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Supersonic Neutral Winds in an Outer Atmosphere. I Isothermal Conditions

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Abstract

The concept of steady nozzle flow along a narrow tube is applied to the motion of neutral gas constituents in a locally heated isothermal region of the outer atmosphere of a planet or star. The artificial nozzle throat at which the flow becomes supersonic is achieved by using one of two streamline functions within a vertical plane suggested by the possible shapes of convection cells. One streamline spirals above the surface of the planet at constant elevation, and the other 'bends over' asymptotically towards the horizontal. The former achieves the nozzle throat by the gravitational force decreasing with distance from the centre of the planet, while the latter relies on the gravitational component along the streamline decreasing as the streamline approaches the horizontal. Conditions under which the effects of viscosity and frictional interactions in the Earth's atmosphere may be neglected from the assumed hydrodynamic description are considered. For the proposed models and boundary conditions appropriate to an intensely heated region of the Earth's atmosphere, the dependences of the critical distance, temperature boundary values, and velocity and heating profiles on variations in the parameters are shown.

1. Introduction

It is possible that supersonic neutral winds may occur in regions of a planet's atmosphere when there is intense heating. In the case of the Earth, such heating may be caused by auroral processes (Cole 1966) or nuclear explosions. During a period of extremely high geomagnetic activity, above 120 km altitude, large lateral neutral velocity components in excess of $0\cdot5\ \mathrm{km\,s^{-1}}$ have been recorded that may be supersonic (Rees 1971, 1972; Lloyd *et al.* 1972). The possibility of supersonic winds has since been further established by accelerometer measurement of lateral winds in excess of $1\cdot5\ \mathrm{km\,s^{-1}}$ (DeVries 1971). Winds of such speeds were predicted earlier (Cole 1962*a*).

The general problem of heating and movement of the atmosphere during geomagnetic activity is complex (Cole 1966, 1971). In addition to the heating and force components present during quiet periods, there are heating components due to Joule dissipation, viscosity and corpuscular bombardment, and force components caused by the heating itself and the Lorentz term associated with electric currents. The latter are known to be able to produce winds of such speeds (Cole 1962*a*); however, it is not intended to discuss this source of movement here. The present investigation is concerned with a very much simplified problem in which only an atmospheric heat source is taken to be the driver of the motion and no interaction with an ionosphere is considered. To this end, the concept of steady nozzle flow along a narrow tube, as developed for the solar wind, is further explored.

Clauser (1960) first suggested the analogy that the gravitational field of the Sun plays the same role in coronal expansions as does the throat in a deLaval nozzle. Assuming the basic steady state hydrodynamic equations, Parker (1960, 1963) later extensively developed the idea to explain how stellar winds in general, and solar winds in particular, may become supersonic (for a review, see Parker 1965; Dessler 1967). The plasma is assumed to radiate above the star's surface with spherical symmetry. The concept has also been applied to the Earth's polar wind, where ionospheric plasma flows away from the Earth along divergent open geomagnetic field lines (Axford 1968; Banks and Holzer 1968); the effects of ion production and loss and also friction have been included in the polar wind models.

Holzer and Axford (1970) have reviewed the application of the above concept to a number of types of wind. They have shown that the results obtained from the general theory used previously for stellar and polar winds may also be applied to the galactic winds, comet winds and the classical accretion problem. Another recent application has been to the possibility of supersonic plasma flow in a collapsing post-sunset ionosphere (Fontheim and Banks 1972). A rapid downward flow of hydrogen plasma from the protonosphere takes place as a result of the rapidly decreasing pressure in the topside ionosphere during twilight hours. Given steady frictionless flow, a critical point may be shown to exist for suitable temperature gradients, but not for the isothermal case.

The above authors have used the hydrodynamic approximation with success in describing the movement of rarefied gases in the outer atmosphere of the Sun and Earth. Other authors (e.g. Dessler and Michel 1966; Hollweg 1970; Lemaire and Scherer 1970) have used the kinetic approximation, which has the disadvantage, however, that it lacks flexibility in ignoring some effects and including others. Holzer *et al.* (1971) have compared kinetic and hydrodynamic models of an expanding ion-exosphere and conclude that the hydrodynamic description provides a good approximation throughout the polar ionosphere and is preferable because of its greater flexibility in allowing the model to be considerably simplified. One may argue, as does Parker (1965), that if the atmosphere is sufficiently dense for the mean free path to be small compared with the scale height and/or the radial distance from the parent body then the ordinary hydrodynamic equations are appropriate. However, the hydrodynamic approximation, which is adopted in this paper, will in general become less appropriate the further one proceeds away from the body. In the discussion in Section 4 on the effects of viscosity in the Earth's atmosphere, the inseparable link with the mean free path is outlined. It is shown that viscosity may be neglected from the hydrodynamic equations provided the hydrodynamic description applies, and that this description becomes increasingly less justifiable for altitudes above 1200 km.

The motion of neutral gas constituents in response to a source of intense heating in an isothermal atmosphere is explored here when only pressure gradient and gravitational forces are assumed. Although the effects of viscosity and frictional interactions are not specifically included in the models, conditions under which they may be neglected are presented for the case of the Earth's outer atmosphere using data from a static model (Jacchia 1971). To include these effects, as well as chemical reactions, in the models presented would greatly complicate the analysis and result in a loss of generality because one would have to relate the analysis to a specific region of a specific planet's atmosphere.

