"PARALLEL" VISCOUS MODIFICATION OF THE RESISTIVE "TEARING" INSTABILITY IN A CYLINDRICAL MODEL OF THE HARD–CORE PINCH

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Abstract

The influence of "parallel" viscosity on the resistive "tearing" mode in a cylindrical model of a plasma is investigated for a hard-core pinch. Iterative solutions of the basic equation indicate that the destabilization suggested by a previous cartesian model is not substantiated, and in fact under extreme shear there is slight stabilization. It is clear that geometrical effects must be properly included in order to obtain an accurate description of the role of parallel viscosity.

There has been widespread investigation into plasma behaviour in pinch-type devices where the simple geometries make for relatively straightforward theoretical analyses. Interest is still maintained in suppressing the resistive instabilities (Furth *et al.* 1963), which are not stabilized by strong magnetic field shear. In this note, the influence of "parallel" viscosity (Stringer 1970) on the resistive "tearing" mode in a cylindrical geometry is considered for a hard-core pinch. The tearing mode is the long wavelength instability which is particularly noticeable in hard-core pinch experiments (Aitken *et al.* 1964). One is interested in the mode number $m = 1$ since it is well known that this corresponds to the fastest growing mode. It is to be noted that the $m = 0$ mode is precluded as its wavelength would be much greater than laboratory dimensions (Aitken *et al.* 1965), and that higher m modes may be important in larger diameter vessels (Furth *et al.* 1972). Growth rates have been computed previously for a cartesian model (Marinoff 1971) and for an inviscid cylindrical model (Hosking 1967).

The equilibrium configuration adopted here is exactly as described by Hosking (1967), with a magnetic field

$$\boldsymbol{H}_0 = H_{0\theta}(r)\,\boldsymbol{e}_\theta + H_{0z}\,\boldsymbol{e}_z\,,$$

where

$$H_{0\theta}(r) = Ar + C/r \qquad \text{and} \qquad H_{0z} = \text{const.}$$

The notation and parameters used are essentially those of Hosking, but a range of resistivity values appropriate to a temperature range from 10^4 to 10^8 K has been included, rather than the single value $\eta = 50\ \mathrm{m^2\,s^{-1}}$. As before, gravity and resistivity gradients are neglected and hydromagnetic stability is assumed.

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For perturbation quantities of the form

$$f_1(r,\theta,z,t) = f_1(r)\exp(\omega t + im\theta + ikz),$$

and adopting the Lagrangian displacement vector

$$\boldsymbol{\xi}(r) \equiv \boldsymbol{v}_1(r)/\omega,$$

the linearized perturbation equations, including parallel viscosity, for an incompressible plasma are

$$\rho\omega^2\xi_r + D\pi_1 = \mu\{iFH_{1r} - 2(H_{0\theta}/r)H_{1\theta}\} - (3\rho\nu f_2^2/r)s, \tag{1a}$$

$$\rho\omega^2\xi_\theta + (im/r)\pi_1 = \mu\{iFH_{1\theta} + (D^*H_{0\theta})H_{1r}\} + 3\rho\nu f_2(iF/H_0)s, \tag{1b}$$

$$\rho\omega^2\xi_z + ik\pi_1 = \mu\{iFH_{1z} + (DH_{0z})H_{1r}\} + 3\rho\nu f_3(iF/H_0)s; \tag{1c}$$

$$H_{1r} = iF\xi_r + (\eta/\omega)\{(DD^* - K^2)H_{1r} - (2im/r^2)H_{1\theta}\}, \tag{2a}$$

$$H_{1\theta} = iF\xi_\theta - (DH_{0\theta} - H_{0\theta}/r)\xi_r + (\eta/\omega)\{(DD^* - K^2)H_{1\theta} + (2im/r^2)H_{1r}\}, \tag{2b}$$

$$H_{1z} = iF\xi_z - (DH_{0z})\xi_r + (\eta/\omega)(DD^* - K^2)H_{1z}; \tag{2c}$$

$$D^*\xi_r + (im/r)\,\xi_\theta + ik\xi_z = 0, \tag{3}$$

$$D^*H_{1r} + (im/r)\,H_{1\theta} + ikH_{1z} = 0; \tag{4}$$

where

$$s = \omega\left(\frac{iF}{H_0}\frac{\boldsymbol{\xi}\cdot\boldsymbol{H}_0}{H_0} + \frac{f_2^2\xi_r}{r}\right), \qquad K = (m^2/r^2 + k^2)^{\frac{1}{2}},$$

$$F = \boldsymbol{K}\cdot\boldsymbol{H}_0 = (m/r)H_{0\theta} + kH_{0z}, \qquad \pi_1 = p_1 + \mu\boldsymbol{H}_0\cdot\boldsymbol{H}_1 + \rho_0\nu s,$$

$$f_2 = H_{0\theta}/H_0, \qquad f_3 = H_{0z}/H_0,$$

$$D \equiv d/dr, \qquad D^* \equiv D + r^{-1}.$$

The above equations apply in the whole plasma region, but the relations (2) may be replaced in the "outer" region by

