OSCILLATORY CONVECTION IN A VISCOELASTIC FLUID LAYER IN HYDROMAGNETICS

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[Manuscript received 24 May 1972]

Abstract

A study is made of the overstable mode of convection in an infinite horizontal layer of a viscoelastic fluid, heated from below, in the presence of a magnetic field. It is shown first that the problem is characterized by a variational principle. Proper solutions are then obtained for the case of two rigid boundaries using the variational method. It is found that the magnetic field has a stabilizing effect on the thermally induced overstability in a viscoelastic fluid, as in the case of an ordinary viscous fluid. The thermodynamic significance of the variational principle is also considered.

I. Introduction

Chandrasekhar (1961) has given an extensive account of the various investigations, in both hydrodynamics and hydromagnetics, of the problem of the onset of convection in an infinite horizontal viscous fluid layer heated from below. Vest and Arpaci (1969) have considered the stability of a horizontal layer of a *viscoelastic* fluid heated from below and have obtained the conditions under which the thermally induced overstability occurs in a Maxwellian fluid. They found that the effect of elasticity is destabilizing both in the sense that oscillatory convection can occur at a lower critical Rayleigh number R_c (defined below) than does stationary convection, and that R_c for overstability decreases with the increase in elasticity.

Recently the present authors (Bhatia and Steiner 1972) have studied the overstability in a horizontal layer of a Maxwellian fluid heated from below, in the presence of Coriolis forces. The conditions under which oscillatory convection occurs in a rotating Maxwellian fluid layer have been obtained for two different sets of boundary conditions: (1) two *free* boundaries, by exact solution; and (2) two *rigid* boundaries, by a variational method. In both these cases rotation has been found to have a destabilizing influence.

The present authors (1971) have also investigated the problem of overstability in a layer of a Maxwellian fluid heated from below when a uniform magnetic field, acting in a direction parallel to that of gravity, is present. The problem has been solved for the case when the fluid is confined between two *free* boundaries. Although free boundaries are somewhat unrealistic to obtain experimentally, they allow exact solutions. It has been shown there that the magnetic field has a stabilizing influence on the oscillatory mode of convection in a Maxwellian fluid as R_c increases with increase in the strength of the ambient magnetic field. The effect of the magnetic field on R_c is the same as that for an ordinary viscous fluid.

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In this paper we study the same problem for the case when the fluid is contained between two *rigid* boundaries. Firstly we show that a variational principle characterizes the problem, and we then apply the variational method to obtain the values of the critical Rayleigh numbers, wave numbers, and frequencies when both boundaries are rigid. Finally, the thermodynamic significance of the variational principle is considered.

II. Formulation of the Problem

Consider an infinite horizontal layer of a viscoelastic fluid of depth d, which is being heated uniformly from below, and suppose that a uniform vertical magnetic field $\boldsymbol{H} = (0, 0, H)$ is prevalent and that the medium is of finite electrical conductivity. Further assume that the viscoelastic nature of the fluid is described by the Maxwellian constitutive relation

$$p_{ij} + t_0 \, \mathrm{d}p_{ij}/\mathrm{d}t = \mu(\partial v_i/\partial x_j + \partial v_j/\partial x_i), \tag{1}$$

where p_{ij} is the viscous stress tensor, $v_i(u, v, w)$ the velocity vector, $\mathrm{d}/\mathrm{d}t$ the mobile operator, t_0 the Maxwellian relaxation time, and μ the coefficient of viscosity.