If there is a heat source everywhere along the tube of gas without any sinks, the velocity must increase monotonically when the temperature is constant. Therefore, only solutions which possess a critical point at which the velocity passes from the subsonic to supersonic states are considered. To obtain supersonic lateral velocity components, the nozzle throat is achieved by using one of two arbitrarily assumed streamline functions within a vertical plane (see Fig. 1). Both functions may form part of a large convection cell and are chosen for their mathematical convenience in simplifying the equations while allowing the existence of a critical point. The part of the convection cell outside a given narrow tube of gas about a streamline is assumed to form the walls of the tube. It is realised that this is only an approximation to a real case, but it is one which allows direct mathematical treatment. The first function spirals above the surface of the planet such that its angle of elevation is constant, and achieves a nozzle throat in the conventional manner of the solar wind by the gravitational force decreasing with distance from the centre of the planet. The second function 'bends over' asymptotically towards the horizontal in an exponentially decaying manner, and relies instead on a decreasing gravitational component along the streamline as the flow approaches zero elevation. The former, which is especially relevant to the escape of helium (Cole 1962*b*; Axford 1968), corresponds to radial flow when the angle of elevation is 90°.

For both coronal and atmospheric expansion, boundary conditions often exist that ultimately require a 'velocity reversal' whereby the velocity decreases to a subsonic speed having first increased from subsonic to supersonic. The reversal may be achieved through a shock front at which the velocity drops sharply from supersonic to subsonic within a very short distance. Shock fronts have been explored in relation to the solar wind (Axford *et al.* 1963; Dessler 1967; Fahr 1971), polar wind (Banks and Holzer 1969) and collapsing post-sunset winds, where the motion is towards the Earth (Fontheim and Banks 1972). A velocity reversal may also be achieved if another critical point exists. Upon examination of the equations when only pressure gradient and gravitational forces are assumed, it is found that two critical values may result only when the temperature is not constant, and that for a velocity solution to pass through both critical points, one degree of freedom is lost in choosing the parameters on which a solution depends. The use of a variable temperature necessitates a different method of solution of the hydrodynamic equations, and this will be dealt with in the following Part II (Gilbert and Cole 1974; present issue pp. 529–40), as will the conditions for two critical points and a velocity reversal. Indeed, it is not improbable that the flow would become turbulent at some stage when supersonic, and thus upset the assumed nozzle flow dynamics resulting in subsonic speeds.

The present analysis differs from previous ones by its emphasis on atmospheric neutral particles rather than ions and electrons. Other differences are that the flow is considered within a convection cell of a locally heated region of an atmosphere, and not along geomagnetic field lines, as with the polar wind, or radially from the complete surface of a star, as with the solar wind. The dynamics for steady nozzle flow in an isothermal atmosphere are developed from the general equations for a variable temperature. Boundary values for the critical distance are introduced, which result in corresponding temperature boundary values, and the dependence of both the critical distance and temperature boundary values on the parameters is shown. In the absence of any comprehensive data available for velocity and heating profiles, such results are presented only to illustrate the effects of parameter variations. The results for the critical distance and temperature boundary values serve as predictions of the necessary conditions for the flow to become supersonic in a local region of an atmosphere.

2. General Hydrodynamic Equations

Consider the steady state motion of a neutral gas constituent through a narrow expanding tube whose axis is a given streamline in a vertical plane. When the only forces considered are those due to gravity and the pressure gradient, the equations of momentum, continuity, state and energy (e.g. Eckart 1960; Hughes and Brighton 1967) may be written as

$$v\,\mathrm{d}v/\mathrm{d}r = -\rho^{-1}\,\mathrm{d}p/\mathrm{d}r - \mathrm{d}\psi/\mathrm{d}r\,, \tag{1}$$

$$\mathrm{d}(\rho A v)/\mathrm{d}r = 0\,, \tag{2}$$

$$p = R\rho T\,, \tag{3}$$

$$q = p\,\mathrm{d}v/\mathrm{d}r + c_{\mathrm{v}}\,\rho v\,\mathrm{d}T/\mathrm{d}r\,, \tag{4}$$

where r is the distance along a streamline from the apex of the nozzle, v the velocity, p the pressure, ρ the density, A the cross sectional area of a narrow tube of gas, T the temperature, ψ the gravitational potential, c_{v} the specific heat at constant volume, R the gas constant, and q the net accession of heat from all sources and sinks. A characteristic thermal velocity c $(= (RT)^{\frac{1}{2}})$ is defined, which differs from the sonic speed, equal to $(\gamma RT)^{\frac{1}{2}}$, where γ is the ratio of the specific heats. Reference to subsonic and supersonic speeds should be interpreted to mean less than c and greater than c respectively.

