COULOMB INTERFERENCE IN THE HALF-SHELL t-MATRIX DERIVED FROM THE HAMADA–JOHNSTON POTENTIAL

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Abstract

The calculation of the half-shell two-nucleon t-matrix from the wavefunction due to the Hamada–Johnston potential in the presence of the Coulomb interaction is described. The contribution from the long range nature of the Coulomb interaction is included and its effect on the half-shell t-matrix is indicated.

I. Introduction

As the two-nucleon interaction is required off the energy shell in calculations for a many-nucleon system, a great deal of effort has been devoted to the study of the off-shell behaviour of the t-matrix. The experimental constraints on the elastic t-matrix do not uniquely specify the potential and it is possible to construct t-matrices with different off-shell behaviour. The sensitivity of many-body calculations to the off-shell structure of the two-body t-matrix half off the energy shell, referred to as the half-shell t-matrix, has recently emerged as a more fundamental quantity (Baranger *et al.* 1969), as the fully off-shell t-matrix may be calculated from it. The half-shell t-matrix occurs often in nuclear physics, as, for example, in nucleon–nucleon bremsstrahlung (Sobel 1965), the plane wave impulse approximation in knock-out reactions, and quasi-free scattering (Redish, Stephenson, and Lerner 1970). Information concerning the wavefunction as well as the elastic phase shifts are contained in the half-shell t-matrix, and hence it may be parameterized in terms of the short range behaviour of the wavefunction (Picker, Redish, and Stephenson 1971).

Off-shell behaviour has usually been considered in the absence of the Coulomb interaction which is not restricted to a finite number of partial waves. However, the Coulomb scattering amplitude makes a large contribution to the p–p scattering amplitude especially at small angles, and in this paper the nature of the Coulomb contribution to the p–p t-matrix is considered in a typical off-shell situation, i.e. the half-shell t-matrix. The form of the half-shell t-matrix due to a realistic local potential, the Hamada–Johnston (1962) potential, is given for a general partial wave and in the presence of the Coulomb interaction. The total t-matrix is separated into the "nuclear" term and the pure Coulomb half-shell t-matrix. The latter term does not allow accurate results to be obtained from a finite partial wave expansion and an approximation formula is used.

The results of the calculation enable an estimation of the consequences of either omitting the Coulomb contribution from the half-shell t-matrix altogether or of

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including the Coulomb potential in just those partial waves for which the nuclear force is effective. It is found that Coulomb interference is large and constructive around the near on-shell region for small scattering angles and that it results in a reduction of a few per cent elsewhere.

II. Half-shell t-matrix

In this section we consider the calculation of the half-shell t-matrix due to a realistic local potential when the Coulomb interaction is present. The present work is restricted to the isospin $T = 1$ two-nucleon system, but the results are easily applied to the $T = 0$ system by omitting the Coulomb contribution and making the necessary changes to the statistics. Without the Coulomb interaction the final expressions reduce to those obtained by Sobel (1965). We wish to emphasize here those aspects of the formulation which differ from the latter work due to the Coulomb interaction, especially in the coupled states.

The half-shell t-matrix is related to the wavefunction by (Goldberger and Watson 1964)

$$\langle \boldsymbol{k}' \,|\, t(k^2) \,|\, \boldsymbol{k} \rangle = \langle \boldsymbol{k}' \,|\, V \,|\, \psi^{(+)}(\boldsymbol{k}, \boldsymbol{r}) \rangle, \tag{1}$$

where $\psi^{(+)}$ is the solution of the Schrödinger equation for the interaction V with outgoing boundary conditions. When V includes the Coulomb potential, a partial wave expansion of (1) will omit important long range contributions. To overcome this difficulty we separate the matrix element in a way which is exact only on-shell. From the theory of scattering by two potentials (Goldberger and Watson 1964)

$$\begin{aligned}\langle \boldsymbol{k}' \,|\, V_{\mathrm{C}} + V_{\mathrm{n}} \,|\, \psi^{(+)}(\boldsymbol{k}, \boldsymbol{r}) \rangle &= \langle \phi_{\mathrm{C}}^{(-)} \,|\, V_{\mathrm{n}} \,|\, \psi^{(+)} \rangle + \langle \phi_{\mathrm{C}}^{(-)} \,|\, V_{\mathrm{C}} \,|\, \boldsymbol{k} \rangle \\ &= \langle \phi_{\mathrm{C}}^{(-)} \,|\, V_{\mathrm{n}} \,|\, \psi^{(+)} \rangle + \langle \boldsymbol{k} \,|\, T_{\mathrm{C}}(k'^2) \,|\, \boldsymbol{k}' \rangle,\end{aligned} \tag{2}$$

where $\phi_{\mathrm{C}}^{(-)}$ is the solution of the Coulomb Schrödinger equation with ingoing boundary conditions. The first term in (2), the "nuclear" term, can be expanded in partial waves while an approximate method is used for the second term, which is the half-shell Coulomb t-matrix.