The basic equations governing the motion of a Maxwellian fluid, appropriate to the problem under investigation, in the presence of a magnetic field have been obtained by Bhatia and Steiner (1971), following closely Chandrasekhar (1961) and using the linearized form of relation (1). The details are therefore omitted here, but the procedure adopted in deriving these equations is briefly indicated. From the curl of the equations of motion taken once and twice, the equation governing the perturbation $\boldsymbol{h}(h_x, h_y, h_z)$ in the magnetic field $\boldsymbol{H}$ and the curl of this equation, and finally the equation governing the perturbation θ in temperature T, a set of five linear differential equations are obtained after employing normal mode analysis and using the Boussinesq approximation. Among the resulting equations the two governing Z (where Z is related to the z component, ζ, of the vorticity vector $\boldsymbol{\omega}$ defined by $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v}$ through (5) below) and X (where X is related to the z component, ξ, of the vector $\boldsymbol{M}$ defined by $\boldsymbol{M} = \operatorname{curl} \boldsymbol{h}$ through (5) below) can be combined to lead to the conclusion that both X and Z vanish identically, as in the case of a Newtonian fluid. Thus we obtain finally the set of three equations (formulae (23), (24), and (26) of Bhatia and Steiner 1971)

$$(1+\Gamma\sigma)\{\sigma(\mathrm{D}^2-a^2)W + (g\alpha d^2/\nu)a^2\Theta - (\mu_e Hd/4\pi\rho\nu)\mathrm{D}(\mathrm{D}^2-a^2)K\} = (\mathrm{D}^2-a^2)^2 W, \tag{2}$$

$$(\mathrm{D}^2-a^2-\sigma p_2)K = -(Hd/\eta)\mathrm{D}W, \tag{3}$$

$$(\mathrm{D}^2-a^2-\sigma p_1)\Theta = -(\beta d^2/\kappa)W. \tag{4}$$

In these equations α is the coefficient of volume expansion, β $(= -\mathrm{d}T/\mathrm{d}z)$ is the adverse temperature gradient, κ is the coefficient of thermometric conductivity, η is the magnetic resistivity, μ_e is the magnetic permeability, and the kinematic viscosity $\nu = \mu/\rho$ where ρ is the density. The time and space dependence of the various quantities is assumed to be of the form

$$\{w, \theta, h_z, \zeta, \xi\} = \{W(z), \Theta(z), K(z), Z(z), X(z)\}\exp(\mathrm{i}k_x x + \mathrm{i}k_y y + nt), \tag{5}$$

where $k = (k_x^2+k_y^2)^{\frac{1}{2}}$ is the wave number and n, which may be complex, is the frequency of the perturbation. Also the following transformations have been made to frame the equations in dimensionless form:

$$a = kd\,, \quad \sigma = nd^2/\nu\,, \quad p_1 = \nu/\kappa\,, \quad p_2 = \nu/\eta\,, \quad \Gamma = t_0\,\nu/d^2\,, \tag{6}$$

where p_1 is the Prandtl number, p_2 is the magnetic Prandtl number, and Γ is the elastic parameter. The operator D is equivalent to d/dz, where z is now measured in terms of d.

Equations (2)–(4) must be solved subject to the appropriate boundary conditions. In order now to investigate the practical case of rigid boundaries, we first show that the problem is characterized by a variational principle.

III. Variational Principle

Let us write

$$F = (\mathrm{D}^2-a^2)\{\mathrm{D}^2-a^2-\sigma(1+\Gamma\sigma)\}W+(\mu_e Hd/4\pi\rho\nu)(1+\Gamma\sigma)\mathrm{D}(\mathrm{D}^2-a^2)K \tag{7}$$

and eliminate Θ between equations (2) and (4) to obtain

$$(\mathrm{D}^2-a^2-p_1\,\sigma)F = -(1+\Gamma\sigma)Ra^2W\,, \tag{8}$$

where the Rayleigh number R is defined by

$$R = g\alpha\beta d^4/\kappa\nu\,. \tag{9}$$

The conditions to be satisfied on the boundaries depend on whether the bounding surfaces are free or rigid and perfectly conducting or nonconducting. On different boundaries the conditions that must be satisfied by W, Θ, and K are (Chandrasekhar 1961):

both rigid and free boundaries	$W = 0$,	Θ (or F) $= 0$;	(10a)
rigid boundary	$\mathrm{D}W = 0$,	$Z = 0$;	(10b)
free boundary	$\mathrm{D}^2W = 0$,	$\mathrm{D}Z = 0$;	(10c)

and

perfectly conducting boundary	$\mathrm{D}X = 0$,	$K = 0$;	(10d)
boundary adjoining nonconducting medium	$X = 0$.		(10e)