Upon elimination of p and ρ from equations (1)–(3), the 'general nozzle flow equation' (Parker 1965) is given by

$$\left(v - \frac{c^2}{v}\right)\frac{\mathrm{d}v}{\mathrm{d}r} = -A\frac{\mathrm{d}(c^2/A)}{\mathrm{d}r} - \frac{\mathrm{d}\psi}{\mathrm{d}r} = F(r)\,. \tag{5}$$

The gravitational potential may be written as

$$\psi = -g_0\,a^2/(a+y)\,, \tag{6}$$

where a is the distance from the centre of the planet to the base reference position r_0, g_0 the gravitational acceleration at r_0, and y the streamline function $y(x)$ defined in Fig. 1. For coronal expansion, $r_0 = a$ when the flow radiates with spherical symmetry from the centre of the Sun. Parker (1958, p. 669) has applied the case $r_0 < a$ to the hypothetical outflow of gas from an active region. It may be seen from Fig. 1 that

$$\cos\Phi = \mathrm{d}y/\mathrm{d}s\,, \tag{7}$$

where $s = r - r_0$ and Φ is the angle between the streamline and the vertical. Upon integration of equation (7), the distance along the streamline from the reference point is given by

$$s = \int_0^y \sec\Phi\,\mathrm{d}y\,. \tag{8}$$

Since $\mathrm{d}\psi/\mathrm{d}r = \mathrm{d}\psi/\mathrm{d}s = (\mathrm{d}\psi/\mathrm{d}y)(\mathrm{d}y/\mathrm{d}s)$, from equations (6) and (7),

$$\mathrm{d}\psi/\mathrm{d}r = g_0 \cos\Phi\,(1+y/a)^{-2}. \tag{9}$$

Assuming $A \propto r^n$ for some integer n, the right-hand side of equation (5) may then be written more explicitly as

$$F(r) = nc^2/r - \mathrm{d}(c^2)/\mathrm{d}r - g_0 \cos\Phi\,(1+y/a)^{-2}.$$

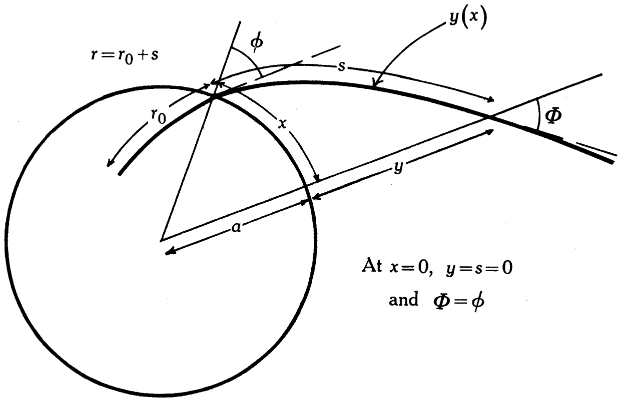


Fig. 1. Definition of the coordinate system for a streamline in a plane through the centre of a planet. The results in the subsequent graphs are expressed in terms of the dimensionless parameters $X = x/a$, $Y = y/a$, $S = s/a$ and $R_0 = r_0/a$.

A critical point exists when both sides of equation (5) are zero, $\mathrm{d}v/\mathrm{d}r$ being nonzero. This occurs when $v = c$ and $r = r_c$ (the subscript c denoting a critical value), where r_c is a zero of $F(r)$; Parker has demonstrated the existence of solutions in the neighbourhood of a critical point (e.g. Parker 1960, p. 830; 1963, p. 59). A necessary condition for $v = c$ at $r = r_c$ is $F'(r_c) > 0$, where the prime denotes differentiation with respect to r. If $F'(r_c)$ is negative, then r_c is not a critical value but a stationary one where v remains either greater than or less than c. This is shown by considering the two possible cases in the vicinity of a critical point where the velocity is either monotonically increasing or decreasing. In the case of the former, $\mathrm{d}v/\mathrm{d}r$ is positive and $v < c^2/v$ when $r < r_c$, and $v > c^2/v$ when $r > r_c$, so that $F(r)$ changes sign from negative to positive as the critical point is passed. For the latter case, $\mathrm{d}v/\mathrm{d}r$ is negative and v is greater or less than c^2/v when r is less or greater than r_c respectively, which results in the same condition for $F(r)$ on passing through the critical point.

3. Constant Temperature Analysis

When the temperature remains constant, equation (5) may be integrated to give

$$v^2 - c^2 \ln(v^2) = v_a^2 - c^2 \ln(v_a^2) + 2\int_{r_a}^{r} F(r)\,\mathrm{d}r, \tag{10}$$

where

$$F(r) = nc^2/r - g_0 \cos\Phi\,(1+y/a)^{-2}.$$

The limits of integration r_a and v_a are such that, for solutions which pass through the critical point, $r_a = r_c$ and $v_a = c$; otherwise, r_a is the value of r when $v = c$ (dv/dr being undefined) or v_a is the value of v when $r_a = r_c$ (dv/dr being zero). This is illustrated by the well-known velocity topology (e.g. Parker 1963). Generally $F(r)$ cannot be integrated exactly, in which case it is integrated numerically using the trapezoidal rule. Equation (10) is then solved for v using either Newton's method or the method of false position. For solutions passing through the critical point, equation (10) becomes

$$v^2 - c^2 \ln(v^2) = c^2\{1-\ln(c^2)\} + 2\int_{r_c}^{r} F(r)\,dr. \tag{11}$$

The energy equation (4) reduces to

$$q = p\,dv/dr,$$

which upon substitution for p becomes

$$q = \frac{\rho_0 v_0 c^2}{v}\left(\frac{r_0}{r}\right)^n \frac{dv}{dr}, \tag{12}$$

where ρ_0 and v_0 are values at r_0. Sources of energy in a heated region of an atmosphere are likely to far exceed any possible sinks so that, for an isothermal atmosphere, dv/dr must be positive for q to be positive. Hence, for v to increase monotonically from subsonic to supersonic, $F(r)$ must have only one zero value for $r > r_0$ and $F(r_0)$ must be negative. Also, the assumption of a constant temperature excludes heat conduction as a contribution to q.