Since we wish to consider realistic local potentials containing tensor interaction, the total Hamiltonian commutes only with $\boldsymbol{S}^2$, $\boldsymbol{J}^2$, J_z, and parity, thus causing orbital angular momentum states $l = j \pm 1$ to be mixed. The wavefunction must be expanded in simultaneous eigenfunctions of $\boldsymbol{S}^2$, $\boldsymbol{J}^2$, and J_z and the Pauli principle restricts the states such that $(-1)^{l+s+T} = -1$. For $T = 1$ we have singlet–even and triplet–odd states. We use the notation of Ashkin and Wu (1948) here for partial wave expansion of the wavefunction due to tensor force scattering. The non-conservation of orbital angular momentum is implied by the fact that the radial wavefunction will satisfy coupled Schrödinger equations. All angles unless otherwise indicated are referenced from the incident momentum $\boldsymbol{k}$.

The expansion of the total wavefunction for a spin $\boldsymbol{S}$ is

$$\psi_{m_s}^{(+)}(\boldsymbol{k}, \boldsymbol{r}) = \sum_{jlm_j} \{4\pi(2l+1)\}^{\frac{1}{2}}\, \mathrm{i}^l \exp(\mathrm{i}\delta_{lj}) \{u_{lj}(r)/kr\} \mathscr{Y}_{jls}^{m_j}(\hat{\boldsymbol{r}}, \boldsymbol{\sigma})\, C_{lsj}^{0m_s m_j} \tag{3}$$

and the corresponding expansion for the Coulomb wavefunction is

$$\overset{*}{\phi}{}^{(-)}_{\mathrm{C}m_s}(\boldsymbol{k}',\boldsymbol{r}) = \sum_{jlm_j} \{4\pi(2l+1)\}^{\frac{1}{2}}\,\mathrm{i}^l \exp(\mathrm{i}\sigma_l')\{F_l(\eta,k'r)/k'r\}\mathscr{Y}^{m_j}_{jls}(-\hat{\boldsymbol{r}}_{k'},\boldsymbol{\sigma})\,C^{0m_sm_j}_{lsj}, \tag{4}$$

where in this notation $C^{m_lm_sm_j}_{lsj}$ are the usual Clebsch–Gordan coefficients and we have introduced spin–angle eigenfunctions defined by

$$\mathscr{Y}^{m_j}_{jls}(\Omega,\boldsymbol{\sigma}) = \sum_{m_lm_s} Y^{m_l}_l(\Omega)\,\chi^{m_s}_s(\boldsymbol{\sigma})\,C^{m_lm_sm_j}_{lsj}. \tag{5}$$

Also, $F_l(\eta,k'r)$ is the regular Coulomb wavefunction, $\eta = \mu e^2/\hbar^2k'$, and σ_l and δ_{lj} are the Coulomb and total scattering phase shifts respectively. In order to facilitate the angular integration, we use the addition theorem for spherical harmonics (Edmonds 1960) to express $\phi^{*(-)}_{\mathrm{C}m_s}(\boldsymbol{k}',\boldsymbol{r})$ as a function of $\hat{\boldsymbol{r}}$, at the same time introducing the scattering angle $\theta = \hat{\boldsymbol{k}}'.\hat{\boldsymbol{k}}$. With the aid of the expansions (3) and (4) the nuclear term in the t-matrix (2) can be expanded, the spin–angle integrations performed, and expressions obtained for the t-matrix due to a particular total spin S and with initial and final spin projections m_{i} and m_{f} respectively.

The coupled channels require special treatment. Here we use the Blatt and Biedenharn (1952) parameterization in which the S-matrix for the coupled channels $l = j\pm 1$ can be represented by three parameters, the mixing parameter ϵ_j and the eigenphases δ^α_j and δ^β_j. This is equivalent to constructing a total scattering wavefunction with mixtures of $l = j\pm1$ states which are eigenstates of the scattering. To do this we must modify the expansion of the total wavefunction $\psi^{(+)}$ by setting

$$A^{m_j}_{j-1,j}\{u_{j-1,j}(r)/r\}\mathscr{Y}^{m_j}_{j,j-1,1} + A^{m_j}_{j+1,j}\{u_{j+1,j}(r)/r\}\mathscr{Y}^{m_j}_{j,j+1,1} = \alpha^{m_j}_j\,\phi^{m_j}_{j,\alpha} + \beta^{m_j}_j\,\phi^{m_j}_{j,\beta}, \tag{6}$$

where the $A^{m_j}_{l,j}$ represent the constants in the expansion (3) of $\psi^{(+)}$. If we take the eigenstates $\phi^{m_j}_{j\{\alpha,\beta\}}$ to be of the form

$$\begin{pmatrix}\phi^{m_j}_{j,\alpha}\\ \phi^{m_j}_{j,\beta}\end{pmatrix} = r^{-1}\begin{pmatrix}\exp(\mathrm{i}\delta^\alpha_j)\,v^\alpha_{j-1,j}(r) & -\exp(\mathrm{i}\delta^\alpha_j)\,v^\alpha_{j+1,j}(r)\\ \exp(\mathrm{i}\delta^\beta_j)\,v^\beta_{j-1,j}(r) & -\exp(\mathrm{i}\delta^\beta_j)\,v^\beta_{j+1,j}(r)\end{pmatrix}\begin{pmatrix}\mathrm{i}^{j-1}\,\mathscr{Y}^{m_j}_{j,j-1,1}\\ \mathrm{i}^{j+1}\,\mathscr{Y}^{m_j}_{j,j+1,1}\end{pmatrix}, \tag{7}$$

