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Tetrads and the Gravitational-inertial Field

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Abstract

The tetrad formulation of general relativity allows a non-tensorial decomposition of the gravitational field into two components which have been thought to represent the permanent and inertial parts. It is shown here that this division does not hold for arbitrary motions in a flat space–time, and therefore cannot be expected to hold in more general spaces.

A Riemannian manifold is one in which Euclidean geometry has been imposed in the tangent space at each point so as to permit the introduction of a smooth inner product. The natural frames, which arise in the study of such manifolds, lead in general to oblique coordinates in each of the tangent spaces. The relation between the Riemannian metric $g_{\mu\nu}$ at a given point of the manifold and the metric induced into the tangent space at the same point is

$$g_{\mu\nu} = {}^a h_\mu {}^b h_\nu n_{ab}\,, \tag{1}$$

where n_{ab} is the flat space metric of the tangent space. By requiring that $n_{ab} = \eta_{ab} = \eta^{ab} \equiv \mathrm{diag}(1, -1, -1, -1)$ the usual (orthonormal) tetrad formulation results (Pirani 1957). The tetrads ${}^a h_\mu$ then in addition to

$$g_{\mu\nu} = {}^a h_\mu {}^b h_\nu \eta_{ba} \tag{2}$$

obey the set of relations:

$$ {}^a h_\mu {}_b h^\mu = \delta^a_b\,, \qquad {}^a h^\mu {}^b h_\mu = \eta^{ab}\,, \tag{3a}$$

$$ {}^a h_\mu {}_a h^\nu = \delta^\nu_\mu\,, \qquad {}_a h^\mu {}_b h_\mu = \eta_{ab}\,. \tag{3b}$$

The principle of equivalence guarantees that ${}^a h_\mu$ can be found at any point satisfying equation (2). That it be possible to find a tetrad satisfying (2) everywhere requires that the curvature tensor as calculated from the $g_{\mu\nu}$ vanish everywhere.

The unit vector fields ${}^a h_\mu$ are associated with four generally arbitrary congruences, one of which will be taken as timelike and identical with the world lines of the physical problem at hand. The structure introduced corresponds to a set of everywhere orthonormal but in general anholonomic coordinates (Schouten 1954). It should be noted that the 10 components of the $g_{\mu\nu}$ are not sufficient to determine the 16 components of the ${}^a h_\mu$. The ${}^a h_\mu$ are determined only up to a Lorentz transformation which may vary arbitrarily with position (Utiyama 1956; Kibble 1961).

In consequence of using orthonormal rather than natural frames the connection coefficients are no longer given in terms of the metric by the Christoffel symbols. Instead the Ricci rotation coefficients defined by

$$\gamma_{abc} \equiv {}_a h_{\mu;\nu}\, {}_b h^{\mu}\, {}_c h^{\nu} \tag{4}$$

play a fundamental role. By covariantly differentiating either of equations (3b), γ_{abc} can be seen to be antisymmetric in the first two indices and has therefore only 24 components.

The following interpretation of the definition (4) is based in part on the work of Levi-Civita (1961): At any point P we have ${}^a h_{\mu}\, {}_b h^{\mu} = \delta^a_b$. Let the vector associated with the a-congruence be displaced from P by the local displacement law determined by the c-congruence, and let the vector associated with the b-congruence be displaced from P along the c-congruence by parallel transport. Then γ_{abc} is the rate of change of the cosine of the angle between the two vectors.

One may also define the tensors associated with the Ricci coefficients, the mixed form of which is (see equations (2) and (3))

$$\gamma^{\phi}_{\mu\nu} \equiv \gamma_{abc}\, {}^a h_{\mu}\, {}^b h^{\phi}\, {}^c h_{\nu} = {}_a h_{\mu;\nu}\, {}^a h^{\phi}\,. \tag{5}$$

In the next section we will be concerned primarily with these associate tensors.

Decomposition of the Field

Many authors (e.g. Davis 1970; Gatha and Dutt 1971) have noted that the field

$$\left\{ {\mu \atop \nu\omega} \right\} = \tfrac{1}{2} g^{\mu\sigma} \{ g_{\sigma\nu,\omega} + g_{\omega\sigma,\nu} - g_{\nu\omega,\sigma} \} \tag{6}$$

with the relation (2) yields

$$\left\{ {\mu \atop \nu\omega} \right\} = {}_a h^{\mu}\, {}^a h_{(\nu,\omega)} - \tfrac{1}{2} \{ g^{\mu\sigma} (g_{\phi\nu} \Lambda^{\phi}_{\omega\sigma} + g_{\phi\omega} \Lambda^{\phi}_{\nu\sigma}) \}\,, \tag{7}$$

where

$$\Lambda^{\mu}_{\nu\omega} \equiv 2\, {}_a h^{\mu}\, {}^a h_{[\nu,\omega]}\,. \tag{8}$$

