COHERENT GYROMAGNETIC EMISSION AS A RADIATION MECHANISM

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Abstract

The properties of coherent gyromagnetic emission of radiation which can escape from a source of astrophysical interest by a bi-Maxwellian distribution of nonrelativistic electrons (thermal velocities $\beta_{\perp 0} c$ and $\beta_{\parallel 0} c$, and streaming velocity $\beta_s c$) are considered. It is found that: (1) Coherent emission occurs only for $\beta_{\perp 0}^2 \gtrsim \beta_{\parallel 0}$, and only for a more extreme anisotropy for waves close to their cutoff frequencies. (2) Coherent emission in the x-mode at the second harmonic, i.e. at $\omega \approx 2\Omega_e > \omega_p$, is the dominant process except for $\beta_s > \omega_p^2/\Omega_e^2$ when coherent emission in the x-mode at the first harmonic, i.e. at $\omega \sim \Omega_e \gg \omega_p$, is possible and dominates. It is argued that these results are not strongly dependent on the choice of distribution function. It is pointed out that the use of the quasilinear equations can be justified in any astrophysical context but that the above properties apply even when the quasilinear treatment is not valid. Limitations imposed on the mechanism by the presence of absorption in the thermal plasma and by competing instabilities are shown to be relevant except for emission in the x-mode at the first harmonic.

I. INTRODUCTION

The fact that certain nonthermal distributions of electrons could lead to the growth of waves with frequencies close to the electron gyrofrequency (and its harmonics) was suggested by Harris (1959) and Bekefi *et al.* (1961) (for other references see the review by Timofeev and Pistunovich 1970) and was first demonstrated in the laboratory by Hirshfield and Wachtel (1964). Four distinct types of astrophysical application have been suggested for this mechanism: (1) the scattering (in pitch angle) of suprathermal particles, e.g. electrons in the Van Allen belts or solar and galactic cosmic rays, in which anisotropic distributions of particles can cause the waves involved in the scattering to grow; (2) a related situation to (1), in which certain v.l.f. emissions from the magnetosphere are interpreted as resulting from the growth of waves in the whistler mode due to the presence of an anisotropic distribution of electrons; (3) coherent synchrotron radiation (see e.g. McCray 1966; Zheleznіakov 1967), which involves emission of ultra-relativistic electrons; and (4) coherent gyromagnetic emission by nonrelativistic electrons of waves which escape from the source, as has been suggested for the Jovian dekametric radiation (see e.g. Ellis 1962, 1963, 1965; Hirshfield and Bekefi 1963; Fung 1966; Goldreich and Lynden-Bell 1969; Goldstein and Eviatar 1972) and for solar bursts of spectral type I (Fung and Yip 1966) and type III (Kuckes and Sudan 1971). For all the suggested applications in (4),

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alternative radiation mechanisms based on some form of the plasma emission mechanisms, as defined by Melrose and Sy (1972*a*), have been proposed: see Zhelezniakov (1965, 1970, p. 616) on the Jovian dekametric radiation and Takakura (1963) on type I bursts; plasma emission is the widely accepted mechanism for type III bursts.

The purpose of this paper is to present a detailed discussion of the theory behind the above class (4) of suggested astrophysical applications. An important restriction imposed is that the emitted waves have a refractive index less than unity. The argument behind this is simply that the only waves able to escape directly from a source to infinity (without requiring nonlinear conversion processes) are those at $\omega > \omega_p$ in the o-mode and at $\omega > \omega_x = \frac{1}{2}\Omega_e + \frac{1}{2}(\Omega^2 + 4\omega_p^2)^{\frac{1}{2}}$ in the x-mode of magnetoionic theory. These waves do have a refractive index less than unity. For $\mu_\sigma < 1$, in the notation used by Melrose and Sy (1972*a*), wave–particle interactions are possible only via the normal Doppler effect, i.e. only for harmonics $s > 0$, in the present notation. If isotropic distributions $f(p)$ with $\partial f(p)/\partial p > 0$ are excluded, coherent emission at $s > 0$ is possible only if the distribution is anisotropic in the sense of having an excess of perpendicular (to the magnetic field lines) momentum. Such a distribution could be generated by a stream of suprathermal electrons propagating in the direction of increasing magnetic field strength, this being the mechanism used by Hirshfield and Wachtel (1964) in obtaining such coherent gyromagnetic emission in the laboratory.

Throughout the present paper the distribution of (nonrelativistic) electrons is assumed to be of the form

$$f(p,\alpha) = \{(2\pi)^{3/2} m^3 c^3 \beta_{\perp 0}^2 \beta_{\parallel 0}\}^{-1} \exp\left(-\frac{\beta^2 \sin^2\alpha}{2\beta_{\perp 0}^2} - \frac{(\beta\cos\alpha - \beta_s)^2}{2\beta_{\parallel 0}^2}\right), \tag{1}$$

where βc is the speed of the electron, α is the pitch angle, and $\beta_{\perp 0}$, $\beta_{\parallel 0}$, and β_s are dimensionless constants. This is the bi-Maxwellian streaming distribution, which was used extensively by Stix (1962, Chapter 9). There are a number of formal advantages in this choice of distribution function: in particular, all the relevant integrals can be performed explicitly in evaluating growth rates, the condition for coherent emission to occur takes on a particularly simple form, and the limit of a δ function distribution can be approached continuously (for the relevance of this fact see Section V). In the Appendix a different choice of distribution is examined to check that the results of the present paper are not strongly dependent on the specific form of the distribution function.

In Section II the quasilinear equations describing the interaction between electrons and waves in an arbitrary magnetoionic mode are written down and applied to evaluate the absorption coefficient for a distribution of the form (1). In Section III the growth rates at harmonics $s = 1$ and $s \geqslant 2$ are evaluated for both o-mode and x-mode waves. It is found that the extreme anisotropy $\beta_{\perp 0}^2 \gtrsim \beta_{\parallel 0}$ is required for coherent emission of the waves of interest to occur. In Section IV limitations imposed by the requirements that the waves escape and that other instabilities do not dominate (thereby preventing the coherent emission of interest from occurring) are shown to be severe. In Section V the alternative reactive-medium instabilities (in the nomenclature of Briggs 1964) and their relevance are discussed. The conclusions are summarized in Section VI.

II. Quasilinear Treatment

(*a*) *Emissivity*

The emissivity in the magnetoionic waves is defined here to mean the power lost by an electron per unit solid angle about the wave normal direction per unit (angular) frequency range. It has the functional dependence $\eta^{\sigma}(s,\omega,\theta)$, on separation into contributions from each harmonic $s = 0, \pm 1, \pm 2, \ldots$. The wave properties used are the refractive index $\mu_{\sigma}(\omega,\theta)$ and the two quantities $K_{\sigma}(\omega,\theta)$ and $T_{\sigma}(\omega,\theta)$ which describe the polarization (see Appendix I of Melrose and Sy 1972*a*). The electron is described in terms of its speed $c\beta$, pitch angle α, and gyrofrequency $\Omega_{\mathrm{e}}(1-\beta^2)^{\frac{1}{2}}$, where $\Omega_{\mathrm{e}} = |e||\boldsymbol{B}|/mc$.