In order to more specifically examine the conditions under which the velocity may pass from being subsonic to supersonic, the streamline function must now be defined. Two functions are specified: the first is termed a 'linear' or 'constant-Φ' function where the nozzle throat is achieved in the conventional manner by the gravitational force which decreases with distance from the centre of the planet, while the second is termed an 'exponential' function where the gravitational component along a streamline decreases as the streamline bends over asymptotically towards the horizontal.

The dimensionless quantities $X = x/a$, $Y = y/a$, $S = s/a$, $R_0 = r_0/a$ and $\lambda = g_0 a/nc^2$ are introduced with the subscripts already used retaining their meaning for the first three quantities. The equation $F(r_c) = 0$ may be written in terms of these quantities as

$$(1+Y_c)^2 = \lambda(S_c+R_0)\cos\Phi. \tag{13}$$

In order to restrict the range of possible parameter values, critical distance boundary values are specified for each model. A general value k (corresponding to S_c for the constant-Φ model and to X_c for the exponential model) is introduced such that $0 \leqslant k \leqslant K$, where K is a specified constant. The value k may represent either a lower or upper boundary for the critical distance, except of course at $k = 0$ and K, which are the extreme lower and upper boundaries respectively. The condition $F(r_0) < 0$ necessary for v to increase monotonically from subsonic to supersonic is equivalent to the extreme lower boundary condition $S_c > 0$. It is desired to obtain expressions for the corresponding boundary values of λ from equation (13), and hence of T, where the latter quantity is given in terms of the former from the definition of λ by

$$T = (g_0\, a/nR)\lambda^{-1}. \tag{14}$$

It can be seen that a lower boundary for λ corresponds to an upper boundary for T and vice versa.

(a) Constant-Φ Function, Linear for $Y \ll 1$

In the curvilinear coordinate system defined in Fig. 1, it may be shown that

$$\mathrm{d}Y/\mathrm{d}X = (1+Y)\cot\Phi. \tag{15}$$

On integrating this equation when Φ is constant ($=\phi$), the streamline function for a constant-Φ spiral is given by

$$Y = \exp(X\cot\phi)-1 \qquad \text{for} \qquad 0 < \phi < \tfrac{1}{2}\pi. \tag{16}$$

For $\phi = 0$, the model simply represents outward radial flow with $S = Y$ and $X = 0$. For $Y \ll 1$, which is a realistic condition for the Earth's neutral atmosphere, equation (16) may be approximated by the function $Y = X\cot\phi$. Hence the model or function is referred to as being 'linear'.

The mathematical convenience of choosing a constant-Φ function is apparent in the evaluation of S in equation (8), which results simply in

$$S = Y\sec\phi.$$

Hence, when $\Phi = \phi$, the general boundary value for λ corresponding to $S_c = k$ is given directly from equation (13) by

$$\lambda_k = (1+k\cos\phi)^2\sec\phi\,/(k+R_0).$$

Equation (13) may also be written as a quadratic in Y_c with solution

$$Y_c = \tfrac{1}{2}\lambda-1+\{\tfrac{1}{4}\lambda^2-\lambda(1-R_0\cos\phi)\}^{\frac{1}{2}}. \tag{17}$$

Provided the extreme lower boundary condition for S_c is satisfied (which requires only one possible positive value of S_c), the solution for Y_c is real and the root given by equation (17) is the required one, since it is positive while the other root is negative. To be consistent with the defined lower and upper boundaries at $k = 0$ and K respectively, the condition $\lambda_0 < \lambda_K$ must be satisfied, where the subscript for λ is the value of k. This may be expressed by the inequality

$$R_0 > \sec\phi\,/(2+K\cos\phi).$$

The function $F(r)$ in equation (11) may be integrated analytically to give

$$\int_{r_c}^{r} F(r)\,\mathrm{d}r = nc^2\ln(r/r_c)+g_0\,a^2\{(a+s\cos\phi)^{-1}-(a+s_c\cos\phi)^{-1}\}.$$