After multiplication of equation (8) by F and integration over the range of z (i.e. from 0 to 1), we finally obtain by integrating by parts once or repeatedly and using the boundary conditions (10)

$$R = I_1/a^2(1+\Gamma\sigma)I_2\,, \tag{11}$$

where

$$I_1 = \int_0^1 \{(\mathrm{D}F)^2 + (a^2 + p_1\sigma)F^2\}\,\mathrm{d}z \tag{12a}$$

and

$$I_2 = \int_0^1 \{(\mathrm{D}^2 - a^2)W\}^2\,\mathrm{d}z + \sigma(1+\Gamma\sigma)\int_0^1 \{(\mathrm{D}W)^2 + a^2W^2\}\,\mathrm{d}z$$
$$+\frac{(1+\Gamma\sigma)\mu_\mathrm{e}\,\eta}{4\pi\rho\nu}\left(\int_0^1 \{(\mathrm{D}^2-a^2)K\}^2\,\mathrm{d}z + \sigma p_2\int_0^1 \{(\mathrm{D}K)^2 + a^2K^2\}\,\mathrm{d}z\right). \tag{12b}$$

Now consider the effect on R of respective variations δW and δK in W and K which are compatible with the boundary conditions. Retaining only the linear terms we get

$$\delta R = \{\delta I_1 - Ra^2(1+\Gamma\sigma)\delta I_2\}/a^2(1+\Gamma\sigma)I_2\,. \tag{13}$$

Integration by parts once or repeatedly and use of the corresponding boundary conditions on δW, δF, and δK then lead, after lengthy but straightforward calculations, to the result

$$\delta R = -\frac{2}{a^2(1+\Gamma\sigma)I_2}\int_0^1 \delta F\{(\mathrm{D}^2 - a^2 - p_1\sigma)F + Ra^2(1+\Gamma\sigma)W\}\,\mathrm{d}z\,. \tag{14}$$

The condition that the variation δR in R vanishes, for all small variations δF compatible with the boundary conditions, is that the quantity within the braces in the integrand of equation (14) must vanish. This condition is precisely the governing differential equation (8). Hence equation (11) provides the basis for a variational procedure for solving the present problem of convection in a viscoelastic fluid layer in the presence of a magnetic field.

IV. Solutions for Two Rigid Boundaries

We can now apply the variational method to obtain the solutions for the overstable mode of convection in a viscoelastic fluid layer contained between two rigid perfectly conducting boundaries. It has been pointed out by Chandrasekhar (1961) that in problems of this kind the first approximation should give the required characteristic values to an accuracy of a few per cent. As we are primarily interested here in gaining an insight into the tendencies of the effects of the magnetic field, it thus will be proper to obtain solutions that are accurate to this approximation.

Since the lowest characteristic value of R will occur among the even solutions in W and the odd solutions in K, we shall take (to first approximation) the solution for F appropriate to two rigid boundaries as

$$F = \cos\pi z\,, \tag{15}$$

where (in view of the symmetry with respect to the bounding planes) the origin of z has now been shifted so that its limits are $\pm\frac{1}{2}$. Eliminating K between equations (3) and (7) we find that W is a solution of

$$(\mathrm{D}^2-a^2)[(\mathrm{D}^2-a^2-\mathrm{i}\sigma p_2)\{\mathrm{D}^2-a^2-\mathrm{i}\sigma(1+\mathrm{i}\Gamma\sigma)\} - Q\mathrm{D}^2(1+\mathrm{i}\Gamma\sigma)]W = -c_1\cos\pi z\,, \tag{16}$$

where

$$c_1 = \pi^2+a^2+\mathrm{i}\sigma p_2 \tag{17}$$

and Q is a dimensionless number

$$Q = \mu_e H^2 d^2/4\pi\rho\nu\eta\,. \tag{18}$$

The parameter σ in these equations has been replaced by $\mathrm{i}\sigma$. We shall now suppose that σ is real, as we are interested in specifying the critical Rayleigh number for the onset of instability via a state of *purely oscillatory motion.*