For $s \neq 0$ (this restriction is not maintained throughout the paper) one has (Eidman 1958, 1959; Melrose 1968*a*, 1968*b*; Trulsen and Fejer 1970; Sakurai 1972)

$$\eta^{\sigma}(s,\omega,\theta) = \frac{e^2\Omega_{\mathrm{e}}^2(1-\beta^2)^{\frac{1}{2}}y_{\sigma}}{2\pi c\mu_{\sigma}(1+T_{\sigma}^2)\sin^2\theta}$$

$$\times\left|\left(\frac{K_{\sigma}y_{\sigma}\sin\theta}{(1-\beta^2)^{\frac{1}{2}}}+z_{\sigma}T_{\sigma}\right)sJ_s(sx_{\sigma})+sx_{\sigma}J_s'(sx_{\sigma})\right|^2\delta(\omega-s\Omega_{\mathrm{e}}y_{\sigma}), \tag{2}$$

where, in the notation of Wild and Hill (1971),

$$x_{\sigma} = \mu_{\sigma}\beta\sin\alpha\sin\theta/(1-\mu_{\sigma}\beta\cos\alpha\cos\theta), \tag{3a}$$

$$y_{\sigma} = (1-\beta^2)^{\frac{1}{2}}/(1-\mu_{\sigma}\beta\cos\alpha\cos\theta), \tag{3b}$$

$$z_{\sigma} = (\cos\theta-\mu_{\sigma}\beta\cos\alpha)/(1-\mu_{\sigma}\beta\cos\alpha\cos\theta). \tag{3c}$$

(*b*) *Quasilinear Equations*

The quasilinear equations can be written down in terms of the emissivity and a quantity describing the distribution of waves by rewriting equations given by, for example, Melrose (1968*a*). Let the distribution of waves be described by the energy density $W^{\sigma}(\omega,\theta)$ per unit solid angle about the wave normal direction per unit frequency range. If all variations are assumed to be temporal, one has

$$\partial W^{\sigma}(\omega,\theta)/\partial t = \varepsilon^{\sigma}(\omega,\theta)-\gamma^{\sigma}(\omega,\theta)\,W^{\sigma}(\omega,\theta) \tag{4}$$

and

$$\frac{\partial f(p,\alpha)}{\partial t} = \frac{\partial}{\partial(\cos\alpha)}\Big(D_{\alpha}f(p,\alpha)\Big)+p^{-2}\frac{\partial}{\partial p}\Big(p^2D_pf(p,\alpha)\Big)$$

$$+\frac{\partial}{\partial(\cos\alpha)}\left\{\left(D_{\alpha\alpha}\frac{\partial}{\partial(\cos\alpha)}+D_{\alpha p}\frac{\partial}{\partial p}\right)f(p,\alpha)\right\}$$

$$+p^{-2}\frac{\partial}{\partial p}\left\{p^2\left(D_{p\alpha}\frac{\partial}{\partial(\cos\alpha)}+D_{pp}\frac{\partial}{\partial p}\right)f(p,\alpha)\right\}. \tag{5}$$

If it is assumed that $f(p,\alpha)$ is normalized by

$$2\pi \int_{-1}^{1} \mathrm{d}(\cos\alpha) \int_{0}^{\infty} \mathrm{d}p\, p^2 f(p,\alpha) = 1 \tag{6}$$

with number density n_1 then the volume emissivity in equation (4) is given by

$$\varepsilon^{\sigma}(\omega,\theta) = \sum_{s=-\infty}^{\infty} n_1 2\pi \int_{-1}^{1} \mathrm{d}(\cos\alpha) \int_{0}^{\infty} \mathrm{d}p\, p^2\, \eta^{\sigma}(s,\omega,\theta) f(p,\alpha) \tag{7}$$

and the absorption coefficient by

$$\gamma^{\sigma}(\omega,\theta) = -\sum_{s=-\infty}^{\infty} n_1 2\pi \int_{-1}^{1} \mathrm{d}(\cos\alpha) \int_{0}^{\infty} \mathrm{d}p\, p^2 \frac{(2\pi)^3 c^3\, \eta^{\sigma}(s,\omega,\theta)}{\omega^3 \mu_{\sigma}^2\, \partial(\omega\mu_{\sigma})/\partial\omega}$$
$$\times \frac{\omega}{\beta c}\left(\frac{\partial}{\partial p} - p^{-1}(\cos\alpha - \mu_{\sigma}\beta\cos\theta)\frac{\partial}{\partial(\cos\alpha)}\right) f(p,\alpha). \tag{8}$$

In equation (5) the coefficients describing the effects of spontaneous emission are

$$\begin{bmatrix} D_{\alpha} \\ D_{p} \end{bmatrix} = \sum_{s=-\infty}^{\infty} 2\pi \int_{-1}^{1} \mathrm{d}(\cos\theta) \int_{0}^{\infty} \frac{\mathrm{d}\omega}{\omega}\, \eta^{\sigma}(s,\omega,\theta) \frac{\omega}{\beta c} \begin{bmatrix} -p^{-1}(\cos\alpha - \mu_{\sigma}\beta\cos\theta) \\ 1 \end{bmatrix} \tag{9}$$

while those describing the net effect of the induced processes are

$$\begin{bmatrix} D_{\alpha\alpha} \\ D_{\alpha p} = D_{p\alpha} \\ D_{pp} \end{bmatrix} = \sum_{s=-\infty}^{\infty} 2\pi \int_{-1}^{1} \mathrm{d}(\cos\theta) \int_{0}^{\infty} \frac{\mathrm{d}\omega}{\omega} \frac{(2\pi)^3 c^3\, \eta^{\sigma}(s,\omega,\theta)\, W^{\sigma}(\omega,\theta)}{\omega^3 \mu_{\sigma}^2\, \partial(\omega\mu_{\sigma})/\partial\omega}$$
$$\times \left(\frac{\omega}{\beta c}\right)^2 \begin{bmatrix} p^{-2}(\cos\alpha - \mu_{\sigma}\beta\cos\theta)^2 \\ -p^{-1}(\cos\alpha - \mu_{\sigma}\beta\cos\theta) \\ 1 \end{bmatrix}. \tag{10}$$

When all variations are assumed to be spatial, equation (4) is often more convenient in the form of a transfer equation. Let l be a distance along the ray path and let $I^{\sigma}(\omega,\theta_{\mathrm{r}})$ be the intensity, i.e. the power crossing unit surface area per unit solid angle about the ray direction per unit frequency range. One then has (Bekefi 1966, p. 38; Énomé 1969; Zhelezniakov 1971)

$$\mu_{\mathrm{r}\sigma}^2 \frac{\partial}{\partial l}\left(\frac{I^{\sigma}(\omega,\theta_{\mathrm{r}})}{\mu_{\sigma\mathrm{r}}^2}\right) = \varepsilon_{\mathrm{r}}^{\sigma}(\omega,\theta_{\mathrm{r}}) - \kappa^{\sigma}(\omega,\theta_{\mathrm{r}})\, I^{\sigma}(\omega,\theta_{\mathrm{r}}), \tag{11}$$

with

$$\left.\begin{aligned} \varepsilon_{\mathrm{r}}^{\sigma}(\omega,\theta_{\mathrm{r}}) &= \varepsilon^{\sigma}(\omega,\theta)\frac{\partial(\cos\theta)}{\partial(\cos\theta_{\mathrm{r}})}, \qquad \kappa^{\sigma}(\omega,\theta_{\mathrm{r}}) = \frac{\gamma^{\sigma}(\omega,\theta)}{v_g^{\sigma}(\omega,\theta)}, \\ \mu_{\sigma\mathrm{r}}^2(\omega,\theta_{\mathrm{r}}) &= \mu_{\sigma}^2(\omega,\theta)\frac{v_g^{\sigma}(\omega,\theta)}{c}\frac{\partial(\omega\mu_{\sigma})}{\partial\omega}\frac{\partial(\cos\theta)}{\partial(\cos\theta_{\mathrm{r}})}. \end{aligned}\right\} \tag{12}$$

The relevant ray properties $v_g^\sigma(\omega,\theta)$ and θ_r were given explicitly for the magnetoionic waves by Melrose and Sy (1972*b*).

For the waves with $\mu_\sigma < 1$ rough calculations indicate that the ray angle θ_r differs little from the wave normal angle θ, for instance, $|\theta_r - \theta| < 10°$, while the group velocity is substantially less than c only for $\mu_\sigma \ll 1$. In view of this, the more cumbersome transfer equation (11) could be replaced by the simpler equation (4), with $l = ct$, $I^\sigma = cW^\sigma$, and $\kappa^\sigma = \gamma^\sigma/c$, for semiquantitative purposes.

(*c*) *Bi-Maxwellian Streaming Distribution*

On making the nonrelativistic approximation in expressions (7) and (8) and inserting the distribution (1), the integrals can be performed analytically. The integrals over the products of Bessel functions in the emissivity (equation (2)) lead to modified Bessel functions $I_s(\lambda_\sigma)$ and their derivatives $I_s'(\lambda_\sigma)$ with the argument

$$\lambda_\sigma = \{(\omega/\Omega_e)\mu_\sigma \beta_{\perp 0}\sin\theta\}^2 . \tag{13}$$

No approximations to the wave properties (not even the magnetoionic approximation made throughout this paper) need be made. The results apply not only for any $s \neq 0$ but also for $s = 0$.