(b) Exponential Function

The exponential function is defined as

$$Y = Y_\infty\{1-\exp(-X/H)\},$$

where Y_∞ is the value of Y approached asymptotically as X tends to infinity, and H is the dimensionless scale height. From equation (15),

$$\cos\Phi = \frac{\mathrm{d}Y/\mathrm{d}X}{\{(\mathrm{d}Y/\mathrm{d}X)^2+(1+Y)^2\}^{\frac{1}{2}}} \tag{18}$$

and, from equation (8),

$$S = \int_0^X (\mathrm{d}Y/\mathrm{d}X)\sec\Phi\,\mathrm{d}X,$$

so that S may be written in terms of X as

$$S = \int_0^X \left[\left\{1+Y_\infty - Y_\infty\exp\left(-\frac{X}{H}\right)\right\}^2 + \left\{\frac{Y_\infty}{H}\exp\left(-\frac{X}{H}\right)\right\}^2\right]^{\frac{1}{2}} \mathrm{d}X$$

or, upon substituting $E = \exp(-X/H)$, as

$$S = \int_1^E -\frac{H}{E}\left((1+Y_\infty)^2 - 2Y_\infty E(1+Y_\infty) + (Y_\infty E)^2(1+H^{-2})\right)^{\frac{1}{2}} \mathrm{d}E, \tag{19}$$

which is evaluated as a standard integral (Spiegel 1968). Also in terms of E, equation (18) becomes

$$\cos\Phi = \frac{Y_\infty E/H}{\{(1+Y_\infty - Y_\infty E)^2+(Y_\infty E/H)^2\}^{\frac{1}{2}}} \tag{20}$$

and equation (13) results in

$$(1+Y_\infty - Y_\infty E_c)^2\{(1+Y_\infty - Y_\infty E_c)^2+(Y_\infty E_c/H)^2\}^{\frac{1}{2}} = \lambda(R_0+S_c)(Y_\infty E_c/H). \tag{21}$$

Substitution of S_c in this equation using (19) results in a complicated expression in the unknown variable E_c, which in general can only be solved numerically, as must the integral of $F(r)$.

It is desired with this model to achieve a critical condition by virtue of the streamline bending over asymptotically towards the horizontal, rather than by moving a sufficient distance above the surface of the planet as with the linear model. Therefore, the general boundary value k for the critical distance is chosen to correspond to X_c rather than S_c, which simplifies some expressions, and the conditions $Y_\infty \leqslant K$ and $K \leqslant 0{\cdot}3$ are also imposed. The boundary value for λ is given from equation (21) by

$$\lambda_k = \frac{(1+Y_\infty - Y_\infty E_k)^2\{(1+Y_\infty - Y_\infty E_k)^2+(Y_\infty E_k/H)^2\}^{\frac{1}{2}}}{(R_0+S_k)(Y_\infty E_k/H)}, \tag{22}$$

where the subscript k here denotes the value at $X = k$. The minimum value of R_0 such that $\lambda_0 < \lambda_K$ is found from equation (22).

To obtain an analytical expression for X_c or S_c, in addition to the existing conditions it is necessary to assume $Y_\infty \ll 1$ and $H \geqslant bY_\infty$, where b must be greater than 1 and is here set equal to 2. In effect, this allows the approximations

$$S \approx X \qquad \text{and} \qquad \cos\Phi \approx Y_\infty E/H = (Y_\infty/H)\exp(-X/H).$$

For a given value of Y_∞ ($\leqslant 0\cdot 05$), the maximum relative errors in the approximations are found when $H = 2Y_\infty$ and X tends to zero (equals zero for $\cos\Phi$). The condition $H \geqslant 2Y_\infty$ may be removed without loss of accuracy provided a lower limit X_L is introduced for X. The limits for S and $\cos\Phi$ are calculated separately as follows.

For $H = 2Y_\infty$, as X tends to zero, $X/S = 2/\sqrt{5}$, thus resulting in a maximum relative error in S of $|1-2/\sqrt{5}|$, i.e. about $0\cdot 1$. The value of X_L required to give the same error is readily found in the extreme case when H tends to zero. S may be written as $X+Y_\infty$, thus giving an error in S of $1-X/(X+Y_\infty)$. Equating the two values gives $X_L = 9Y_\infty$. In non-extreme cases when H is larger, X_L may be found numerically (e.g. $X_L \approx 8Y_\infty$ when $H = 0\cdot 4\,Y_\infty$, $X_L \approx 3\cdot 5\,Y_\infty$ when $H = Y_\infty$).

There are two contributing sources of error in the approximation for $\cos\Phi$. The first source is obtained by neglecting the term $(Y_\infty E/H)^2$ in equation (20). Since E is greatest ($=1$) when $X = 0$, for $H = 2Y_\infty$, the maximum relative error in $\cos\Phi$ is therefore $|1-\sqrt{5}/2|$, i.e. again about $0\cdot 1$. For $H < 2Y_\infty$, the same error results when $(Y_\infty E/H) = 0\cdot 5$, i.e. $X_L = H\ln(2Y_\infty/H)$. When for example $Y_\infty = 5H$, then $X_L = 2\cdot 3\,H$ or $0\cdot 46\,Y_\infty$. The other source of error is obtained by approximating the term $(1+Y_\infty - Y_\infty E)^2$ by unity in equation (20). When X is large (i.e. as E approaches zero), the maximum relative error in $\cos\Phi$ is then Y_∞, where $Y_\infty \leqslant 0\cdot 05$, and is therefore smaller than the former source.