Appropriate to the problem, W is given by

$$W = c_1\gamma_1 \cos\pi z + \sum_{j=1}^{3} B_j \cosh q_j z\,, \tag{19}$$

where

$$\gamma_1^{-1} = c_1(\pi^2+a^2)\{\pi^2+a^2+\mathrm{i}\sigma(1+\mathrm{i}\Gamma\sigma)\}+Q(1+\mathrm{i}\Gamma\sigma)\pi^2(\pi^2+a^2)\,, \tag{20}$$

the B_j ($j = 1,2,3$) are constants of integration, and q_1^2, q_2^2, and q_3^2 are the roots of the cubic equation

$$(q^2-a^2)[(q^2-a^2-\mathrm{i}\sigma p_2)\{q^2-a^2-\mathrm{i}\sigma(1+\mathrm{i}\Gamma\sigma)\}-Q(1+\mathrm{i}\Gamma\sigma)q^2] = 0\,. \tag{21}$$

Substituting in equation (8) the solution for W obtained from equation (19), with the B_j determined from the boundary conditions for W, DW, and K, and with σ replaced by $\mathrm{i}\sigma$, we follow Chandrasekhar (1961) and obtain to first approximation

$$R = \frac{\pi^2+a^2+\mathrm{i}p_1\sigma}{a^2(1+\mathrm{i}\Gamma\sigma)\{c_1\gamma_1+2(0/0)\}}\,, \tag{22}$$

where the value of (0/0) is given by Chandrasekhar as

$$\begin{aligned}(0/0) = 2\pi^2\gamma_1^2\Delta \prod_{j=1}^{3} \coth(\tfrac{1}{2}q_j)\{&x_1^{-1}(x_3-x_2)(c_1+x_1)^2 q_1\tanh(\tfrac{1}{2}q_1)\\ &+x_2^{-1}(x_1-x_3)(c_1+x_2)^2 q_2\tanh(\tfrac{1}{2}q_2)+x_3^{-1}(x_2-x_1)(c_1+x_3)^2 q_3\tanh(\tfrac{1}{2}q_3)\}\,,\end{aligned} \tag{23}$$

with

$$\begin{aligned}\Delta^{-1} = &(q_2 q_3/x_2 x_3)(x_3-x_2)\coth(\tfrac{1}{2}q_1)+(q_3 q_1/x_3 x_1)(x_1-x_3)\coth(\tfrac{1}{2}q_2)\\ &+(q_1 q_2/x_1 x_2)(x_2-x_1)\coth(\tfrac{1}{2}q_3)\end{aligned} \tag{24}$$

and

$$x_j = q_j^2-a^2-\mathrm{i}\sigma p_2\,. \tag{25}$$

Numerical calculations have been performed to determine the critical values R_c, a_c, and σ_c for selected values of the other parameters. Since evaluation of equation (22) for assigned values of Q, p_1, p_2, Γ, a, and σ leads in general to a physically meaningless complex value of R, a numerical search was conducted to find, firstly, the values of σ for which the imaginary part of R vanished, and then the σ that gave the lowest positive real value of R. A neutral curve was traced after repeating the procedure for several values of a near the critical region by requiring σ to converge with high accuracy to the value for which the imaginary part of R vanished. The minimum of this curve then gave the point (a_c, R_c).