On separation into contributions from each value of s,

$$\varepsilon^\sigma(\omega,\theta) = \sum_{s=-\infty}^{\infty} \varepsilon^\sigma(s,\omega,\theta) \quad \text{and} \quad \gamma^\sigma(\omega,\theta) = \sum_{s=-\infty}^{\infty} \gamma^\sigma(s,\omega,\theta) . \tag{14}$$

One then has

$$\varepsilon^\sigma(s,\omega,\theta) = \frac{n_1 e^2 \omega \beta_{\perp 0}^2 A^\sigma(s,\omega,\theta)}{(2\pi)^{3/2}\beta_{\parallel 0}\, c\,|\cos\theta|}\exp\left(-\frac{\{\omega(1-\mu_\sigma\beta_s\cos\theta)-s\Omega_e\}^2}{2\omega^2\beta_{\parallel 0}^2\mu_\sigma^2\cos^2\theta}\right) \tag{15}$$

and

$$\gamma^\sigma(s,\omega,\theta) = \frac{(2\pi)^3 c}{\omega^2\mu_\sigma^2\{\partial(\omega\mu_\sigma)/\partial\omega\}m\beta_{\perp 0}^2}\, g^\sigma(s,\omega,\theta)\,\varepsilon^\sigma(s,\omega,\theta) , \tag{16}$$

with

$$g^\sigma(s,\omega,\theta) = \frac{s\Omega_e}{\omega} + \frac{\beta_{\perp 0}^2}{\beta_{\parallel 0}^2}\left(1 - \frac{s\Omega_e}{\omega} - \mu_\sigma\beta_s\cos\theta\right). \tag{17}$$

The modified Bessel functions appear only in the quantity

$$\begin{aligned} A^\sigma(s,\omega,\theta) = \frac{\exp(-\lambda_\sigma)}{1+T_\sigma^2}\Bigg\{&\left(\frac{\omega}{\Omega_e}(K_\sigma\cos\theta - T_\sigma\sin\theta)\tan\theta + sT_\sigma\sec\theta\right)^2\frac{I_s(\lambda_\sigma)}{\lambda_\sigma} \\ &+2\left(\frac{\omega}{\Omega_e}(K_\sigma\cos\theta - T_\sigma\sin\theta)\tan\theta + sT_\sigma\sec\theta\right)\Big(I_s'(\lambda_\sigma) - I_s(\lambda_\sigma)\Big) \\ &+\left(\frac{s^2}{\lambda_\sigma} + 2\lambda_\sigma\right)I_s(\lambda_\sigma) - 2\lambda_\sigma I_s'(\lambda_\sigma)\Bigg\} . \end{aligned} \tag{18}$$

It should be noted that equation (15) predicts an interaction between particles and waves with $\mu_\sigma \leqslant 1$ at $s \leqslant 0$ only because the distribution (1), being a nonrelativistic distribution, includes unphysical particles with $\beta > 1$.

(*d*) *Approximations*

For $\omega \approx s\Omega_e$, $\beta_{\perp 0} \ll 1$, $\sin\theta \ll 1$, and $\mu_\sigma < 1$ one has $\lambda_\sigma \ll s^2$. This is just the condition for the power series expansion of the modified Bessel functions to converge rapidly. On retaining only the lowest order terms in λ_σ for each value s and keeping $s \leqslant 0$ for completeness, the expression (18) reduces to

$$A^\sigma(s,\omega,\theta) \approx \frac{1}{2(1+T_\sigma^2)}\frac{(\frac{1}{2}\lambda_\sigma)^{|s|-1}}{|s|\,!}$$
$$\times\left(\frac{\omega}{\Omega_e}(K_\sigma\cos\theta - T_\sigma\sin\theta)\tan\theta + sT_\sigma\sec\theta + |s|\right)^2. \tag{19}$$

For $s \geqslant 2$ one can usually justify setting $\omega = s\Omega_e$, $K_\sigma = 0$,

$$T_o = -\cos\theta/(|\cos\theta|), \quad \text{and} \quad T_x = \cos\theta/(|\cos\theta|)$$

in equation (19). This gives

$$A^{o,x}(s \geqslant 2,\omega,\theta) \approx (s^2/4s\,!)(\tfrac{1}{2}\lambda_\sigma)^{s-1}(1 \mp |\cos\theta|)^2, \tag{20}$$

where the minus sign refers to the o-mode. (This also applies to the x-mode at $s = 1$ when such emission is allowed; see the inequality (37) in Section III(*c*).)

For $s = 1$ one can rewrite equation (19) using

$$K_\sigma = \frac{XY\sin\theta}{(1-X)}\frac{T_\sigma}{T_\sigma - Y\cos\theta} \quad \text{and} \quad T_\sigma^2 = 1 - \frac{Y\sin^2\theta}{(1-X)\cos\theta}T_\sigma, \tag{21}$$

with $Y = \Omega_e/\omega$ and $X = \omega^2/\omega^2$. However, as shall emerge, the next order terms in λ_σ are not necessarily negligible for the o-mode at $s = 1$. Retaining these terms one has

$$A^o(s=1,\omega,\theta) \approx \frac{1}{2(1+T_o{}^2)}$$
$$\times\left((1-Y)\frac{T_o(1-X\cos^2\theta)-(1-X)\cos\theta(\sin^2\theta - Y\cos^2\theta)}{Y(1-X)(T_o - Y\cos\theta)\cos^2\theta} - \frac{\mu_0^2\beta_{\perp 0}^2\sin^2\theta}{Y^2}\right)^2. \tag{22}$$

This result further simplifies in the following three limiting cases, each for $Y \approx 1$,

$$A^o(s=1,\omega,\theta) \approx \left(\frac{3(1-Y)}{8(1-X)} - \tfrac{1}{4}\beta_{\perp 0}^2\right)^2(2-X)^2\theta^4, \qquad \theta^2 \ll 1-X, \tag{23a}$$

$$\approx \frac{1}{2\cos^2\theta}\left(\frac{(1-X\cos^2\theta)(1-Y)}{\sin^2\theta} - (1-X)^2\beta_{\perp 0}^2\cos^2\theta\right)^2,$$
$$\sin^2\theta \gg 2(1-X)|\cos\theta|, \tag{23b}$$

$$\approx \frac{\cos^2\theta\,\sin^4\theta}{2(1+\cos^2\theta)}\left(\frac{(1+2\cos^2\theta)(1-Y)}{\cos^2\theta(1+\cos^2\theta)} - \beta_{\perp 0}^2\right)^2, \qquad X \ll 1. \tag{23c}$$

These three cases correspond respectively in equation (22) to

$$T_{\mathrm{o}} = -\frac{\cos\theta}{|\cos\theta|}\left(1+\frac{\theta^2}{2(1-X)}\right), \quad T_{\mathrm{o}} = -\frac{\sin^2\theta}{(1-X)\cos\theta}, \quad T_{\mathrm{o}} = -\frac{1}{\cos\theta}. \tag{24}$$

III. Maximum Growth Rates

For semiquantitative purposes, the detailed expressions for the growth rates (or, equivalently, minus the absorption coefficients) derived in Section II can be approximated by the maximum growth rate together with the bandwidth of the growing waves (i.e. the half-width in frequency about the maximum) and an estimation of the angular dependence. In this section the maximum growth rates and bandwidths are estimated and it is argued that the dependence on θ differs little from the well-known case of gyromagnetic absorption in a thermal plasma.

As indicated in the Introduction, the restriction to waves which can escape implies that only $s > 0$ need be considered. (It should be commented that the absence of particle–wave interactions at $s \leqslant 0$ for $\mu_\sigma \leqslant 1$ implies that $\varepsilon^\sigma(s,\omega,\theta)$ and $\gamma^\sigma(s,\omega,\theta)$ should be zero for $s \leqslant 0$, $\mu_\sigma \leqslant 1$. The finiteness of expressions (15) and (16) for $s \leqslant 0$, $\mu_\sigma \leqslant 1$ results entirely from the inclusion of unphysical particles with $\beta > 1$ in the nonrelativistic distribution (1).)

The frequency range in which $\gamma^\sigma(s,\omega,\theta)$ is negative (and hence the growth rate positive) follows directly from the fact that the sign of $\gamma^\sigma(s,\omega,\theta)$ is determined by the sign of $g^\sigma(s,\omega,\theta)$, as may be seen from expressions (16) and (17). If the "centre of the line" for emission or absorption at the sth harmonic is defined by the zero of the exponent in expression (15), i.e. by

$$\omega = s\Omega_{\mathrm{e}}/(1-\mu_\sigma\beta_s\cos\theta) \approx s\Omega_{\mathrm{e}}(1+\mu_\sigma\beta_s\cos\theta), \tag{25}$$

one concludes from (17) that coherent emission can occur *only on the low frequency side* of the centre of the line. (This is necessarily the case only for $\mu_\sigma \leqslant 1$.)

