β is sufficiently small the main contribution to the power comes from the term $|\boldsymbol{a}_1|^2$. In this case

$$\mathrm{d}P_1/\mathrm{d}\Omega = \omega^4\mu_0^2(1+\cos^2\theta)/8\pi c^3. \tag{24}$$

The mean radiated power is then

$$w = 2\mu_0^2\,\omega^4/3c^3, \tag{25}$$

to this approximation. The expression (25) for w is exactly the same as the corresponding expression for the mean power radiated by a spinning point dipole with its centre at rest.

In the general case the mean power radiated per unit solid angle is

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega} = \frac{\omega^4\mu_0^2}{2\pi c^3\gamma^2}\left(\sum_{s=1}^{\infty} s^4\{\mathrm{J}_s'(z)\}^2 + \frac{\cot^2\theta}{\beta^2}\sum_{s=1}^{\infty} s^4\mathrm{J}_s^2(z)\right). \tag{26}$$

It is shown in the Appendix that

$$\sum_{s=1}^{\infty} s^4\mathrm{J}_s^2(s\epsilon) = \frac{\epsilon^2(64+592\epsilon^2+472\epsilon^4+27\epsilon^6)}{256(1-\epsilon^2)^{13/2}} \tag{27}$$

and

$$\sum_{s=1}^{\infty} s^4\{\mathrm{J}_s'(s\epsilon)\}^2 = \frac{64+624\epsilon^2+632\epsilon^4+45\epsilon^6}{256(1-\epsilon^2)^{11/2}}, \tag{28}$$

and accordingly

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega} = \frac{\omega^4\mu_0^2}{512\pi c^3\gamma^2}\left(\frac{\cos^2\theta(64+592\epsilon^2+472\epsilon^4+27\epsilon^6)}{(1-\epsilon^2)^{13/2}} + \frac{64+624\epsilon^2+632\epsilon^4+45\epsilon^6}{(1-\epsilon^2)^{11/2}}\right), \tag{29}$$

where $\epsilon = \beta\sin\theta$. It is evident from (29) that the radiation is beamed strongly in the plane of the circle when the motion is highly relativistic. The average radiated power may be obtained most conveniently by integrating $\mathrm{d}w/\mathrm{d}\Omega$ numerically. The resulting values of w for various values of β are given in column 2 of Table 1. It should be noted from the numerical results that the spinning dipole formula (25) becomes a very poor approximation for $\beta \gtrsim 0\cdot5$.

(*b*) $\boldsymbol{\mu}$ *Parallel to* $\dot{\boldsymbol{\beta}}$

With the coordinate system as in (*a*), and making use of the properties of Bessel functions, the following expression for $|\boldsymbol{a}_s|^2$ is readily found

$$|\boldsymbol{a}_s|^2 = \frac{s^4\omega^4\mu_0^2}{4\pi c^3}\left\{\left(\frac{1-\beta^2\sin^2\theta}{\beta\sin\theta}\right)^2\mathrm{J}_s^2+\cos^2\theta\,(\mathrm{J}_s')^2\right\}. \tag{30}$$

The argument of the Bessel functions occurring in (30) is $s\beta\sin\theta$. When $\beta \ll 1$ the mean power radiated per unit solid angle is given approximately by

$$\mathrm{d}w/\mathrm{d}\Omega = \omega^4\mu_0^2(1+\cos^2\theta)/8\pi c^3, \tag{31}$$

so that in the low β approximation the radiated power varies as ω^4 as for (*a*) above.

In the general case w may be found for various values of β by numerical integration. The results are given in column 3 of Table 1.

As in the previous case the radiated power increases very rapidly with β. When $1-\beta$ is small we find

$$w = (7\mu_0^2\omega^4/24c^3)(1-\beta)^{-4}.$$

Furthermore, by the use of equations (27) and (28), the expression for $\mathrm{d}w/\mathrm{d}\Omega$ can be shown to have a sharp maximum in the orbital plane when β is close to $1 \cdot 0$.