with the radial eigenfunctions having the asymptotic forms

$$\begin{Bmatrix}v^\alpha_{j-1,j}(r)\\ v^\beta_{j-1,j}(r)\end{Bmatrix} \sim \begin{Bmatrix}\cos\epsilon_j\\ -\sin\epsilon_j\end{Bmatrix}\sin\left(kr-\tfrac{1}{2}\pi(j-1)-\eta\ln(2kr)+\begin{Bmatrix}\delta^\alpha_j\\ \delta^\beta_j\end{Bmatrix}\right), \tag{8a}$$

$$\begin{Bmatrix}v^\alpha_{j+1,j}(r)\\ v^\beta_{j+1,j}(r)\end{Bmatrix} \sim \begin{Bmatrix}\sin\epsilon_j\\ \cos\epsilon_j\end{Bmatrix}\sin\left(kr-\tfrac{1}{2}\pi(j+1)-\eta\ln(2kr)+\begin{Bmatrix}\delta^\alpha_j\\ \delta^\beta_j\end{Bmatrix}\right), \tag{8b}$$

then the constants $\alpha^{m_j}_j$ and $\beta^{m_j}_j$ will be given by

$$\begin{pmatrix}\alpha^{m_j}_j\\ \beta^{m_j}_j\end{pmatrix} = \begin{pmatrix}\cos\epsilon_j & \sin\epsilon_j\\ -\sin\epsilon_j & \cos\epsilon_j\end{pmatrix}\begin{pmatrix}a^{m_j}_{j-1,j}\\ a^{m_j}_{j+1,j}\end{pmatrix}, \tag{9}$$

where

$$a^{m_j}_{j\pm1,j} = A^{m_j}_{j\pm1,j}/\mathrm{i}^{j\pm1}.$$

The radial eigenfunctions are obtained by solving the coupled Schrödinger equation with two independent sets of boundary conditions at $r = 0$ (or $r = r_{\mathrm{c}}$ if the potential has a hard core) and taking a linear combination of the two sets of solutions, then requiring that the states $l = j \pm 1$ have the same total phase shift.

The matching procedure in the asymptotic region is

$$a\, v^{(1)}_{j-1,j}(r) + b\, v^{(2)}_{j-1,j}(r) \underset{r\to\infty}{\sim} F_{j-1}(\eta, kr)\cos\theta_j + G_{j-1}(\eta, kr)\sin\theta_j\,, \tag{10a}$$

$$a\, v^{(1)}_{j+1,j}(r) + b\, v^{(2)}_{j+1,j}(r) \underset{r\to\infty}{\sim} F_{j+1}(\eta, kr)\cos\phi_j + G_{j+1}(\eta, kr)\sin\phi_j\,, \tag{10b}$$

and we require that

$$\sigma_{j-1} + \theta_j = \sigma_{j+1} + \phi_j\,.$$

This procedure determines ϵ_j, δ^{α}_j, and δ^{β}_j, and when the Coulomb interaction is present the eigenphases contain the Coulomb phase whereas in the uncoupled states the pure Coulomb phase σ_l is easily separated out. It is convenient to define the radial eigenfunctions $u^{\{\alpha;\beta\}}_{j\pm 1,j}(r)$ as having the asymptotic forms (8) without the coupling parameters. The Schrödinger equations for the radial wavefunctions are, for the uncoupled states $l = j$,

$$\left\{\frac{\hbar^2}{2\mu}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - \frac{l(l+1)}{r^2}\right) - V_{\mathrm{C}}(r)\right\} u_l(r) = V_l(r)\, u_l(r) \tag{11}$$

and, for the coupled states $l = j \pm 1$,

$$\left\{\frac{\hbar^2}{2\mu}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - \frac{j(j-1)}{r^2}\right) - V_{\mathrm{C}}(r)\right\} u_{j-1,j}(r) = V_{j-1,j}(r)\, u_{j-1,j}(r) + V^{\mathrm{t}}_j(r)\, u_{j+1,j}(r)\,, \tag{12a}$$

$$\left\{\frac{\hbar^2}{2\mu}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - \frac{(j+1)(j+2)}{r^2}\right) - V_{\mathrm{C}}(r)\right\} u_{j+1,j}(r) = V_{j+1,j}(r)\, u_{j+1,j}(r) + V^{\mathrm{t}}_j(r)\, u_{j-1,j}(r)\,, \tag{12b}$$

where

$$V_l(r) = \langle l, sjm \,|\, V_{\mathrm{n}} \,|\, l, sjm\rangle\,, \qquad V^{t}_j(r) = \langle j-1, sjm \,|\, V_{\mathrm{n}} \,|\, j+1, sjm\rangle\,,$$

$$V_{j-1,j}(r) = \langle j-1, sjm \,|\, V_{\mathrm{n}} \,|\, j-1, sjm\rangle\,, \quad V_{j+1,j}(r) = \langle j+1, sjm \,|\, V_{\mathrm{n}} \,|\, j+1, sjm\rangle\,.$$