As the calculations and numerical search were long and tedious and the present aim was mainly to gain an insight into the effects of the magnetic field, R_c has been obtained for only a few selected values of the dimensionless number Q. The results are shown in Table 1 for $\Gamma = 0\cdot5$ and $1\cdot0$ with $p_1 = 1\cdot0$ and $p_2 = 0\cdot1$.

TABLE 1

CRITICAL VALUES OF WAVE NUMBERS, FREQUENCIES, AND RAYLEIGH NUMBERS FOR ONSET OF OVERSTABILITY

The values are for $p_1 = 1\cdot0$ and $p_2 = 0\cdot1$. For comparison, the last column gives the corresponding values of R_c for two free boundaries from Bhatia and Steiner (1971*b*)

Q	$\Gamma = 0\cdot5$			$\Gamma = 1\cdot0$			
	a_c	σ_c	R_c	a_c	σ_c	R_c	R_c (free)
0	3·9185	8·3806	$1\cdot1284\times10^2$	3·6966	6·0601	$5\cdot1562\times10$	$4\cdot3386\times10$
10	5·8474	9·7342	$3\cdot0230\times10^2$	6·5536	7·6065	$2\cdot2304\times10^2$	$1\cdot9175\times10^2$
100	9·6269	13·998	$1\cdot4732\times10^3$	11·062	11·461	$1\cdot3087\times10^3$	$1\cdot2246\times10^3$
500	14·069	19·619	$5\cdot9142\times10^3$	16·238	16·267	$5\cdot5938\times10^3$	$5\cdot4374\times10^3$
1000	16·631	22·938	$1\cdot1211\times10^4$	19·210	19·074	$1\cdot0775\times10^4$	$1\cdot0571\times10^4$
5000	24·630	33·416	$5\cdot2181\times10^4$	28·471	27·903	$5\cdot1264\times10^4$	$5\cdot0889\times10^4$
10 000	29·200	39·438	$1\cdot0263\times10^5$	33·757	32·968	$1\cdot0135\times10^5$	$1\cdot0087\times10^5$

It is clear from Table 1 that R_c increases as Q increases, thereby showing a stabilizing effect of the magnetic field. This was just the result obtained earlier (Bhatia and Steiner 1971) for the case of two free boundaries, and we may thus conclude that the presence of a magnetic field has a stabilizing influence on the overstable mode of convection in a viscoelastic fluid layer whether the two boundaries are rigid or free.

V. THERMODYNAMIC SIGNIFICANCE OF VARIATIONAL PRINCIPLE

It has been shown by Chandrasekhar (1961) that in the presence of a magnetic field the variational principle for the onset of instability as stationary convection is equivalent to the thermodynamic relation

$$\epsilon_\sigma + \epsilon_\nu = \epsilon_g , \tag{26}$$

where ϵ_ν is the rate of dissipation of energy by the viscous stresses, ϵ_g is the rate of liberation of energy by the buoyancy forces, and ϵ_σ is the rate of dissipation of energy by Joule heating. For the case when the marginal state is oscillatory we must make proper allowance in (26) for the time dependence of kinetic energy and the perturbation in the magnetic field. Denoting the change in kinetic energy by ϵ'_ν and in the perturbations in the magnetic field by ϵ'_σ, we can write the energy balance equation as

$$\epsilon_\sigma + \epsilon'_\sigma + \epsilon_\nu + \epsilon'_\nu = \epsilon_g . \tag{27}$$

We now proceed to show that the same statement can be made for the present problem. To do this we first consider the term ϵ_ν in equation (27):

$$\epsilon_\nu = -\rho\nu \int_0^d \langle v_i\, \partial p_{ji}/\partial x_j \rangle\, \mathrm{d}z\,, \tag{28}$$

where the angle brackets indicate an average over the horizontal plane. After using the linearized form for p_{ij} under the Boussinesq approximation, the expression for ϵ_ν becomes

$$\epsilon_\nu = -\frac{\rho\nu}{(1+\Gamma\sigma)d^2}\int_0^1 \{\langle w(\mathrm{D}^2-a^2)w\rangle + \langle u(\mathrm{D}^2-a^2)u\rangle + \langle v(\mathrm{D}^2-a^2)v\rangle\}\, \mathrm{d}z\,. \tag{29}$$