Table 1

POWER RADIATED BY MAGNETIC DIPOLE IN UNIFORM MOTION FOR THREE ORIENTATIONS

The values of power are given in units of $\mu_0^2\,\omega^4/c^3$

(1) β	(2) (a) $\boldsymbol{\mu} \parallel \boldsymbol{\beta}$	(3) Energy for dipole orientation (b) $\boldsymbol{\mu} \parallel \dot{\boldsymbol{\beta}}$	(4) (c) $\boldsymbol{\mu} \perp \boldsymbol{\beta},\dot{\boldsymbol{\beta}}$
0·01	$6 \cdot 672 \times 10^{-1}$	$6 \cdot 672 \times 10^{-1}$	$2 \cdot 668 \times 10^{-5}$
0·05	$6 \cdot 805 \times 10^{-1}$	$6 \cdot 808 \times 10^{-1}$	$6 \cdot 759 \times 10^{-4}$
0·10	$7 \cdot 233 \times 10^{-1}$	$7 \cdot 247 \times 10^{-1}$	$2 \cdot 818 \times 10^{-3}$
0·15	$7 \cdot 999 \times 10^{-1}$	$8 \cdot 031 \times 10^{-1}$	$6 \cdot 794 \times 10^{-3}$
0·20	$9 \cdot 190 \times 10^{-1}$	$9 \cdot 251 \times 10^{-1}$	$1 \cdot 331 \times 10^{-2}$
0·25	$1 \cdot 096 \times 10^{0}$	$1 \cdot 106 \times 10^{0}$	$2 \cdot 360 \times 10^{-2}$
0·30	$1 \cdot 354 \times 10^{0}$	$1 \cdot 370 \times 10^{0}$	$3 \cdot 972 \times 10^{-2}$
0·35	$1 \cdot 733 \times 10^{0}$	$1 \cdot 757 \times 10^{0}$	$6 \cdot 522 \times 10^{-2}$
0·40	$2 \cdot 301 \times 10^{0}$	$2 \cdot 337 \times 10^{0}$	$1 \cdot 063 \times 10^{-1}$
0·45	$3 \cdot 171 \times 10^{0}$	$3 \cdot 225 \times 10^{0}$	$1 \cdot 740 \times 10^{-1}$
0·50	$4 \cdot 556 \times 10^{0}$	$4 \cdot 635 \times 10^{0}$	$2 \cdot 897 \times 10^{-1}$
0·55	$6 \cdot 859 \times 10^{0}$	$6 \cdot 978 \times 10^{0}$	$4 \cdot 955 \times 10^{-1}$
0·60	$1 \cdot 091 \times 10^{1}$	$1 \cdot 109 \times 10^{1}$	$8 \cdot 812 \times 10^{-1}$
0·65	$1 \cdot 856 \times 10^{1}$	$1 \cdot 885 \times 10^{1}$	$1 \cdot 655 \times 10^{0}$
0·70	$3 \cdot 439 \times 10^{1}$	$3 \cdot 489 \times 10^{1}$	$3 \cdot 351 \times 10^{0}$
0·75	$7 \cdot 155 \times 10^{1}$	$7 \cdot 245 \times 10^{1}$	$7 \cdot 549 \times 10^{0}$
0·80	$1 \cdot 756 \times 10^{2}$	$1 \cdot 775 \times 10^{2}$	$1 \cdot 992 \times 10^{1}$
0·85	$5 \cdot 592 \times 10^{2}$	$5 \cdot 637 \times 10^{2}$	$6 \cdot 770 \times 10^{1}$
0·90	$2 \cdot 856 \times 10^{3}$	$2 \cdot 872 \times 10^{3}$	$3 \cdot 671 \times 10^{2}$
0·95	$4 \cdot 616 \times 10^{4}$	$4 \cdot 629 \times 10^{4}$	$6 \cdot 268 \times 10^{3}$
0·99	$2 \cdot 910 \times 10^{7}$	$2 \cdot 912 \times 10^{7}$	$4 \cdot 117 \times 10^{6}$

(c) $\boldsymbol{\mu}$ *Perpendicular to the Circle*

With the coordinate system as before we find

$$|\boldsymbol{a}_s|^2 = (s^4\omega^4\mu_0^2/4\pi c^3)\cos^2\theta\{\mathrm{J}_s^2\cot^2\theta + \beta^2(\mathrm{J}_s')^2\}, \tag{32}$$

where the argument of the Bessel functions is $s\beta\sin\theta$. When $\beta \ll 1$ the mean power radiated per unit solid angle is given approximately by

$$\mathrm{d}w/\mathrm{d}\Omega = (\omega^6\mu_0^2\,a^2/16\pi c^5)\cos^2\theta(1+\cos^2\theta). \tag{33}$$

The ω^6 dependence is a special feature of this orientation. Column 4 of Table 1 shows the values of w for various values of β for the perpendicular orientation of the dipole. The radiated power is always less in this case than for the other two orientations.

V. Keplerian Orbits

The case of Keplerian orbits is relevant to stars with a dipole field moving in a binary orbit. The orbit, in the x–y plane, may be defined parametrically by the equations

$$x = a(\cos\xi - \epsilon)\,, \qquad y = a(1-\epsilon^2)^{\frac{1}{2}}\sin\xi\,, \qquad \omega t' = \xi - \epsilon\sin\xi\,, \tag{34}$$

where the period is $2\pi/\omega$, a is the semi-major axis, ϵ is the eccentricity, and ξ is a parameter which goes from zero to 2π in one revolution.