$$H_{1r} \simeq iF\xi_r, \tag{2a$'$}$$

$$H_{1\theta} \simeq iF\xi_\theta - (DH_{0\theta} - H_{0\theta}/r)\xi_r, \tag{2b$'$}$$

$$H_{1z} \simeq iF\xi_z - (DH_{0z})\xi_r, \tag{2c$'$}$$

which are asymptotically valid as $S = \tau_R/\tau_H \to \infty$ except in the vicinity of $F = 0$, that is, in the "inner" or resistive region. The system of equations in the outer region may be replaced by the equivalent differential equation

$$\mathrm{D}\left(\frac{R(1+N-G)}{K^2}\mathrm{D}^*\xi_r\right)-\left\{R-\frac{Q^2k^2m^2I}{RK^2r^2G}+\left(D-\frac{1}{r}\right)\frac{Qm^2(1+I-G)}{K^2r^2}\right.$$

$$\left.-\frac{2\mu H_{0\theta}}{r}\left(\mathrm{D}H_{0\theta}-\frac{H_{0\theta}}{r}\right)+\frac{m^2H_{0\theta}^2Q(I-G)}{r^3F^2}\right\}\xi_r=0, \quad (5)$$

where

$$R=\rho\omega^2+\mu F^2, \qquad Q=2\mu FH_{0\theta}/m,$$

$$G(r)=\frac{1}{1-K^2H_{0z}/kF}, \qquad N(r)=\frac{G(1+rf_2J/m)}{1-(f_3-krf_2/m)kJ/K^2},$$

$$J(r)=\frac{3\rho\nu\omega}{RG}\left(\frac{F}{H_0}\right)^3, \qquad I(r)=\frac{G(1+rf_2^2H_0JR/m^2QF)}{1-(f_3-krf_2/m)kJ/K^2}.$$

In the inviscid limit $\nu\to 0$, equation (5) reduces to equation (11) of Hosking (1967) for $\rho\omega^2 \ll \mu F^2$.

From the approximation (2a′), equation (5) may be re-expressed in the form

$$\mathrm{D}^2H_{1r}+f(r)\mathrm{D}H_{1r}+g(r)H_{1r}=0, \quad (6)$$

where

$$f(r)=\frac{1}{r}-\frac{2\mathrm{D}F}{F}+\frac{\mathrm{D}(RM)}{RM}+\frac{2m^2}{r^2K^2},$$

$$g(r)=-\frac{\mathrm{D}^2F}{F}+\left(\frac{1}{r}-\frac{\mathrm{D}F}{F}\right)\left(f(r)-\frac{2}{r}\right)-\frac{K^2}{M}$$

$$+\frac{K^2}{RM}\left\{\frac{Q^2k^2m^2I}{RK^2r^2G}-\left(\mathrm{D}-\frac{1}{r}\right)\frac{Qm^2(1+I-G)}{K^2r^2}\right.$$

$$\left.+\frac{2\mu H_{0\theta}}{r}\left(\mathrm{D}H_{0\theta}-\frac{H_{0\theta}}{r}\right)-\frac{m^2H_{0\theta}^2Q(I-G)}{r^3F^2}\right\},$$

$$M(r)=1+N(r)-G(r).$$

All these expressions are valid for $H_{0z}\equiv H_{0z}(r)$.

Following Marinoff (1971), equation (6) was solved as two initial value problems by the Gill-modified Runge-Kutta technique and the logarithmic derivative Δ' so obtained was matched in an iterative procedure to that value of Δ' in the inviscid inner region derived by Furth *et al.* (1963). Only the estimated maximum growth rates for the viscous problem are presented in Table 1(a) as the inviscid results agree to the two significant figures shown, except that slightly larger values of R_0, the radial coordinate at $F=0$, are applicable. Again retention of $\rho\omega^2$ relative to μF^2 was necessary.

TABLE 1(a)

MAXIMUM GROWTH RATES AT VARIOUS TEMPERATURES FOR RESISTIVE "TEARING" INSTABILITY IN AN INCOMPRESSIBLE PLASMA

$a = 0 \cdot 09$ m, $b = 0 \cdot 11$ m, $H_{0z} = 1 \cdot 075 \times 10^5$ A m^{-1}, $\varepsilon = 1$, $m = 1$