It may be observed that the value of X_L required to maintain the accuracy of the approximation for S is in general much larger than that required for $\cos\Phi$. However, provided that S_c is not very much greater than R_0, inspection of equation (13) reveals that the accuracy of $\cos\Phi$ is more significant than that of S in the determination of Y_c. This is as one would expect, since the critical value is achieved in this model by virtue of a decrease in $\cos\Phi$ rather than an increase in S.

The above approximations result in

$$\lambda_k = \frac{H}{Y_\infty}\frac{\exp(k/H)}{R_0+k},$$

$$\int_{r_c}^{r} F(r)\,\mathrm{d}r = nc^2\ln\!\left(\frac{r_0+x}{r_0+x_c}\right) + g_0\,y_\infty\left\{\exp\!\left(-\frac{x}{h}\right) - \exp\!\left(-\frac{x_c}{h}\right)\right\},$$

$$R_0 > K\{\exp(K/h)-1\},$$

and S_c is then given by the solution of

$$\exp(X_c/H) = \lambda(R_0+X_c)(Y_\infty/H). \tag{23}$$

In general, equation (23) must be solved numerically. However, when $H > X_c$ (always when $H > K$), the Taylor series expansion for $\exp(X_c/H)$ may be used and, on neglecting powers of X_c/H greater than two, the solution is given by

$$X_c = H(\lambda Y_\infty - 1) + \{H^2(\lambda Y_\infty - 1)^2 - 2H(H-\lambda Y_\infty R_0)\}^{\frac{1}{2}}.$$

4. Viscosity and Frictional Interactions

Effects of viscosity and frictional interactions are not included in the preceding analysis because of the loss of generality when dealing with a specific part of a specific planet's atmosphere and also because of the enormous complexities introduced. However, an attempt is made here to determine conditions under which these effects may be reasonably neglected in the Earth's outer atmosphere.

Different neutral species are dominant in different regions of the Earth's upper atmosphere, N_2, O, He and H all being principal constituents in particular height regions. Rather than attempting to deal with the composite neutral gas, it is simpler to consider a single constituent. However, it is not intended that the other neutral constituents then exert a frictional force on the particular species being investigated. By examining the effect of each principal constituent as if the other neutral constituents were absent, limits of the overall effect may be inferred. Results for the principal constituent for a particular height region will most closely represent the true physical behaviour of the composite neutral gas.

Collisions of both neutral particles and ions with electrons may be neglected because the electrons are so much lighter (although the electrons are themselves greatly affected). The neutral particles and ions may be assumed to have the same mass (for the same species) so that, unless the ions move with the neutral gas, frictional forces should be considered. Above 150 km altitude, ions flow readily along magnetic field lines but not across them (see e.g. Rishbeth and Garriott 1969). Assuming that the ions are initially stationary, the maximum frictional force occurs when the magnetic field lines are perpendicular to the neutral gas flow. Otherwise, the ions would flow along the field lines under steady state conditions, thus decreasing the difference in velocity between the neutral gas and ions.

Let m_i and m_n be the physical mass of an ion and neutral particle respectively (for the same constituent, it is assumed that $m_i = m_n$), M_i and M_n the corresponding molecular mass ($m_i = 1{\cdot}66\times 10^{-24} M_i$ etc.) and n_i and n_n the corresponding concentrations. The frictional force per unit volume due to collisions between neutral particles and stationary ions is given by (e.g. Cowling 1956; Rishbeth and Garriott 1969) $m_i n_i \nu_i v$, where ν_i is the collision frequency between ions and neutral particles, and v the velocity of neutral particles as previously defined. As pointed out by Rishbeth and Garriott (1969), the collision frequency ν_i does not represent the real frequency of collisions between ions and neutral particles, but it is regarded as a coefficient indicating the rate of transfer of momentum and is sometimes called the 'effective collision frequency' or the 'frictional frequency'. However, some authors use a different collision parameter, which is more closely related to the actual kinetic theory frequency of collisions, and replace m_i by the 'reduced mass' $m_i m_n/(m_i+m_n)$. The viscous drag force per unit volume of the neutral gas constituent is given approximately by (e.g. Hughes and Brighton 1967) $\eta\, d^2v/dr^2$, assuming that the coefficient of viscosity η is constant.

Inclusion of the above forces in the nozzle flow equation, when the temperature is constant, results in

$$F(r) = \frac{nc^2}{r} - \frac{g_0 \cos\Phi}{(1+y/a)^2} - \frac{m_i n_i \nu_i v}{m_n n_n} + \frac{\eta}{m_n n_n}\frac{d^2 v}{dr^2}.$$

Each of these two additional acceleration terms will be treated independently, neglecting the presence of the other. As a characteristic estimate of these accelerations, it is convenient to choose the critical point, where $(r, v) = (r_c, c)$ and $F(r_c) = 0$. Then, in the absence of the additional acceleration terms, the 'expansion' term nc^2/r must be equal to the 'gravitational' term $g_0 \cos\Phi/(1+y/a)^2$. Hence, if either additional acceleration term is to be neglected, the absolute value of both must be much less than either the expansion or gravitational term.