It is convenient to measure azimuthal angles from the x axis and to take the z axis as the polar axis of a spherical polar coordinate system. The polar and azimuthal angles of $\boldsymbol{n}$ are taken to be θ and ψ respectively and we assume that $\boldsymbol{\mu}$ is perpendicular to the plane of the orbit. By using the variable ξ instead of t', and defining A and δ by

$$A^2 = \{\omega a(\sin\theta\cos\psi)/c\}^2 + \{\epsilon + \omega a(1-\epsilon^2)^{\frac{1}{2}}(\sin\theta\sin\psi)/c\}^2 \tag{35}$$

and

$$\tan\delta = \frac{\epsilon + \omega a(1-\epsilon^2)^{\frac{1}{2}}(\sin\theta\sin\psi)/c\,,}{\omega a(\sin\theta\cos\psi)/c} \tag{36}$$

$\boldsymbol{a}_s$ is found to depend on integrals of the form

$$\int_0^{2\pi} \exp[-\mathrm{i}s(\xi - A\cos(\xi-\delta))]\begin{Bmatrix}1\\ \cos\xi\\ \sin\xi\end{Bmatrix}\mathrm{d}\xi\,,$$

where one of the terms in the braces occurs in the integrand. Such integrals may be written in terms of Bessel functions, and in this way $|\boldsymbol{a}_s|^2$ is found to be given by

$$|\boldsymbol{a}_s|^2 = \frac{\omega^6 s^4 \mu_0^2 \cos^2\theta}{4\pi c^3}\left[\mathrm{J}_s^2\left\{\left(\frac{a}{cA}\right)^2(1-\epsilon^2\sin^2\delta) - \frac{1}{\omega^2}\left(1-\frac{\epsilon\sin\delta}{A}\right)^2\right\}\right.$$
$$\left. + (\mathrm{J}_s')^2\left\{\left(\frac{a}{c}\right)^2(1-\epsilon^2\cos^2\delta) - \left(\frac{\epsilon\cos\delta}{\omega}\right)^2\right\}\right]\,, \tag{37}$$

where the argument of the Bessel functions is sA. When ϵ is zero, equation (37) reduces to (32), as it should. For $\beta \ll 1$ and $\ll \epsilon$ we find

$$|\boldsymbol{a}_s|^2 = \frac{\omega^6 s^4 \mu_0^2 a^2 \cos^2\theta}{4\pi c^5}\left[\mathrm{J}_s^2\left(\frac{1-\epsilon^2}{\epsilon^2}\right)(1-\sin^2\theta\sin^2\psi) + (\mathrm{J}_s')^2(1-\sin^2\theta\cos^2\psi)\right]\,, \tag{38}$$

where the argument of the Bessel functions is now $s\epsilon$. The general expression for $\mathrm{d}w/\mathrm{d}\Omega$ may be obtained using equation (37) together with (27) and (28).

The case where $\boldsymbol{\mu}$ has any other orientation is more difficult since a Lorentz factor of γ^{-2} appears and this varies during the orbit. The complications introduced seem hardly worth assessing in detail unless a specific model is being explored.

VI. Implications for Pulsars

Consider firstly a spinning neutron star carrying a dipole magnetic field shifted from the centre. The point dipole is carried on a circular trajectory and traverses it at uniform velocity if the gentle slowing down due to radiation reaction is neglected. If the velocity is always much less than the velocity of light, which is true for all observed pulsars interpreted with the spinning neutron star model, then the energy radiated is dominated by terms which vary as ω^4, as for the centred dipole. This is so even in a general orientation unless $\boldsymbol{\mu}$ is extremely close to the perpendicular to the circle, when the energy radiated varies as ω^6.

During the early stages of formation of the star larger values of β can be envisaged, and provided $\beta \lesssim 0\cdot5$ the typical relativistic effects on the structure of the star and on the metric may be ignored. It is clear from Table 1 that during the early stages the energy radiated can be much greater than the spinning centred dipole formula would suggest. Accordingly, estimates of the age of pulsars must take into account a short period of rapid energy loss (and therefore rapid slowing down) followed by a period of much slower variation.

For the case of Keplerian motion with $\beta \gtrsim 0\cdot5$, the dipole radiation is substantial. However, when β is very small, and $\boldsymbol{\mu}$ is perpendicular to the orbit, the energy radiated by the dipole is always less than that radiated by gravitational radiation (Landau and Lifshitz 1962) by a factor of approximately

$$\frac{1}{8}\left(\frac{\text{magnetic energy of star}}{\text{gravitational energy of star}}\right)\left(\frac{\text{radius of star}}{\text{orbital radius}}\right)^2,$$

where the eccentricity of the orbit has been assumed to be small ($\ll 1$). Each factor in the foregoing expression is very much less than unity and the gravitational radiation dominates. If the dipole is parallel to the radius vector the equivalent factor is approximately

$$\frac{1}{10}\left(\frac{\text{magnetic energy}}{\text{gravitational energy}}\right)\left(\frac{\text{rest energy}}{\text{gravitational energy}}\right)\left(\frac{\text{radius of star}}{\text{orbital radius}}\right),$$

where the energies refer to the star carrying the dipole and β and the eccentricity have been assumed to be small. In general, gravitation is again the dominant factor. The electromagnetic radiation from the dipole magnetic field is therefore dynamically important only when $\beta \gtrsim 0\cdot5$. This can occur as the star loses energy and spirals into the other member of the binary, although eventually for $\beta \sim 1$ the present analysis becomes invalid.