(*a*) *Angular Dependences*

The variation of $\gamma^\sigma(s,\omega,\theta)$ with θ, as shown by equations (16) and (17), is only weakly dependent on the value of β_s ($|\beta_s| \ll 1$ necessarily) for $\mu_\sigma < 1$, while the θ dependence is independent of $\beta_{\perp 0}$ when the approximation (19) is made. Consequently, the θ dependence would be the same as that for a thermal plasma with $\beta_{\perp 0} = \beta_{\parallel 0} \equiv \beta_{\mathrm{th}}$ and $\beta_s = 0$. (Gyromagnetic absorption in a thermal plasma has been discussed by e.g. Zheleznyakov 1970, pp. 447–52.)

If $\gamma_0^\sigma(s,\omega,\theta)$ is the absorption coefficient in the thermal plasma then this is given by expression (16) with n_1 replaced by n_{e} and with $\beta_{\perp 0} = \beta_{\parallel 0} \equiv \beta_{\mathrm{th}}$ and $\beta_s = 0$. Granted that the θ dependence of the maximum value of $\gamma_0(s,\omega,\theta)$ is known, on defining the function $\xi_s^\sigma(\theta)$ by writing

$$\{\gamma_0(s,\omega,\theta)\}_{\max} = \xi_s^\sigma(\theta)\,(\omega_{\mathrm{p}}^2/s\Omega_{\mathrm{e}})\beta_{\mathrm{th}}^{2s-3}, \tag{26}$$

the above arguments imply that the same function $\xi_s^\sigma(\theta)$ should describe the θ dependence of the growth rates of interest. This argument applies only for $s > 1$ as equation (26) is not valid for $s = 1$.

The angular dependence strongly favours coherent emission *in the x-mode* over that in the o-mode, just as preferential absorption of the x-mode occurs in a thermal plasma (see e.g. Zhelezniakov 1970, p. 456). For $s = 1$ one may identify the function $\xi_1^x(\theta)$ with $A^x(s = 1, \omega, \theta)$ given by the relation (20) for the x-mode. For the o-mode at $s = 1$ the angular dependence is determined by the relations (23).

(*b*) *Coherent Emission at* $s \geqslant 2$

For $s \geqslant 2$ the absorption coefficient (16) can be written in the form

$$\gamma^\sigma(s \geqslant 2, \omega, \theta) = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{n_1}{n_e} \frac{\omega_p^2}{\omega} \frac{A^\sigma(s, \omega, \theta)}{\mu_\sigma \partial(\omega\mu_\sigma)/\partial\omega} \frac{\beta_{\perp 0}^2}{\beta_{\parallel 0}^2} f(u_s), \tag{27}$$

with

$$f(u) = \{u(1 - d\mu_\sigma \beta_{\parallel 0} |\cos\theta|) - b + d\} \exp\{-\tfrac{1}{2}(u-b)^2\}$$

$$\approx (u - b + d)\exp\{-\tfrac{1}{2}(u-b)^2\} \tag{28}$$

and

$$u_s = \frac{\omega - s\Omega_e}{\omega\mu_\sigma \beta_{\parallel 0} |\cos\theta|}, \qquad b = \frac{\beta_s \cos\theta}{\beta_{\parallel 0} |\cos\theta|}, \qquad d = \frac{\beta_{\parallel 0}}{\beta_{\perp 0}^2 \mu_\sigma |\cos\theta|}. \tag{29}$$

The maximum negative value of $\gamma^\sigma(s \geqslant 2, \omega, \theta)$ as a function of frequency occurs at the negative extremum of $f(u)$, namely,

$$u = b - \tfrac{1}{2}d - \tfrac{1}{2}(d^2 + 4)^{\frac{1}{2}}. \tag{30}$$

If this value of u were much less than unity, any coherent emission would be negligible. For the negative maximum not to be exponentially small, it is required that $d \lesssim 1$, a condition which reduces to

$$\beta_{\perp 0}^2 \mu_\sigma |\cos\theta| \gtrsim \beta_{\parallel 0} \tag{31}$$

irrespective of β_s. Condition (31) requires $\beta_{\perp 0}^2 \gtrsim \beta_{\parallel 0}$ and becomes all the more extreme for $\mu_\sigma \ll 1$, i.e. near the cutoff frequencies, or for $|\cos\theta| \ll 1$, i.e. for emission at large angles relative to the ambient magnetic field.

For $s \geqslant 2$ the maximum growth rate is of the form (for $\mu_\sigma \sim 1$, $\beta_{\perp 0}^2 \gtrsim \beta_{\parallel 0}$)

$$|\gamma^\sigma(s \geqslant 2, \omega, \theta)|_{\max} \approx \xi_s^\sigma \frac{n_1}{n_e} \frac{\omega_p^2}{s\Omega_e} \frac{\beta_{\perp 0}^{2s}}{\beta_{\parallel 0}^2}. \tag{32}$$

The maximum as a function of θ occurs at an intermediate angle $0 \ll \theta \ll \frac{1}{2}\pi$.

Because the condition (31) is so very extreme for $\mu_\sigma \ll 1$, coherent emission close to the cutoff frequencies is implausible. It is reasonable to suppose that the inequalities $\omega > \omega_p$ (o-mode) and $\omega > \omega_x$ (x-mode) are well satisfied. Supposing that any coherent emission at the sth harmonic occurs at frequencies satisfying

$$|\omega - s\Omega_e| \ll |\omega - \omega_p| \quad \text{or} \quad |\omega - \omega_x|,$$

one has the requirements

$$s\Omega_e > \omega_p \qquad \text{and} \qquad s\Omega_e > \omega_x \tag{33}$$

for coherent emission to be permitted for the o-mode and x-mode respectively.

The conditions (33) imply that there is a range of values of Ω_e/ω_p for which emission into the o-mode is allowed while emission into the x-mode is forbidden. This range of values is given by

$$\omega_p < s\Omega_e < \omega_x, \qquad \text{that is,} \qquad s(s-1)\Omega_e^2 < \omega_p^2 < s^2\Omega_e^2. \tag{34}$$

Even for $s = 2$ this is an implausibly restrictive range, in that it seems unlikely that coherent emission at any $s \geqslant 2$ would be restricted to the o-mode due to the ratio Ω_e/ω_p falling in the appropriate range.

Finally, the bandwidth of the growing waves can be estimated from the width of the negative peak in $\gamma^\sigma(s, \omega, \theta)$. To within a factor of order unity it follows from the first of equations (29) that the width is given roughly by $\Delta u_s \sim 1$, or

$$\Delta\omega_s \approx s\Omega_e \mu_\sigma \beta_{\parallel 0} |\cos\theta|. \tag{35a}$$

However, relativistic effects cannot necessarily be neglected in estimating the bandwidth for large s. In fact the relativistic modification to the gyrofrequency leads to an effective bandwidth of

$$\Delta\omega_s \approx s^{3/2}\Omega_e \beta_{\perp 0}^2 \tag{35b}$$

(see Bekefi 1966, p. 202). If the evaluation of expression (35b) exceeds that of (35a), the relativistic effects are significant and the effective bandwidth of the growing waves would then be determined by (35b). Of course, there would be an associated reduction of the maximum growth rate given by equation (32) due to this smearing out of the instability over a wider bandwidth.

(*c*) *Coherent Emission at $s = 1$ in x-mode*

Coherent emission at $s = 1$ in the x-mode has a maximum growth rate given by equation (32) with $\xi_1^x \approx 1$ whenever such emission is possible. The requirement that such coherent emission occur at $\omega > \omega_x$ is severe. For $\omega_p \ll \Omega_e$ one has

$$\omega_x \approx \Omega_e + \omega_p^2/\Omega_e. \tag{36}$$

For a large streaming velocity and for emission of waves in the forward streaming direction, i.e. for $\beta_s \cos\theta > 0$, the centre of the line as defined by equation (25) is above ω_x for

$$\mu_\sigma \beta_s \cos\theta > \omega_p^2/\Omega_e^2. \tag{37}$$

This is the condition that must be satisfied for coherent emission at $s = 1$ in the x-mode to be possible. The bandwidth of the growing waves is again given by one of the equations (35). In this case the maximum as a function of θ occurs very close to $|\cos\theta| = 1$.