In terms of the solutions for w and h_z let us suppose that a simple cellular motion is represented by

$$\left.\begin{aligned} w &= W(z)\cos a_x x \cos a_y y\,, & h_z &= K(z)\cos a_x x\cos a_y y\,,\\ u &= -a^{-2}a_x \mathrm{D}W \sin a_x x\cos a_y y\,, & v &= -a^{-2}a_y \mathrm{D}W\cos a_x x\sin a_y y\,,\\ h_x &= -a^{-2}a_x \mathrm{D}K\sin a_x x\cos a_y y\,, & h_y &= -a^{-2}a_y\mathrm{D}K\cos a_x x\sin a_y y\,. \end{aligned}\right\} \tag{30}$$

Using the relations (30) to carry out explicitly the averaging process and integrating by parts, making use of the boundary conditions (10), we finally obtain

$$\epsilon_\nu = \frac{\rho\nu}{4(1+\Gamma\sigma)a^2d^2}\int_0^1 \{(\mathrm{D}^2-a^2)W\}^2\, \mathrm{d}z\,. \tag{31}$$

The other terms in equation (27) can be evaluated in the same manner to give

$$\epsilon'_\nu = \frac{\sigma\rho\nu}{4a^2d^2}\int_0^1 \{a^2W^2 + (\mathrm{D}W)^2\}\, \mathrm{d}z\,, \tag{32}$$

$$\epsilon_\sigma = \frac{\mu_e\,\eta}{4\pi d^2}\int_0^1 \langle |\operatorname{curl} \boldsymbol{h}|^2\rangle\, \mathrm{d}z = \frac{\mu_e\,\eta}{16\pi a^2d^2}\int_0^1 \{(\mathrm{D}^2-a^2)K\}^2\, \mathrm{d}z\,, \tag{33}$$

$$\epsilon'_\sigma = \frac{\mu_e}{4\pi}\frac{\partial}{\partial t}\left(\int_0^d \langle h_i^2\rangle\, \mathrm{d}z\right) = \frac{\mu_e\,\sigma}{16\pi a^2d^2}\int_0^1 \{(\mathrm{D}K)^2 + a^2K^2\}\, \mathrm{d}z\,, \tag{34}$$

$$\epsilon_g = \frac{\rho\kappa\nu^2}{4(1+\Gamma\sigma)^2 g\alpha\beta d^6a^4}\int_0^1 \{(\mathrm{D}F)^2 + (a^2+p_1\sigma)F^2\}\, \mathrm{d}z\,. \tag{35}$$

Substitution of the relations (31)–(35) into equation (27) leads to an expression that is identical with equation (11) for R, whose minimum was taken to represent the critical state of marginal stability. In the case of overstable motion, which is of interest here, $\sigma = \mathrm{i}\sigma_i$ and we may thus state: overstability in a viscoelastic fluid layer heated from below in the presence of a magnetic field will occur at the lowest possible adverse temperature gradient at which the rates of change of kinetic energy and magnetic energy can balance, in a synchronous manner, the periodically varying rates of energy dissipation by the shear stresses and Joule heating and the energy release by the buoyancy forces, assuming that stationary convection has not occurred.

VI. Acknowledgments

The computations were performed on the CDC 3200 computer at the Monash University Computer Centre. One of us (J.M.S.) received support during the course of the work from a Commonwealth Postgraduate Award. The other author (P.K.B.) held a visiting appointment (August 1969 to November 1970) at Monash University, and he is grateful to that University for financial support and to Jodhpur University, India, for the leave of absence. The authors thank Miss B. Rodda for typing the manuscript.

VII. References

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