R_0 (m)	J_0 (A)	$H_{0\theta}$ at R_0 (A m^{-1})	$S = \tau_R/\tau_H$	$\frac{b-a}{R_0}$	k (m^{-1})	T (K)	n_i (m^{-3})	η (m^2 s^{-1})	ν (m^2 s^{-1})	δ	Δ'	ω_{max} (s^{-1})
0·1084	$9 \cdot 5 \times 10^4$	$1 \cdot 21 \times 10^3$	$1 \cdot 46 \times 10^5$	0·184	−1·04	10^6	10^{21}	1·27	$9 \cdot 21 \times 10^3$	$1 \cdot 1 \times 10^{-1}$	32	$3 \cdot 0 \times 10^5$
0·1087	$9 \cdot 5 \times 10^4$	$9 \cdot 88 \times 10^3$	$3 \cdot 90 \times 10^4$	0·183	−0·846	10^7	10^{21}	$4 \cdot 78 \times 10^{-2}$	$2 \cdot 45 \times 10^6$	$3 \cdot 8 \times 10^{-2}$	95	$1 \cdot 0 \times 10^5$
0·1090	$9 \cdot 5 \times 10^4$	$7 \cdot 59 \times 10^3$	$1 \cdot 06 \times 10^6$	0·183	−0·648	10^8	10^{21}	$1 \cdot 75 \times 10^{-3}$	$6 \cdot 72 \times 10^8$	$1 \cdot 3 \times 10^{-2}$	290	$3 \cdot 4 \times 10^5$
0·1068	$9 \cdot 5 \times 10^4$	$2 \cdot 45 \times 10^4$	$6 \cdot 24 \times 10$	0·187	−2·13	10^4	10^{18}	$9 \cdot 67 \times 10^2$	$1 \cdot 21 \times 10^2$	$3 \cdot 1 \times 10^{-1}$	11	$2 \cdot 7 \times 10^7$
0·1084	$9 \cdot 5 \times 10^4$	$1 \cdot 21 \times 10^4$	$1 \cdot 41 \times 10^3$	0·184	−1·04	10^5	10^{18}	$4 \cdot 18 \times 10$	$2 \cdot 81 \times 10^4$	$1 \cdot 1 \times 10^{-1}$	32	$9 \cdot 7 \times 10^6$
0·1088	$9 \cdot 5 \times 10^4$	$9 \cdot 12 \times 10^3$	$3 \cdot 62 \times 10^4$	0·183	−0·779	10^6	10^{18}	1·63	$7 \cdot 21 \times 10^6$	$3 \cdot 8 \times 10^{-2}$	94	$3 \cdot 3 \times 10^6$
0·1092	$9 \cdot 5 \times 10^4$	$6 \cdot 07 \times 10^3$	$1 \cdot 00 \times 10^6$	0·183	−0·517	10^7	10^{18}	$5 \cdot 90 \times 10^{-2}$	$1 \cdot 99 \times 10^9$	$1 \cdot 3 \times 10^{-2}$	280	$1 \cdot 1 \times 10^6$
0·1095	$9 \cdot 5 \times 10^4$	$3 \cdot 78 \times 10^3$	$2 \cdot 80 \times 10^7$	0·182	−0·321	10^8	10^{18}	$2 \cdot 10 \times 10^{-3}$	$5 \cdot 59 \times 10^{11}$	$4 \cdot 2 \times 10^{-3}$	870	$3 \cdot 6 \times 10^5$

TABLE 1(b)

MAXIMUM GROWTH RATES FOR EXTREME SHEAR

$a = 0 \cdot 099$ m, $b = 0 \cdot 101$ m, $H_{0z} = 1 \cdot 075 \times 10^5$ A m^{-1}, $\varepsilon = 1$, $m = 1$

R_0 (m)	J_0 (A)	$H_{0\theta}$ at R_0 (A m^{-1})	$S = \tau_R/\tau_H$	$\frac{b-a}{R_0}$	k (m^{-1})	T (K)	n_i (m^{-3})	η (m^2 s^{-1})	ν (m^2 s^{-1})	δ	Δ'	ω_{max} (s^{-1})
0·1009	$9 \cdot 5 \times 10^4$	$7 \cdot 56 \times 10^3$	$1 \cdot 06 \times 10^5$	0·019	−0·697	10^8	10^{21}	$1 \cdot 75 \times 10^3$	$\begin{cases} 6 \cdot 72 \times 10^8 \\ 0 \end{cases}$	$5 \cdot 75 \times 10^{-2}$ $5 \cdot 77 \times 10^{-2}$	67·0 67·7	$1 \cdot 70 \times 10^5$ $1 \cdot 71 \times 10^5$
0·1009	$9 \cdot 5 \times 10^5$	$7 \cdot 56 \times 10^4$	$1 \cdot 30 \times 10^5$	0·019	−6·97	10^8	10^{21}	$1 \cdot 75 \times 10^3$	$\begin{cases} 6 \cdot 72 \times 10^8 \\ 0 \end{cases}$	$2 \cdot 39 \times 10^{-2}$ $2 \cdot 40 \times 10^{-2}$	82·3 85·2	$5 \cdot 03 \times 10^5$ $5 \cdot 17 \times 10^5$

The destabilization suggested by the cartesian model (Marinoff 1971) is not substantiated here, and indeed in the extreme shear cases (expected maximum stabilization) recorded in Table 1(*b*) there is slight stabilization with about a 2% decrease in maximum growth rate. It is clear that geometrical effects must be properly included in order to obtain a more accurate description of the role of parallel viscosity. Finally it is emphasized that the present results are for an incompressible plasma and parallel viscosity has not been included in the inner region.

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