(*d*) *Coherent Emission at* $s = 1$ *in o-mode*

Coherent emission at $s = 1$ in the o-mode would be possible for $\Omega_e > \omega_p$ under less restrictive conditions than those required for coherent emission at $s = 1$ in the x-mode. Ignoring the corrections of order $\beta_{\perp 0}^2$ in equation (22), the absorption coefficient at $s = 1$ in the o-mode would reduce to

$$\gamma^o(s = 1, \omega, \theta) = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{n_1}{n_e} \frac{\omega_p^2}{\omega} \frac{\beta_{\perp 0}^2 \mu_0 \cos^2\theta}{\partial(\omega\mu_0)/\partial\omega} G(\omega, \theta)\, h(u_1), \tag{38}$$

where $G(\omega, \theta)$ is the factor multiplying $(1-Y)^2$ in (22) and where, by comparison with equations (28) and (29),

$$h(u) = u^2(u-b+d)\exp\{-\tfrac{1}{2}(u-b)^2\}. \tag{39}$$

The neglected terms in (22) are important for $u^2 \lesssim d^{-2}G(\omega, \theta)$.

The function $h(u)$ has one negative maximum for $b < d$, which lies at $\omega > \Omega_e$, while for $b > d$ it has two negative maxima, one at $\omega < \Omega_e$ and the other at $\omega > \Omega_e$. Again the negative maxima are not exponentially small only for $d \lesssim 1$. With $u^2 \lesssim 1$ in the range of interest, the terms involving $\beta_{\perp 0}^2$ in equation (22) are of the same order as those retained in equation (39). The maximum negative value of $h(u)$ is of order unity and the maximum growth rate is given roughly by

$$|\gamma^o(s = 1, \omega, \theta)|_{\max} \approx \xi_1^o \frac{n_1}{n_e} \frac{\omega_p^2}{\Omega_e} \times \begin{cases} \beta_{\perp 0}^2 & \text{(40a)} \\ \beta_{\perp 0}^6/\beta_{\parallel 0}^2, & \text{(40b)} \end{cases}$$

where (40a) or (40b) apply according to whether the term involving $\beta_{\perp 0}^2$ in equation (22) is neglible or dominant respectively For $\beta_{\perp 0}^2 \approx \beta_{\parallel 0}$ the alternative expressions (40a) and (40b) are approximately equal. The factor ξ_1^o is of order unity here.

It should be noted that equations (40) imply that the growth rate for the o-mode at $s = 1$ is less than the growth rate for the o-mode at $s = 2$, which is in turn less than that for the x-mode at $s = 2$.

The bandwidth of the growing waves is again given roughly by one of equations (35). The function (39), as compared with (28), leads to a somewhat broader negative maximum at $\omega < \Omega_e$ than is predicted by equations (35), while for $b > d$, that is, for

$$\mu_\sigma \beta_s \cos\theta \gtrsim \beta_{\parallel 0}(\beta_{\perp 0}^2/\beta_{\parallel 0})^{-1}, \tag{41}$$

the secondary negative maximum at $\omega > \Omega_e$ has a somewhat narrower bandwidth.

IV. Limitations on Maser Mechanism

Coherent emission could be a significant effect only when the number of e-folding growth lengths (or times) is much greater than unity. This number of growth lengths, called the effective optical depth for negative absorption, also needs to be greater than the optical depth due to absorption in the overlying plasma if the radiation is to escape. Besides these limitations there are further ones associated with the saturation of the maser, particularly when other instabilities should occur under milder conditions than those required for the coherent emission of escaping radiation.

(*a*) *Optical Depth Effects*

As pointed out in Section II(*b*) it is reasonable to ignore the details of the ray properties and treat absorption per unit length by dividing γ^σ by c. Thus if L is the characteristic distance over which wave growth occurs one can define the effective optical depth $(\tau_s^-)^\sigma$ for negative absorption at the sth harmonic for waves in the mode σ by

$$(\tau_s^-)^\sigma = |\gamma^\sigma(s,\omega,\theta)|_{\max} L/c\,. \tag{42}$$

The length L is limited by the dimensions of the region containing the distribution (1), a linear dimension L_1 say, and also by $\Delta\omega_s/s\Omega_e$ times the characteristic length $L_B = B/(|\operatorname{grad} B|)$ over which Ω_e changes. Thus one has

$$L < \min[L_1, \beta_{\parallel 0} L_B]\,. \tag{43}$$

The occurrence of significant wave growth at the sth harmonic requires $(\tau_s^-)^\sigma \gg 1$. From equations (32) and (42) this implies

$$\frac{n_1}{n_e} \gg \frac{1}{\xi_s^\sigma}\frac{s\Omega_e\,\beta_{\parallel 0}^2\,c}{\omega_p^2\,\beta_{\perp 0}^{2s}\,L}, \tag{44}$$

except for the o-mode at $s = 1$ (when (40) is to be used in place of (32)). Thus this requirement places a lower limit on the number density of the nonthermal particles. The inequality (44) becomes more restrictive on n_1 with increasing s.

(*b*) *Gyromagnetic Absorption*

Gyromagnetic absorption in the thermal plasma occurs in layers where the frequency passes through $\omega = s\Omega_e$. (Such absorption at $s = 1$ is possible for the o-mode but requires a special treatment; the present author believes the well-known treatment of this case by Gershman (1960) to be unphysical.) The optical depth associated with absorption at the sth harmonic, τ_s^σ say, is given by

$$\frac{(\tau_s^-)^\sigma}{\tau_s^\sigma} \approx \frac{n_1}{n_e}\frac{\beta_{\perp 0}^{2s}}{\beta_{\parallel 0}^2\,\beta_{\mathrm{th}}^{2s-2}}\frac{L}{L_B}\,. \tag{45}$$

It is useful to distinguish between two cases and to discuss them separately:

(A) Radiation coherently emitted at the sth harmonic needs to pass through the layer with $\omega = s\Omega_e$ in order to escape to infinity.

(B) Radiation coherently emitted at the sth harmonic does not encounter the layer with $\omega = s\Omega_e$ in escaping to infinity.

In case (A) one requires $(\tau_s^-)^\sigma \gg \tau_s^\sigma$ in order for the radiation to escape. On setting $L = \beta_{\parallel 0} L_B$ in equation (45) this reduces to the condition

$$n_1/n_e \gg \beta_{\mathrm{th}}^{2s-2}\,\beta_{\parallel 0}/\beta_{\perp 0}^{2s} \approx (\beta_{\mathrm{th}}/\beta_{\parallel 0})^{s-1}, \tag{46}$$

where $\beta_{\perp 0}^2 \approx \beta_{\parallel 0}$ is assumed. This condition applies only to $s \geq 2$. It should be noted that because the ratio (45) is the same for both modes, preferential gyromagnetic absorption of waves in the x-mode cannot overcome preferential emission and thereby allow the preferential escape of waves in the o-mode.

Special conditions are required for case (B) to obtain. Because coherent emission occurs on the low frequency side of the line defined by equation (25), one has $\omega < s\Omega_e$ at the point of emission except when the centre of the line is Doppler shifted to well above $s\Omega_e$. From equation (25) it is seen that this requires $\beta_s \cos\theta > 0$, with the magnitude of the Doppler shift such that one has $b \gtrsim 1$ (see the second of equations (29)), that is, $|\beta_s| \gtrsim \beta_{\parallel 0}$. For waves generated at a frequency slightly in excess of $s\Omega_e$ not to encounter the layer with $\omega = s\Omega_e$ requires either that the waves be propagating outwards, i.e. in the direction of decreasing Ω_e, or be propagating inwards but be reflected before reaching this layer. Except for $s = 1$ in the x-mode such a reflection requires an implausibly restrictive range of the ratio Ω_e/ω_p. Consequently, except for $s = 1$ in the x-mode, case (B) requires $|\beta_s| \gtrsim \beta_{\parallel 0}$, $\beta_s > 0$, and $\cos\theta > 0$ (an outward propagating stream emitting in the forward streaming direction). An immediate implication is that in case (B) the observed frequency drift is from high to low frequencies.

Case (A) is forbidden for $s = 1$ in the x-mode, but case (B) applies either with $\beta_s > 0$, $\cos\theta > 0$ or with $\beta_s < 0$, $\cos\theta < 0$ because inward propagating waves necessarily reach the layer $\omega = \omega_x$, where they are reflected, before reaching the layer $\omega = \Omega_e$.