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APPENDIX

Let

$$F = \sum_{n=1}^{\infty} n^4 \mathrm{J}_n^2(n\epsilon)\,. \tag{A1}$$

From the following identity established by Watson (1958, p. 31)

$$\mathrm{J}_n^2(z) = \frac{1}{\pi}\int_0^{\pi} \mathrm{J}_{2n}(2z\sin\theta)\,\mathrm{d}\theta\,, \tag{A2}$$

equation (A1) may therefore be written as

$$F = \frac{1}{\pi}\int_0^{\pi}\left(\sum_{n=1}^{\infty} n^4 \mathrm{J}_{2n}(2z\sin\theta)\right)\mathrm{d}\theta\,, \tag{A3}$$

with $z = n\epsilon$. Then, using the result (Watson 1958, p. 556)

$$\tfrac{1}{2}(-)^m\left[\frac{\mathrm{d}^{2m}}{\mathrm{d}M^{2m}}\left(\frac{1}{1-\epsilon\cos E}\right)\right]_{M=0} = \sum_{n=1}^{\infty} n^{2m}\mathrm{J}_n(n\epsilon)\,,$$

with

$$\frac{\mathrm{d}}{\mathrm{d}M} = \frac{1}{1-\epsilon\cos E}\frac{\mathrm{d}}{\mathrm{d}E}$$

it is easy, though tedious, to show that

$$\sum_{n=1}^{\infty} n^4\mathrm{J}_n(n\epsilon) = \epsilon(1+9\epsilon)/2(1-\epsilon)^7\,.$$

Interchanging the signs of ϵ and adding the results shows that

$$\sum_{n=1}^{\infty} n^4\mathrm{J}_{2n}(2n\epsilon) = \frac{1}{64}\left(\frac{\epsilon(1+9\epsilon)}{(1-\epsilon)^7} - \frac{\epsilon(1-9\epsilon)}{(1+\epsilon)^7}\right), \tag{A4}$$

and hence equation (A3) becomes

$$F = \frac{z}{64\pi}\int_0^{2\pi}\frac{\cos\phi\,(1+9z\cos\phi)\,\mathrm{d}\phi}{(1-z\cos\phi)^7}\,. \tag{A5}$$

The integrals involved in (A5) may be evaluated easily by appropriate differentiation of the definite integral

$$\int_0^{2\pi} \frac{\mathrm{d}\phi}{(a-b\cos\phi)} = \frac{2\pi}{(a^2-b^2)^{\frac{1}{2}}}.$$

In this way we find

$$\sum_{n=1}^{\infty} n^4 \mathrm{J}_n^2(n\epsilon) = \frac{\epsilon^2(64+592\epsilon^2+472\epsilon^4+27\epsilon^6)}{256(1-\epsilon^2)^{13/2}}, \tag{A6}$$

which is the result (27) quoted in Section IV.

To evaluate

$$\sum_{n=1}^{\infty} n^4\{\mathrm{J}_n'(n\epsilon)\}^2,$$

consider the relation

$$K = \sum_{n=1}^{\infty} n^2 \mathrm{J}_n^2(n\epsilon) = \epsilon^2(4+\epsilon^2)/16(1-\epsilon^2)^{7/2},$$

where the summation has been evaluated by the method indicated above. By partial differentiation with respect to ϵ it can be shown, with the help of the differential equation satisfied by Bessel functions, that

$$\sum_{n=1}^{\infty} n^4\{\mathrm{J}_n'(n\epsilon)\}^2 = \frac{1}{2}\left(\frac{\partial^2 K}{\partial\epsilon^2} + \frac{1}{\epsilon}\frac{\partial K}{\partial\epsilon} - \frac{2(1-\epsilon^2)}{\epsilon^2}\sum_{n=1}^{\infty} n^4 \mathrm{J}_n^2(n\epsilon)\right).$$

Then using (A5) and the expression for K in terms of ϵ we find

$$\sum_{n=1}^{\infty} n^4\{\mathrm{J}_n'(n\epsilon)\}^2 = \frac{64+624\epsilon^2+632\epsilon^4+45\epsilon^6}{256(1-\epsilon^2)^{11/2}}, \tag{A7}$$

which is the required result (28).