For the frictional interaction force, it is convenient to compare the frictional term with the gravitational term, which requires $D \ll 1$, where

$$D = \frac{m_i\, n_i\, \nu_i\, c}{m_n\, n_n\, g_0 \cos\Phi/(1+y/a)^2}.$$

The reference level here is set at sea level so that y corresponds to altitude. In the following calculations, y assumes values between 400 and 2500 km, but Φ may vary between 0 and almost 90°. As Φ approaches 90° the flow is less likely to be perpendicular to the field lines (in the polar rather than equatorial region), so that the decrease in gravitational force component along the streamline is likely to be balanced to some extent by a decrease in v. Hence, average values are chosen such that $y/a = 0{\cdot}2$ and $\Phi = 45°$, which results in $g_0 \cos\Phi/(1+y/a)^2 \approx 500\ \mathrm{cm\,s^{-2}}$.

An expression for the collision frequency between ions and neutral particles has been given by Chapman (1956) as

$$\nu_i = 2{\cdot}6\times10^{-9}(n_n+n_i)M^{-1/2},$$

where M is the mean molecular mass of ions and neutral particles given by

$$M = (M_i\, n_i + M_n\, n_n)/(n_i+n_n).$$

A lower limit for T is chosen to be 2000 K. An upper limit of 8000 K simply doubles c, and hence D. Therefore at $T = 2000$ K, where $c = 4{\cdot}08\times10^5\, M_n^{-1/2}$,

$$D = 2{\cdot}12\times10^{-6}\, M_i\, n_i(1+n_i/n_n)M_n^{-3/2}\, M^{-1/2}.$$

There exists a major problem in obtaining comprehensive and compatible data for both neutral and ionized constituents in the Earth's atmosphere. Any data that are comprehensive apply to a quiet atmosphere. The problem is therefore made worse here in view of the exceptionally disturbed conditions being modelled. Ion concentrations below 400 km (principally NO^+ and $O_2{}^+$) and above 2000 km (principally H^+) are not available. In the present work the assumed ion concentrations are for ion and electron temperatures equal to 1250 K (Hanson 1965), while the neutral concentrations used are for a standard atmosphere with an exospheric temperature of 1900 K (Jacchia 1971). In Table 1 the calculated value of D is included below each value of the neutral constituent concentration; only collisions with the dominant ion at each height level are considered. The ion concentration for H^+ at 2500 km has been obtained by extrapolating Hanson's data. The results in Table 1 show that D is very much less than unity for the dominant neutral constituent at each height value. However, as one would expect, collisions become more significant when the lighter neutral particles collide with heavier ions (e.g. H–O^+ at 800 km altitude, H–He^+ at 1200 km).

The inclusion of viscosity complicates the analysis enormously, since even for an approximated viscous force term the nozzle flow equation is of higher order and cannot be solved analytically. However, the use of dimensional analysis gives a very approximate condition under which viscosity may be neglected. For $v = c$, d^2v/dr^2 may be replaced by c/r_c^2. Comparison of the viscous term with the expansion term leads one to the result obtained by Holzer and Axford (1970), namely that if the characteristic Reynolds number Re ($= m_n n_n c r_c/\eta$) is large compared with unity then viscosity may be neglected. Actually, the condition Re $\gg n$ is obtained from the nozzle flow equation used here (n may assume integral values in the range 1–5).

Table 1. Ion and neutral concentrations in a standard atmosphere

Below each neutral constituent concentration, the calculated value of D (the ratio of the frictional force between the neutral particle and the dominant ion to the gravitational force) is included in square brackets

Height (km)	ion[A]	Particle concentration (cm^{-3}) N_2	O	He	H
400	2×10^5 (O^+)	$1\cdot2\times10^8$	$6\cdot9\times10^8$	—	—
		[$8\cdot7\times10^{-3}$]	[$2\cdot7\times10^{-2}$]	—	—
800	1×10^4 (O^+)	$3\cdot5\times10^5$	$2\cdot4\times10^7$	$2\cdot1\times10^6$	$1\cdot0\times10^3$
		[$4\cdot5\times10^{-4}$]	[$1\cdot3\times10^{-3}$]	[$2\cdot1\times10^{-2}$]	[$9\cdot8\times10^{-1}$]
1200	5×10^3 (He^+)	$1\cdot9\times10^3$	$1\cdot2\times10^6$	$9\cdot9\times10^5$	$8\cdot5\times10^2$
		[$3\cdot2\times10^{-4}$]	[$1\cdot7\times10^{-4}$]	[$2\cdot7\times10^{-3}$]	[$1\cdot5\times10^{-1}$]
1800	2×10^3 (He^+)	—	$2\cdot5\times10^4$	$3\cdot7\times10^5$	$6\cdot7\times10^2$
		—	[$7\cdot4\times10^{-5}$]	[$1\cdot1\times10^{-3}$]	[$3\cdot8\times10^{-2}$]
2500	2×10^3 (H^+)	—	$5\cdot1\times10^2$	$1\cdot4\times10^5$	$5\cdot2\times10^2$
		—	[$1\cdot6\times10^{-4}$]	[$2\cdot7\times10^{-4}$]	[$2\cdot1\times10^{-2}$]

[A] The dominant ion is shown in parentheses.