(*c*) *Saturation of Maser*

The back-reaction on the distribution of particles resulting from coherent emission must be such as to reduce the deviation from thermal equilibrium that causes the coherent emission. In the present case an excess of perpendicular momentum causes the coherent emission and so the back-reaction should effectively reduce $\beta_{\perp 0}^2$. When the intensity of the waves reaches a level such that $\beta_{\perp 0}^2$ decreases on a time scale comparable to the growth time of the waves, the maser saturates. This places a limit on the intensity of the escaping radiation.

The back-reaction can be treated using the quasilinear equations. Although the functional form of $f(p, \alpha)$ changes it is reasonable to take moments of equation (5) to find the way $\beta_{\perp 0}^2$, $\beta_{\parallel 0}^2$, β_s, etc. change. To lowest order in $\beta_{\perp 0}$, $\beta_{\parallel 0}$, and β_s only, $\beta_{\perp 0}^2$ changes as

$$\frac{\mathrm{d}\beta_{\perp 0}^2}{\mathrm{d}t} = 2\pi \int_{-1}^{1} \mathrm{d}(\cos\alpha) \int_0^\infty \mathrm{d}p\, p^2 \beta^2 \sin^2\alpha \frac{\partial f(p, \alpha)}{\partial t}$$
$$\approx -|\gamma^\sigma|_{\max} W^\sigma / n_1 mc^2, \tag{47}$$

where the distribution (1) is inserted in equation (10) and so in (5), where only the contribution from the growing waves is retained, and W^σ is the total energy density in the waves (i.e. integrated over bandwidth and solid angle).

The maser saturates for

$$\frac{\mathrm{d}\beta_{\perp 0}^2}{\mathrm{d}t} \sim -|\gamma^\sigma|_{\max} \beta_{\perp 0}^2, \tag{48}$$

i.e. for

$$W^\sigma \approx n_1 \beta_{\perp 0}^2 mc^2. \tag{49}$$

Thus a significant fraction of the energy in the electrons can be transferred to waves as a result of the coherent emission, a result that may be compared with that of Shapiro (1963) in a related context.

Alternatively, a steady state could result from a balance between the rate of decrease in $\beta_{\perp 0}^2$, given by equation (47), and the rate of increase of $\beta_{\perp 0}^2$ due to a stream propagating in the direction of increasing magnetic field strength, given by

$$\mathrm{d}\beta_{\perp 0}^2/\mathrm{d}t \approx (\beta_s c/L_B)\beta_{\perp 0}^2 . \tag{50}$$

Balancing equations (47) and (50) gives a steady-state integrated (over bandwidth and solid angle) intensity of escaping radiation of

$$cW^\sigma \approx (n_1 \beta_{\perp 0}^2 mc^2)\beta_s c^2/L_B |\gamma^\sigma|_{\max} \quad \mathrm{erg\,cm^{-2}\,s^{-1}} . \tag{51}$$

This method of achieving an excess in perpendicular momentum was the one used by Hirshfield and Wachtel (1964) in obtaining coherent gyromagnetic emission in the laboratory.

(*d*) *Competing Instabilities*

So far only the instabilities which give rise to escaping radiation have been considered. For $\beta_s \gtrsim \beta_{\parallel 0}$ $(>\beta_{\mathrm{th}})$, which is required for case (B) of Section IV(*b*), coherent emission of electron plasma waves should occur. This would reduce β_s and so destroy the condition required for case (B) to apply. More generally, the instabilities of interest require $\beta_{\perp 0}^2 \gtrsim \beta_{\parallel 0}$, while similar instabilities involving waves which cannot escape, e.g. whistlers, should occur under milder conditions so preventing the extreme anisotropy $\beta_{\perp 0}^2 \gtrsim \beta_{\parallel 0}$ from being set up.

The coherent emission of electron plasma waves for $\beta_s > \beta_{\parallel 0}$ (the so-called two-stream instability) is considered first. Retaining only the Cerenkov term, i.e. for $s = 0$, the growth is independent of $\beta_{\perp 0}$ with (see e.g. Tsytovich and Kaplan 1968)

$$|\gamma|_{\max} \approx \frac{n_1}{n_\mathrm{e}}\omega_\mathrm{p}\left(\frac{\beta_s}{\beta_{\parallel 0}}\right)^2 \times \left\{\begin{matrix} 1 & (\Omega_\mathrm{e} < \omega_\mathrm{p}) \\ (\omega_\mathrm{p}/\Omega_\mathrm{e})^3 & (\Omega_\mathrm{e} > \omega_\mathrm{p}) . \end{matrix}\right\} \tag{52}$$

Comparing this with the growth rate (32) for $\beta_{\perp 0}^2 \approx \beta_{\parallel 0}$, $\beta_s \approx \beta_{\parallel 0}$, one finds that the growth rate (52) is the greater for $s \geqslant 3$ and for $s = 1$ in the o-mode, while the two rates are comparable for $s = 2$. This implies that case (B) of Section IV(*b*) may well prove untenable due to the competition from the two-stream instability tending to reduce β_s. For coherent emission at $s = 1$ in the x-mode no such objection applies.

Another instability, which should occur when there is an excess of perpendicular momentum, is coherent emission of waves in the whistler mode. Assuming $\omega \ll (\Omega_\mathrm{e}|\cos\theta|$ or $\omega_\mathrm{p})$ the properties of waves in the whistler mode are given by

$$\mu^2 = \frac{\omega_\mathrm{p}^2}{\omega\Omega_\mathrm{e}|\cos\theta|}, \quad T = \frac{\cos\theta}{|\cos\theta|}, \quad K = \frac{\sin\theta}{|\cos\theta|} . \tag{53}$$

On inserting these wave properties in equation (16) for $s = 1$, coherent emission, which occurs when condition (31) is satisfied, is found to have the maximum growth rate

$$|\gamma|_{\max} \approx (n_1/n_\mathrm{e})\Omega_\mathrm{e}\beta_{\perp 0}^2/\beta_{\parallel 0}^2 \tag{54}$$

occurring for

$$\omega \approx \Omega_\mathrm{e}^3/\omega_\mathrm{p}^2\beta_{\parallel 0}^2 . \tag{55}$$

For $\omega_p \sim \Omega_e$, the frequency (55) does not satisfy the assumed condition $\omega < \Omega_e$ and the relevant growth rates are somewhat smaller than (54). Nevertheless it is apparent that for $s \geqslant 2$ and for $s = 1$ in the o-mode, the growth rate for waves in the whistler mode is comparable with or greater than the growth rates of interest. Because the refractive index is greater than unity for the whistlers, coherent emission of whistlers occurs for a milder anisotropy than $\beta_{\perp 0}^2 \gtrsim \beta_{\parallel 0}$.

It may be concluded that, for all cases other than coherent emission at $s = 1$ in the x-mode, competing instabilities cannot be ignored. The competing instabilities should prevent the conditions required for coherent emission from being set up.

V. Reactive-medium Instabilities

A further limitation arises from the conditions for the quasilinear treatment to apply, namely that the growth rate is less than the bandwidth of the growing waves, which is required for the random phase approximation to be valid. In this section the nature and relevance of the instabilities when the random phase approximation breaks down are discussed. Many authors have implicitly assumed that the growth rate exceeds the bandwidth by adopting a δ function distribution implying zero bandwidth. It is argued that this case is unlikely to be relevant in astrophysical contexts.

(a) *Random Phase Approximation*

According to the above prescription for the validity of the random phase approximation, the results derived above are valid only if the condition

$$n_1/n_e \lesssim (s\Omega_e/\omega_p)^2 \beta_{\parallel 0}^3/\beta_{\perp 0}^{2s} \tag{56}$$

is satisfied. This follows from equations (32) and (35). For $s = 1$ in the o-mode a slightly different condition applies. The condition (56) with $\beta_{\perp 0}^2 \gtrsim \beta_{\parallel 0}$ reduces to

$$n_1/n_e \lesssim (s\Omega_e/\omega_p)^2 \beta_{\parallel 0}^{3-s}. \tag{57}$$

No relevant restriction is implied by either equation (56) or (57) for $s \geqslant 3$ or for $s = 1$ in the o-mode. For $s = 2$ with $\omega_p \sim 2\Omega_e$, equation (57) gives $n_1 \lesssim n_e \beta_{\parallel 0}$. For $s = 1$ in the x-mode, equation (57) gives $n_1 \lesssim n_e (\Omega_e/\omega_p)^2 \beta_{\parallel 0}^2$ but because the inequality (37) implies $|\beta_s| > \omega_p^2/\Omega_e^2$ (and also $|\beta_s| \gtrsim \beta_{\parallel 0}$) this condition is no more severe than for $s = 2$. Thus provided that $\beta_{\parallel 0}$ is not too small one is justified in using the quasilinear equations.