We can now describe the final expressions for the half-shell t-matrix obtained by using the wavefunction expansions (3) and (4) in equation (2). The t-matrix will be a 4×4 matrix in spin space. The antisymmetrized matrix element for the singlet state is

$$^{1}T = 2 \sum_{\text{even } l} (2l+1)\, t_l(k', k; k^2)\, \mathrm{P}_l(\hat{\boldsymbol{k}}' \cdot \hat{\boldsymbol{k}}) + \{T_{\mathrm{C}}(\theta) + T_{\mathrm{C}}(\pi - \theta)\}\,, \tag{13}$$

where $T_{\mathrm{C}}(\theta)$ represents the half-shell Coulomb t-matrix which is discussed below. The triplet matrix is complicated by the coupled states and is described for scattering from initial spin projection m_{i} to final spin projection m_{f} by

$$ {}^{3}_{m_{\mathrm{f}}}T_{m_{\mathrm{i}}} = {}_{m_{\mathrm{f}}}T^{(1)}_{m_{\mathrm{i}}} + {}_{m_{\mathrm{f}}}T^{(2)}_{m_{\mathrm{i}}} + {}_{m_{\mathrm{f}}}T^{(3)}_{m_{\mathrm{i}}} + \delta_{m_{\mathrm{f}}m_{\mathrm{i}}}\{T_{\mathrm{C}}(\theta) - T_{\mathrm{C}}(\pi-\theta)\}, \tag{14} $$

where the expressions for the first three terms on the right-hand side are given in the Appendix. The partial t-matrix elements have the form, for the uncoupled states $l = j$,

$$ t_j(k', k;\, k^2) = -(2\mu/\hbar^2)\{R_j(k', k)/k'k\}\exp[\mathrm{i}(\delta_j + \sigma_j' + \sigma_j)] \tag{15a} $$

and, for the coupled states $l = j\pm 1$,

$$ \begin{Bmatrix} t^{\alpha}_{j\pm1}(k', k;\, k^2) \\ t^{\beta}_{j\pm1}(k', k;\, k^2) \end{Bmatrix} = -(2\mu/\hbar^2)\begin{Bmatrix} R^{\alpha}_{j\pm1}(k', k)/k'k \\ R^{\beta}_{j\pm1}(k', k)/k'k \end{Bmatrix}\exp\left[\mathrm{i}\left(\begin{Bmatrix}\delta^{\alpha}_j \\ \delta^{\beta}_j\end{Bmatrix} + \sigma'_{j\pm1}\right)\right], \tag{15b} $$

where $R_j(k', k)$ and $R^{\{\alpha;\beta\}}_{j\pm1}(k', k)$ are real functions representing the radial integrals which in the coupled states have the forms

$$ \begin{Bmatrix} R^{\alpha}_{j-1}(k', k) \\ R^{\beta}_{j-1}(k', k) \end{Bmatrix} = \int_0^{\infty} F_{j-1}(\eta, k'r)\left[V_{j-1,j}(r)\begin{Bmatrix} u^{\alpha}_{j-1,j}(r) \\ u^{\beta}_{j-1,j}(r) \end{Bmatrix} + \begin{Bmatrix} \tan\epsilon_j \\ -\cot\epsilon_j \end{Bmatrix} V^{\mathrm{t}}_j(r)\begin{Bmatrix} u^{\alpha}_{j+1,j}(r) \\ u^{\beta}_{j+1,j}(r) \end{Bmatrix}\right] \mathrm{d}r, \tag{16a} $$

$$ \begin{Bmatrix} R^{\alpha}_{j+1}(k', k) \\ R^{\beta}_{j+1}(k', k) \end{Bmatrix} = \int_0^{\infty} F_{j+1}(\eta, k'r)\left[V_{j+1,j}(r)\begin{Bmatrix} u^{\alpha}_{j+1,j}(r) \\ u^{\beta}_{j+1,j}(r) \end{Bmatrix} + \begin{Bmatrix} \cot\epsilon_j \\ -\tan\epsilon_j \end{Bmatrix} V^{\mathrm{t}}_j(r)\begin{Bmatrix} u^{\alpha}_{j-1,j}(r) \\ u^{\beta}_{j-1,j}(r) \end{Bmatrix}\right] \mathrm{d}r. \tag{16b} $$

The form of $R_j(k', k)$ is easily obtained as a special case of the above. On the energy shell the radial integrals reduce to

$$ R_j(k', k) \underset{k'\to k}{\to} -\frac{\hbar^2}{2\mu}k\sin\delta_j, \qquad \begin{Bmatrix} R^{\alpha}_{j\pm1}(k', k) \\ R^{\beta}_{j\pm1}(k', k) \end{Bmatrix} \underset{k'\to k}{\to} -\frac{\hbar^2}{2\mu}k\sin\left(\begin{Bmatrix}\delta^{\alpha}_j \\ \delta^{\beta}_j\end{Bmatrix} - \sigma_{j\pm1}\right), \tag{17} $$

and the partial amplitudes take on the usual forms for the nuclear scattering amplitudes,

$$ t_j(k', k;\, k^2) \underset{k'\to k}{\to} \frac{1}{2\mathrm{i}k}\exp(2\mathrm{i}\sigma_j)\{\exp(2\mathrm{i}\delta_j) - 1\}, \tag{18a} $$

$$ \begin{Bmatrix} t^{\alpha}_{j\pm1}(k', k;\, k^2) \\ t^{\beta}_{j\pm1}(k', k;\, k^2) \end{Bmatrix} \underset{k'\to k}{\to} \frac{1}{2\mathrm{i}k}\left[\exp\left(2\mathrm{i}\begin{Bmatrix}\delta^{\alpha}_j \\ \delta^{\beta}_j\end{Bmatrix}\right) - \exp(2\mathrm{i}\sigma_{j\pm1})\right]. \tag{18b} $$