Using the definitions (5) and (8), equation (7) may be written as

$$\left\{ {\mu \atop \nu\omega} \right\} = \Gamma^{\mu}_{\nu\omega} - \gamma^{\mu}_{\nu\omega}\,, \tag{9}$$

where

$$\Gamma^{\mu}_{\nu\omega} \equiv {}_a h^{\mu}\, {}^a h_{\nu,\omega} \tag{10}$$

is an integrable affinity (Schrödinger 1963). As the Riemannian curvature tensor formed from an integrable affinity is known to be zero, and since $\gamma^{\mu}_{\nu\omega}$ is a tensor, equation (9) has been thought to represent a non-tensorial decomposition of the gravitational field into its 'real' (or permanent) and inertial parts (Rosen 1963; Gatha and Dutt 1971).

Note that

$$\tfrac{1}{2}\Lambda^{\mu}_{\nu\omega} = \Gamma^{\mu}_{[\nu\omega]} = \gamma^{\mu}_{[\nu\omega]}\,. \tag{11}$$

That this must hold is also apparent from the symmetry of $\left\{ {\mu \atop \nu\omega} \right\}$ in the lower indices. $\Lambda^{\mu}_{\nu\omega}$ is the associate tensor to the object of anholonomity which is defined as

$$\Lambda^{k}_{ij} \equiv {}_i h^{\mu}\, {}_j h^{\nu}\, {}^k h_{[\mu,\nu]}\,. \tag{12}$$

The object of anholonomity is related to the Ricci coefficients by

$$\gamma_{abc} = \Lambda_{abc} - \Lambda_{bca} - \Lambda_{cab}\,, \tag{13}$$

where

$$\Lambda_{abc} = \Lambda^{d}_{ab}\,\eta_{dc}\,. \tag{14}$$

From equations (7) and (11) we see that $\gamma^{\mu}_{\nu\omega}$ and $\Gamma^{\mu}_{[\nu\omega]}$ depend entirely upon $\Lambda^{\mu}_{\nu\omega}$. This becomes even more apparent if we define

$$\Lambda_{\{\omega\sigma\nu\}} \equiv \Lambda_{\omega\sigma\nu} - \Lambda_{\sigma\nu\omega} + \Lambda_{\nu\omega\sigma} \tag{15}$$

which, with equation (7), yields the decomposition

$$\left\{ {\mu \atop \nu\omega} \right\} = \Gamma^{\mu}_{\nu\omega} - \tfrac{1}{2} g^{\mu\sigma} \Lambda_{\{\omega\sigma\nu\}}\,. \tag{16}$$

If the decomposition (9) or (16) is to be interpreted as meaning that the inertial field is contained entirely in the symmetric part of the first term on the right-hand side then the second term must exclusively contain the permanent gravitational field. By requiring the space–time to be flat, the second term and hence the object of anholonomity must vanish. In this case the first term is required to describe the field for general accelerated motions. That this is not possible can be seen by noting that the vanishing of the object of anholonomity implies that all the congruences associated with the tetrad field are 3-normal *and* geodesic (Estabrook and Wahlquist 1964). This is equivalent to the introduction of holonomic coordinates and is too restrictive a condition to allow for arbitrary accelerated motion in a flat space–time.

As mentioned in the introduction, the vanishing of the Riemannian tensor guarantees that a tetrad satisfying equation (2) everywhere (holonomic coordinates) can be found. However, its vanishing does not necessitate the introduction of such coordinates. For certain classes of problems, in fact, anholonomic coordinates are the most natural to use. An example is the anholonomity of a spatial triad introduced by the Thomas precession in curvilinear accelerated motion.

References

Davis, W. R. (1970). 'Classical Fields, Particles, and the Theory of Relativity' (Gordon & Breach: New York).

Estabrook, F. B., and Wahlquist, H. D. (1964). *J. math. Phys.* **5**, 1629.

Gatha, K. M., and Dutt, R. C. (1971). *Aust. J. Phys.* **24**, 631.

Kibble, T. W. B. (1961). *J. math. Phys.* **2**, 212.

Levi-Civita, T. (1961). 'The Absolute Differential Calculus' (Blackie & Son: London).

Pirani, F. A. E. (1957). *Bull. Acad. pol. Sci.* III **5**, 143.

Rosen, N. (1963). *Ann. Phys.* **22**, 1.

Schouten, J. A. (1954). 'Ricci-calculus', 2nd Ed. (Springer–Verlag: Berlin).

Schrödinger, E. (1963). 'Space–Time Structure', Ch. 6 (Cambridge Univ. Press).

Utiyama, R. (1956). *Phys. Rev.* **101**, 1597.

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