However, many authors in this and related contexts have treated the instability for a distribution function proportional to $\delta(\beta_\parallel - \beta_s)$ (see e.g. Bell and Buneman 1964; Fung 1966; Fung and Yip 1966; Goldreich and Lynden-Bell 1969, equation (22); Yip 1970; Zhelezniakov 1970, p. 492 *et seq.*). The instability in this case is a "reactive-medium" one in the terminology of Briggs (1964) or a "hydrodynamic" one in the terminology used by Soviet authors such as Shapiro (1963).

(b) *Reactive-medium Stage*

If it is supposed that, initially, a stream did have a negligible velocity spread $\beta_{\parallel 0} \approx 0$ so that a reactive-medium instability rather than a quasilinear one developed then according to arguments given by Tsytovich (1970, pp. 183–93) one would expect an initial reactive-medium stage to have as its principal effect a broadening of the velocity spread $\beta_{\parallel 0}$ so that this initial stage would suppress itself. Tsytovich showed for a special case that the reactive-medium stage passes over continuously into the quasilinear stage. Similar conclusions are implicit in the work of Shapiro (1963) and of Singhaus (1964).

Although it is convenient to think of the quasilinear stage as a maser action involving negative absorption and the reactive-medium stage as resulting from an intrinsically growing disturbance with a negative feedback, the work of Singhaus, in particular, indicates that these two stages must be opposite limiting cases of a single instability. This is certainly not indicated by the results of Bell and Buneman (1964) and the other authors cited in subsection (*a*) above. The treatment given below of the reactive-medium instabilities for the distribution (1) does indicate that these are continuations of the quasilinear ones. Granted this, the rapid evolution from the reactive-medium to quasilinear stages implies that any initial reactive-medium stage could be of little final consequence.

(*c*) *Reactive-medium Instabilities*

Let $\varepsilon_{ij}^{(1)}(\boldsymbol{k},\omega)$ be the contribution to the total dielectric tensor from the distribution (1), with the vacuum contribution included only in the part $\varepsilon_{ij}^{(0)}(\boldsymbol{k},\omega)$ due to the background medium. Following the approach summarized in Appendix I of Melrose and Sy (1972*a*) the dispersion relations are solutions of

$$\Lambda(\boldsymbol{k},\omega) = 0 \tag{58}$$

with

$$\Lambda(\boldsymbol{k},\omega) = \Lambda^{(0)}(\boldsymbol{k},\omega) + \lambda_{ij}^{(0)}(\boldsymbol{k},\omega)\,\varepsilon_{ji}^{(1)}(\boldsymbol{k},\omega) + \dots + |\,\varepsilon_{ij}^{(1)}(\boldsymbol{k},\omega)\,|\,. \tag{59}$$

Reactive-medium instabilities appear when two real modes of the combined system become a double solution, say the modes σ and σ' at some $\boldsymbol{k}_0$ satisfy

$$\omega^{\sigma}(\boldsymbol{k}_0) = \omega^{\sigma'}(\boldsymbol{k}_0),$$

and then become a complex conjugate pair of solutions. At the double solution one must have (compare with Melrose and Sy 1972*a*, equation (A5))

$$\left[\frac{W_{\mathrm{T}}(\boldsymbol{k}_0)}{W_E(\boldsymbol{k}_0)}\right]^{\sigma} = \frac{[\omega\,\partial\Lambda(\boldsymbol{k},\omega)/\partial\omega]_{\omega=\omega^{\sigma}(\boldsymbol{k}_0)}}{\lambda_{ss}(\boldsymbol{k},\omega^{\sigma}(\boldsymbol{k}_0))} = 0\,, \tag{60}$$

i.e. the total energy passes through zero and becomes negative ($W_{\mathrm{T}}^{\sigma} < 0$, with $W_E^{\sigma} > 0$ by definition). Both complex ω and negative energy indicate the appearance of an intrinsically growing wave.

If the effect of the stream can be treated as a perturbation on the wave properties, $\omega^{\sigma}(\boldsymbol{k}) \to \omega^{\sigma}(\boldsymbol{k}) + \Delta\omega^{\sigma}(\boldsymbol{k})$ say, one has

$$\mathrm{Re}[\Delta\omega^{\sigma}(\boldsymbol{k})] = -[W_E/W_{\mathrm{T}}]^{\sigma}\omega^{\sigma}e_i^{\sigma *}\,e_j^{\sigma}[\varepsilon_{ij}^{(1)}(\boldsymbol{k},\omega^{\sigma})]^{\mathrm{h}}\,, \tag{61}$$

$$\mathrm{Im}[\Delta\omega^{\sigma}(\boldsymbol{k})] = \mathrm{i}[W_E/W_{\mathrm{T}}]^{\sigma}\omega^{\sigma}e_i^{\sigma *}\,e_j^{\sigma}[\varepsilon_{ij}^{(1)}(\boldsymbol{k},\omega^{\sigma})]^{\mathrm{a}}\,, \tag{62}$$

where h and a refer to the hermitian and antihermitian parts respectively. The absorption coefficient (16) is just minus twice the imaginary part of $\Delta\omega^{\sigma}$ as given by equation (62).

If only the contribution from a given harmonic is retained in equations (61) and (62), the hermitian and antihermitian parts in equations (61) and (62) result from the real and imaginary parts respectively of a function $F(\alpha_s)$ (F is $\mathrm{i}F_0$ in the notation of Stix 1962, p. 179) with

$$\alpha_s = \{\omega(1-\mu\beta_s\cos\theta) - s\Omega_e\}/\sqrt{2}\,\omega\mu\beta_{\|0}\,|\cos\theta\,|\,, \tag{63}$$

$$\mathrm{Im}[F(\alpha_s)] = -\mathrm{i}\sqrt{\pi}\,\{\cos\theta/(|\cos\theta\,|)\}\exp(-\alpha_s^2)\,, \tag{64}$$

and, for $|\,\alpha_s\,| \gg 1$,

$$\mathrm{Re}[F(\alpha_s)] \approx \frac{1}{\alpha_s}\left(1+\frac{1}{2\alpha_s^2}+\ldots\right). \tag{65}$$

The reactive-medium instabilities appear when the term α_s^{-1} in (65) is the dominant term. In this case one solves equation (58) with (59) by first multiplying by α_s and then solving

$$\alpha_s - \alpha_s^{(0)}\alpha_s + [\alpha_s^{(1)}]^2 = 0\,, \tag{66}$$

where

$$\alpha_s^{(0)} = \{\omega(1-\mu_\sigma\beta_s\cos\theta) - s\Omega_e\}/\sqrt{2}\,\omega\mu_\sigma\beta_{\|0}\,|\cos\theta\,| \tag{67}$$

is the unperturbed value for waves in the mode σ and

$$(\alpha_s^{(1)})^2 = -\frac{n_1}{n_e}\frac{\omega_p^2}{\omega^2}\frac{g^\sigma(s,\omega,\theta)\,A^\sigma(s,\omega,\theta)}{\mu_\sigma^2\beta_{\|0}^2\cos^2\theta\,\mu_\sigma\,\partial(\omega\mu_\sigma)/\partial\omega} \tag{68}$$

results from the remaining terms as in equation (16).