If the local potential contains a hard core (e.g. Hamada–Johnston potential), care must be taken in performing the radial integrals (16) as there is a contribution due to the hard core region. We briefly consider now the method of obtaining these contributions.

Those parts of the integrands in (16) enclosed in square brackets are indefinite in the hard core region $0 \leqslant r \leqslant r_{\mathrm{C}}$. It is to be noted that these parts will be the right-hand side expressions in the Schrödinger equations (similar to (12)) satisfied by the coupled radial eigenfunctions. Hence the hard core contribution to R^{α}_{j-1}, for example, is given by

$$(R_{j-1}^{\alpha})_{\text{h.c.}} = \frac{\hbar^2}{2\mu}\int_0^{r_c} F_{j-1}(\eta, k'r)\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-\frac{j(j-1)}{r^2}-V_{\mathrm{C}}(r)\right)u_{j-1,j}^{\alpha}(r)\,\mathrm{d}r\,. \tag{19}$$

Using the facts that $F_l(\eta, k'r) \sim r^{l+1}$ as $r \to 0$ and that $u_{lj}(r) = 0$ for $r < r_{\mathrm{c}}$, the only contribution to the integral in the region $0 \leqslant r \leqslant r_{\mathrm{c}}$ will be at $r = r_{\mathrm{c}}$ where $\mathrm{d}^2\{u_{j-1,j}^{\alpha}(r)\}/\mathrm{d}r^2$ has a delta-function singularity. Integrating by parts we obtain

$$(R_{j-1}^{\alpha})_{\text{h.c.}} = \frac{\hbar^2}{2\mu}F_{j-1}(\eta, k'r_{\mathrm{c}})\left(\frac{\mathrm{d}\{u_{j-1,j}^{\alpha}(r)\}}{\mathrm{d}r}\right)_{r_{\mathrm{c}}}. \tag{20}$$

To complete our calculation of the half-shell $T = 1$ t-matrix with Coulomb corrections we need the half-shell Coulomb t-matrix as indicated in (2). Since we are only interested in estimating its contribution, for ease of computation the approximation due to Ford (1964, 1966) has been chosen. From a study of the screened Coulomb t-matrix off-shell, he found that, except for a region very close to the energy shell, the behaviour of the half-shell Coulomb t-matrix is described by

$$\langle \boldsymbol{k} \,|\, T_{\mathrm{C}}(k'^2) \,|\, \boldsymbol{k}'\rangle \approx -\frac{2\mu}{\hbar^2}\frac{e^2C_0(\eta)\exp(\mathrm{i}\delta_0)\,(k^2-k'^2)^{\mathrm{i}\eta}}{\{(\boldsymbol{k}-\boldsymbol{k}')^2\}^{1+\mathrm{i}\eta}}+O(R^{-1})\,, \tag{21}$$

for $|\,\boldsymbol{k}-\boldsymbol{k}'\,| \gg R^{-1}$ and $k' \gg R^{-1}$, where

$$\delta_0 = \sigma_0-\eta\ln(2k'R)\,, \qquad \sigma_0 = \arg\{\Gamma(1+\mathrm{i}\eta)\}\,, \qquad C_0(\eta) = \exp\{-\tfrac{1}{2}\pi\eta\,|\,\Gamma(1+\mathrm{i}\eta)\,|\}\,.$$

Near to and on the energy shell the contributions from the extremely long range of the Coulomb potential are most important for the Coulomb t-matrix.

In the absence of the Coulomb potential the forms (15) for the partial t-matrix elements, resulting from a realistic local potential, have been shown (Sobel 1967) to satisfy the requirements of off-shell unitarity and time-reversal invariance. These requirements would be preserved in the full t-matrix if the pure Coulomb contribution were not separated out, but, for computational ease and in order to estimate the full Coulomb interference with the nuclear part, the approximation method above has been chosen.

Apart from a normalization constant the real functions $R_j(k', k)$ and $R_{j\pm 1}^{\{\alpha;\beta\}}(k', k)$ appearing in the t-matrix are the so-called quasi-phase parameters used by many authors (e.g. Sobel 1965) to specify the half-shell t-matrix in nucleon–nucleon bremsstrahlung. These functions can be parameterized in terms of the short range behaviour of the wavefunction and the range of variation in the near off-shell behaviour can be estimated (Picker, Redish, and Stephenson 1971). Since we are interested here in interference effects between the nuclear and Coulomb contributions, we form the square modulus of the scattering amplitude

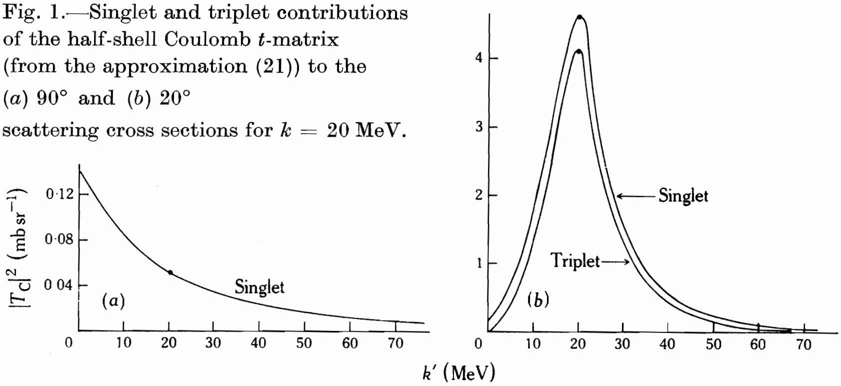
$$|\,T\,|^2 = \tfrac{1}{4}\left\{|\,{}^1T\,|^2+\sum_{m_{\mathrm{f}}m_{\mathrm{i}}}|\,{}_{m_{\mathrm{f}}}^{3}T_{m_{\mathrm{i}}}\,|^2\right\}. \tag{22}$$

On the energy shell, the result can be compared with the elastic p–p cross section.

III. Results and Discussion

The behaviour of the Coulomb half-shell t-matrix resulting from the use of the approximation (21) is indicated in Figure 1. The square modulus of the antisymmetrized t-matrix* is plotted for singlet and triplet states at two typical scattering angles. The triplet amplitude is zero at 90° scattering (Fig. 1(a)) and both the singlet and triplet amplitudes remain small for large scattering angles. For small scattering angles (Fig. 1(b)) a strong peak develops around the on-shell region due to the contributions from the long range of the Coulomb interaction. As the scattering angle tends to zero the singular nature of the Coulomb amplitude is reproduced in the near on-shell region. The on-shell values are obtained from the usual Coulomb scattering amplitude (Goldberger and Watson 1964). At higher energies the Coulomb contribution is significant only for very small angles and close to the on-shell region.

Fig. 1.—Singlet and triplet contributions of the half-shell Coulomb t-matrix (from the approximation (21)) to the (a) 90° and (b) 20° scattering cross sections for $k = 20$ MeV.



The calculation of the nuclear part of the half-shell t-matrix can be checked by ensuring that the radial integrals reduce to the on-shell forms (17) and that the elastic cross section is obtained. The off-shell behaviour of the full t-matrix for the Hamada–Johnston potential at 90° scattering angle is indicated in Figure 2(a). States up to $J = 4$ have been included at the higher energies. As expected the Coulomb contribution is quite small at 90° and is responsible for some destructive interference in the low energy off-shell region (Fig. 2(b)). At smaller scattering angles, however, the Coulomb contribution interferes constructively with the nuclear term and introduces a strong peak at the on-shell point. In Figures 3(a) and 3(b) this small-angle Coulomb interference is indicated for 10 and 20 MeV respectively, where states up to $J = 2$ have been included.

The angular distribution of the half-shell cross section near the on-shell region for 20 MeV is plotted in Figure 4. It is clear that the singular nature of the scattering amplitude at small angles is confined to the near on-shell regions. The elastic amplitudes typically have a small region of destructive interference at angles just outside the strong constructive region, e.g. at 20° in Figure 4. The half-shell Coulomb

* This is the proper description of the quantity plotted in all figures in this paper.

t-matrix undergoes a rapid phase change at these angles, and Figure 5 indicates the off-shell behaviour of the full scattering amplitude obtained in this region. For higher energies the influence of the Coulomb interaction on the half-shell t-matrix is confined to only very small scattering angles.

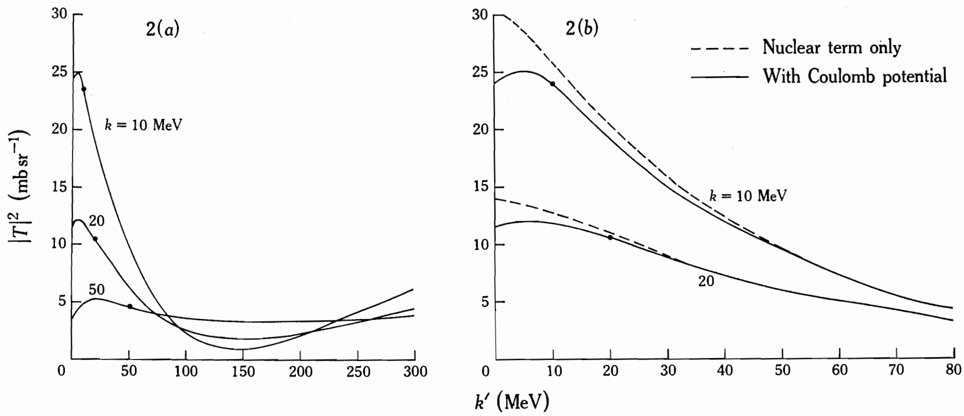


Fig. 2 (*above*).—Showing for the half-shell scattering cross section at 90°:

(*a*) the off-shell behaviour due to the full half-shell t-matrix for the Hamada–Johnston potential, and

(*b*) the influence of the Coulomb potential in the low energy off-shell region.