It is apparent that equation (66) leads to complex solutions only for $[\alpha_s^{(1)}]^2 > 0$, that is, for $g^\sigma(s,\omega,\theta) < 0$. This is just the condition found for the quasilinear instabilities to occur. For $\alpha_s^{(0)} \approx 0$, the complex solutions of equation (66) lead to an imaginary part of the frequencies of the order of

$$|\,\mathrm{Im}[\omega_s^{(1)}]\,| \approx \left(\frac{n_1}{n_e}\frac{\omega_p^2\,|\,g^\sigma(s,\omega,\theta)\,|\,A^\sigma(s,\omega,\theta)}{\mu_\sigma\,\partial(\omega\mu_\sigma)/\partial\omega}\right)^{\frac{1}{2}}. \tag{69}$$

Comparing this with the growth rate $|\,\gamma^\sigma(s,\omega,\theta)\,|_{\max}$ of the quasilinear instabilities one finds the approximate algebraic identity

$$|\,\mathrm{Im}[\omega_s^{(1)}]\,|^2 \approx |\,\gamma^\sigma(s,\omega,\theta)\,|_{\max}\,\omega\mu_\sigma\beta_{\|0}\,|\cos\theta\,|\,. \tag{70}$$

Now equation (69) could only be valid for $|\,\alpha_s\,| > 1$, that is, for

$$|\,\mathrm{Im}[\omega_s^{(1)}]\,| > \omega\mu_\sigma\beta_{\|0}\,|\cos\theta\,|\,.$$

On the other hand the quasilinear treatment presupposes that the random phase approximation is valid, e.g. that one has $|\,\gamma^\sigma(s,\omega,\theta)\,|_{\max} < \omega\mu_\sigma\beta_{\|0}\,|\cos\theta\,|$. At the respective limits of validity, equation (70) indicates that the two instabilities pass over continuously into each other.

The above discussion is heuristic in that a reactive-medium instability which is a continuation of the quasilinear one is sought and found. Nevertheless it is reasonable to conclude that reactive-medium instabilities are in one to one correspondence with quasilinear ones. Contrary to the results of those authors cited above, who chose

δ function distributions, the qualitative properties of the reactive-medium instabilities are the same as those of the corresponding quasilinear instabilities. Even if any initial reactive-medium stage were to occur, this could not affect the qualitative properties of the resulting radiation.

VI. Conclusions and Discussion

The supposition that coherent gyromagnetic emission by nonrelativistic electrons can act as a radiation mechanism in astrophysical contexts has been shown here to be subject to some severe constraints. Assuming that any coherent emission results from an anisotropy, i.e. from $\partial f/\partial \alpha \neq 0$ rather than from $\partial f/\partial p > 0$ as considered by Bekefi *et al.* (1961), for example, it was argued that the actual form of the distribution function chosen should not affect the general properties of the coherent emission and that one should be justified in using the quasilinear equations in the treatment. For the choice of the bi-Maxwellian streaming distribution (1) these properties emerge:

(1) Coherent emission (of waves which can escape) can occur only for $\beta_{\perp 0}^2 \gtrsim \beta_{\parallel 0}$ and, for even more extreme anisotropies, for $\mu_\sigma \ll 1$ or $|\cos\theta| \ll 1$ (see equation (31)).

(2) Coherent emission in the x-mode is favoured over coherent emission in the o-mode at the same harmonic s. For $s \geqslant 2$ the growth rate decreases with increasing s as $\beta_{\perp 0}^{2s}$ (see equation (32)).

(3) Coherent emission at $s = 1$ in the o-mode has a growth comparable with that at $s = 3$ in the o-mode, i.e. less than that at $s = 2$ in the o-mode and so also less than that at $s = 2$ in the x-mode.

(4) Coherent emission at $s = 1$ in the x-mode is possible only for a rapidly streaming distribution, i.e. for $|\beta_s| \gtrsim \beta_{\parallel 0}$, in a very low density plasma with $\omega_p \ll \Omega_e$ (see equation (37)).

These properties imply that (except under the implausibily restrictive condition (34)) this mechanism should produce radiation polarized in the sense of the x-mode. Furthermore the emission should be at the second harmonic, that is, $\omega \approx 2\Omega_e$, except when the condition (37) is satisfied and then the emission should be at the fundamental, that is, $\omega \approx \Omega_e \,(\gg \omega_p)$.

The discussion in Section IV of the limitations placed on the escape of the radiation due to absorption in the thermal plasma and those implied by the existence of competing instabilities shows that the present mechanism should not be considered for $s \geqslant 3$. For $s = 2$ these limitations cannot be ignored in any detailed theory. Only for $s = 1$ in the x-mode are the limitations irrelevant.

The conclusion that coherent emission at $s = 1$ in the x-mode should occur for $\beta_{\perp 0}^2 \gtrsim \beta_{\parallel 0}$, $|\beta_s| \gtrsim \beta_{\parallel 0}$, $|\beta_s| \gtrsim \omega_p^2/\Omega_e^2$ appears to be consistent with the experimental results of Hirshfield and Wachtel (1964). The coherent emission at $s = 2$ described by Blanken and Kuckes (1969) (see also Blanken *et al.* 1969) apparently involves waves in the Bernstein modes, which could not escape directly from a source of astrophysical interest. Waves in the Bernstein modes may be of interest in connection with plasma emission processes in sources with strong magnetic fields but their possible role has yet to be examined in detail (see, however, Kennel *et al.* 1970).

The author intends to discuss in another paper the implications of the present results in relation to the suggested astrophysical applications that were listed in the Introduction. However, it is immediately clear that the suggested applications to type I and type III solar bursts should be questioned if only because the observed emission is in the o-mode.

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APPENDIX

The purpose of this Appendix is to demonstrate that the condition $\beta_{\perp 0}^2 \gtrsim \beta_{\parallel 0}$ has close counterparts for coherent emission of waves with $\mu_\sigma \leqslant 1$ by anisotropic distributions of nonrelativistic electrons other than the bi-Maxwellian streaming distribution (1). For simplicity we consider emission at $s = 1$, $\cos\theta = 1$ *in vacuo* by the distribution (see e.g. Ramaty 1969)

$$f(p,\alpha) \propto \left\{ \begin{array}{ll} \beta^{-r}\sin^m\alpha & (\beta > \beta_0) \\ 0 & (\beta < \beta_0), \end{array} \right\} \tag{A1}$$

with $\beta_0 \leqslant 1$, $r \geqslant 1$, and $m \geqslant 1$. On calculating the absorption coefficient (8) after expanding the Bessel functions in equation (2) and retaining only the lowest order terms, we obtain

$$\gamma(s=1,\omega,\theta) \propto \int_{\beta_0}^{\infty} \mathrm{d}\beta\, \beta^{1-r}\sin^m\alpha\{(r+m)\sin^2\alpha - m\,\Omega_{\mathrm{e}}/\omega\}, \tag{A2}$$

with the condition

$$\omega(1-\beta\cos\alpha) = \Omega_{\mathrm{e}} \tag{A3}$$

left understood. For

$$\{1-\Omega_{\mathrm{e}}/\omega\}^2 \leqslant 2m^{-1}\beta_0^2, \tag{A4}$$

$\sin^m\alpha$ can be expanded in powers of $\cos^2\alpha$ in equation (A2) whence, on inserting the condition (A3), it follows that γ is negative for

$$\Omega_{\mathrm{e}}/\omega - 1 > r/m. \tag{A5}$$

The maximum growth rate occurs close to the frequency given by the equality in (A4), that is, at

$$\Omega_{\mathrm{e}}/\omega - 1 \approx (2/m)^{\frac{1}{2}}\beta_0 \gtrsim r/m. \tag{A6}$$

The counterpart to $\beta_{\perp 0}^2 \gtrsim \beta_{\parallel 0}$ for the distribution (A1) is

$$\langle\beta^2\sin^2\alpha\rangle > (2\langle\beta^2\cos^2\alpha\rangle)^{\frac{1}{2}}, \tag{A7}$$

which reduces to

$$\beta_0^2\frac{r-3}{m+3} > \left(\beta_0^2\frac{r-3}{r-5}\frac{2}{m+3}\right)^{\frac{1}{2}}. \tag{A8}$$

On comparing the relation (A6) in the form

$$m\beta_0^2 \gtrsim \tfrac{1}{2}r^2$$

with (A8) for $r \geqslant 5$, $m \geqslant 3$, that is, with

$$m\beta_0^2 \gtrsim 2,$$

it is apparent that there is a close correspondence between them.

It is not difficult to generalize the above arguments to any distribution with $\langle\beta^2\rangle \leqslant 1$, $\langle\sin^2\alpha\rangle \geqslant \langle\cos^2\alpha\rangle$ to conclude that coherent emission *in vacuo* is a significant effect only for $\langle\beta^2\rangle \gtrsim \langle\cos^2\alpha\rangle$ and that the maximum growth rate occurs at

$$(\Omega_{\mathrm{e}}/\omega) - 1 \approx \langle\beta^2\cos^2\alpha\rangle^{\frac{1}{2}}.$$