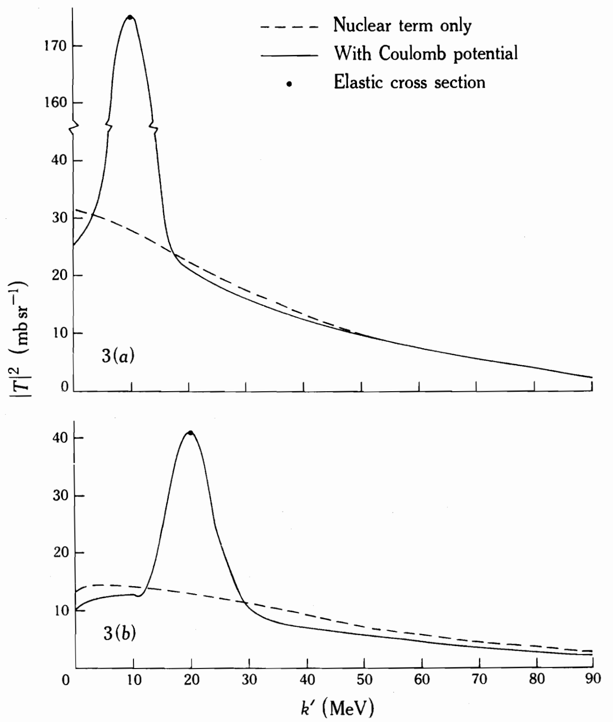


Fig. 3 (*left*).—Small-angle enhancement by the Coulomb contribution to the half-shell scattering cross section. The curves are for a scattering angle of 10° at:

(*a*) $k = 10$ MeV,

(*b*) $k = 20$ MeV.

In conclusion then the results of the present investigation into the effects of the Coulomb interaction on the half-shell t-matrix may be summarized as follows. Except for a region close to the energy shell with small scattering angles, the Coulomb force has little effect, there being a slight destructive interference. The long range contributions from the Coulomb potential result in a strong constructive peak at the near on-shell region for small scattering angles. This latter effect would not be borne out in a finite partial wave expansion of the Coulomb contribution. The approximate expression for the half-shell Coulomb t-matrix employed here serves as an indication of the nature and magnitude of the error involved in omitting the Coulomb contribution from the $T = 1$ half-shell t-matrix.

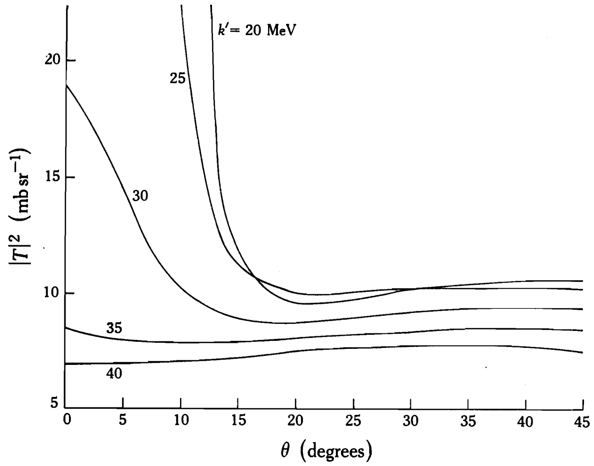


Fig. 4.—Angular distributions of the half-shell cross section near the on-shell region for $k = 20$ MeV and the indicated values of k'.

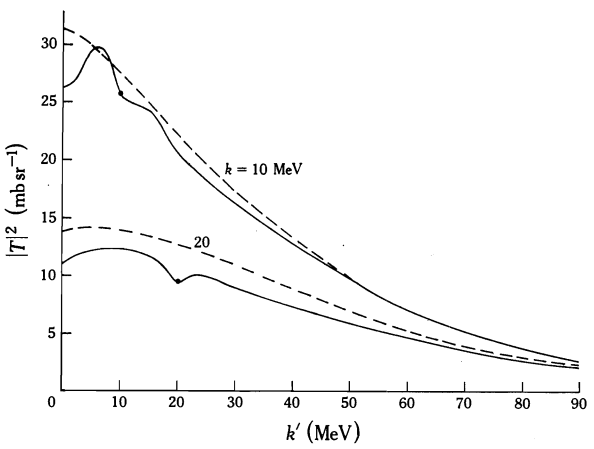


Fig. 5.—Half-shell cross section at 20° for $k = 10$ and 20 MeV where there is destructive interference.

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Appendix

If we take the interaction to be described by a realistic local potential and the Coulomb potential and denote the initial and final spin projections by m_{i} and m_{f} respectively, then the expressions for the antisymmetrized $T=1$ half-shell t-matrix after partial wave expansion are as follows. The singlet term is

$$^{1}T = 2\sum_{\text{even}\,j}\{4\pi(2j+1)\}^{\frac{1}{2}}\,t_j(k',k;k^2)\,Y_j^0(\theta)+\{T_{\mathrm{C}}(\theta)+T_{\mathrm{C}}(\pi-\theta)\}\,, \tag{A1}$$

while the triplet term can be expressed as in equation (14) in Section II, where the terms ${}_{m_{\mathrm{f}}}T^{(1)}_{m_{\mathrm{i}}}$, ${}_{m_{\mathrm{f}}}T^{(2)}_{m_{\mathrm{i}}}$, and ${}_{m_{\mathrm{f}}}T^{(3)}_{m_{\mathrm{i}}}$ correspond to the cases $l=j$; $l=1$, $j=0$; and $l=j\pm1$ respectively. These are given by

$${}_{m_{\mathrm{f}}}T^{(1)}_{m_{\mathrm{i}}} = 2\sum_{\text{odd}\,j}\{4\pi(2j+1)\}^{\frac{1}{2}}\,t_j(k',k;k^2)\,C_{j1j}^{m_{\mathrm{i}}-m_{\mathrm{f}},m_{\mathrm{f}},m_{\mathrm{i}}}\,C_{j1j}^{0,m_{\mathrm{i}},m_{\mathrm{i}}}\,Y_j^{m_{\mathrm{f}}-m_{\mathrm{i}}}(\theta)\,(-1)^{m_{\mathrm{i}}-m_{\mathrm{f}}}\,, \tag{A2}$$

$${}_{m_{\mathrm{f}}}T^{(2)}_{m_{\mathrm{i}}} = 4(3\pi)^{\frac{1}{2}}\,t_{10}(k',k;k^2)\,C_{110}^{m_{\mathrm{i}}-m_{\mathrm{f}},m_{\mathrm{f}},m_{\mathrm{i}}}\,C_{110}^{0,m_{\mathrm{i}},m_{\mathrm{i}}}\,Y_1^{m_{\mathrm{f}}-m_{\mathrm{i}}}(\theta)\,(-1)^{m_{\mathrm{i}}-m_{\mathrm{f}}}\,, \tag{A3}$$

$$\begin{aligned}{}_{m_{\mathrm{f}}}T^{(3)}_{m_{\mathrm{i}}} = 2\sum_{\text{even}\,j}\,&[4\pi\{2(j-1)+1\}]^{\frac{1}{2}}\,C_{j-1,1,j}^{m_{\mathrm{i}}-m_{\mathrm{f}},m_{\mathrm{f}},m_{\mathrm{i}}}\,(E^{\alpha}_{j-1}+E^{\beta}_{j-1})\,Y_{j-1}^{m_{\mathrm{f}}-m_{\mathrm{i}}}\,(-1)^{m_{\mathrm{i}}-m_{\mathrm{f}}}\\ &+[4\pi\{2(j+1)+1\}]^{\frac{1}{2}}\,C_{j+1,1,j}^{m_{\mathrm{i}}-m_{\mathrm{f}},m_{\mathrm{f}},m_{\mathrm{i}}}\,(E^{\alpha}_{j+1}+E^{\beta}_{j+1})\,Y_{j+1}^{m_{\mathrm{f}}-m_{\mathrm{i}}}\,(-1)^{m_{\mathrm{i}}-m_{\mathrm{f}}}\,,\end{aligned} \tag{A4}$$

where

$$E^{\alpha}_{j-1} = \cos^2\epsilon_j\,t^{\alpha}_{j-1}(k',k;k^2)\left(C_{j-1,1,j}^{0,m_{\mathrm{i}},m_{\mathrm{i}}}-\tan\epsilon_j\,C_{j+1,1,j}^{0,m_{\mathrm{i}},m_{\mathrm{i}}}\,\frac{2(j+1)+1}{2(j-1)+1}\right),$$

$$E^{\beta}_{j-1} = \sin^2\epsilon_j\,t^{\beta}_{j-1}(k',k;k^2)\left(C_{j-1,1,j}^{0,m_{\mathrm{i}},m_{\mathrm{i}}}+\cot\epsilon_j\,C_{j+1,1,j}^{0,m_{\mathrm{i}},m_{\mathrm{i}}}\,\frac{2(j+1)+1}{2(j-1)+1}\right),$$

$$E^{\alpha}_{j+1} = \sin^2\epsilon_j\,t^{\alpha}_{j+1}(k',k;k^2)\left(C_{j+1,1,j}^{0,m_{\mathrm{i}},m_{\mathrm{i}}}-\cot\epsilon_j\,C_{j-1,1,j}^{0,m_{\mathrm{i}},m_{\mathrm{i}}}\,\frac{2(j-1)+1}{2(j+1)+1}\right),$$

$$E^{\beta}_{j+1} = \cos^2\epsilon_j\,t^{\beta}_{j+1}(k',k;k^2)\left(C_{j+1,1,j}^{0,m_{\mathrm{i}},m_{\mathrm{i}}}+\tan\epsilon_j\,C_{j-1,1,j}^{0,m_{\mathrm{i}},m_{\mathrm{i}}}\,\frac{2(j-1)+1}{2(j+1)+1}\right).$$