For tenuous gases, η is given by $Bm_n n_n c l$, where l is the mean free path and the constant B has been given as $0\cdot8$ (CIRA 1961), $0\cdot61$ (Richardson 1961) and $0\cdot53$ (Jeans 1959); a value of $0\cdot6$ is assumed here. Then the condition Re $\gg n$ may be written as $l \ll r_c/0\cdot6n$. For a characteristic length L ($= r_c/0\cdot6n$), this is equivalent to the condition Kn $\ll 1$, where the ratio l/L is known as Knudsons number (Kn). The mean free path calculated by Maxwell is given by $(\sqrt{2}\pi\sigma^2 n_n)^{-1}$ (Copeland and Bennett 1961), where σ is the mean collision diameter, and this does not greatly vary according to the different methods of determination and for the different neutral species (Jeans 1959) (e.g. smallest $\sigma \approx 2\times10^{-8}$ cm for He; largest $\sigma \approx 3\cdot3\times10^{-8}$ cm for N_2). Hence, it is assumed here that $\sigma = 3\times10^{-8}$ cm, so that $l \approx 2\cdot5\times10^{14} n_n^{-1}$ cm. Thus the effect of viscosity may be neglected provided $n_n \gg 1\cdot5\times10^{14} n/r_c$. Representative values of n and r_c are chosen to be 3 and 7×10^8 cm respectively, giving $n_n \gg 6\times10^5$ particles cm^{-3}. Table 1 shows that, for the major neutral constituents, this condition is satisfied up to altitudes of about 1200 km. It cannot be concluded, however, that viscosity is necessarily important above this altitude.

Viscosity is inseparably linked with the mean free path, and the condition Kn $\ll 1$ is usually deemed necessary for the hydrodynamic description to be valid. Hence one may conclude that, where the hydrodynamic description closely resembles the physical behaviour of the gas, viscosity may be neglected. When one proceeds further into the region where the hydrodynamic description becomes less valid (above 1200 km), the above basis for examining the effects of viscosity breaks down. In fact, the effects of viscosity would have to be included in an appropriate kinetic description and could not be examined separately as with the hydrodynamic description.

5. Results

For the linear model, the dependence of the critical distance and temperature boundary values on the various parameters is readily determined from the analytical expressions derived in Section 3. By contrast, results for the exponential model in general require numerical solution. The results are therefore presented in graphical form for the exponential model only. Though no further comment is made on the linear model in this paper, it is extended in Part II to deal with a variable temperature.

Table 2. Assumed data for major particle species

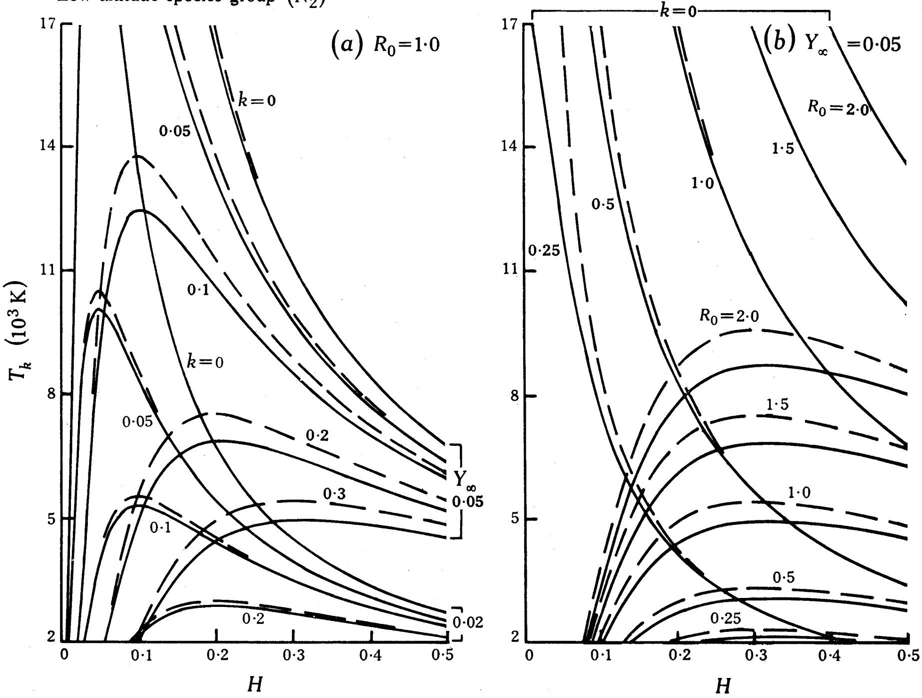
The values in parentheses are the equivalent heights (km) above the Earth's surface. The parameter n is taken to have the value 3

Altitude group	Particle species	R ($\mathrm{cm^2\,s^{-2}\,K^{-1}}$)	a (km)	g_0 ($\mathrm{cm\,s^{-2}}$)	Y_∞ range in exponential model ($K = 0\cdot3$)
Low	N_2	3×10^6	6500 (120)	945	0·02–0·05 (250–445)
	O	$5\cdot2\times10^6$			
High	He	$2\cdot08\times10^7$	6800 (420)	863	0·1–0·3 (1080–2460)
	H	$8\cdot32\times10^7$			

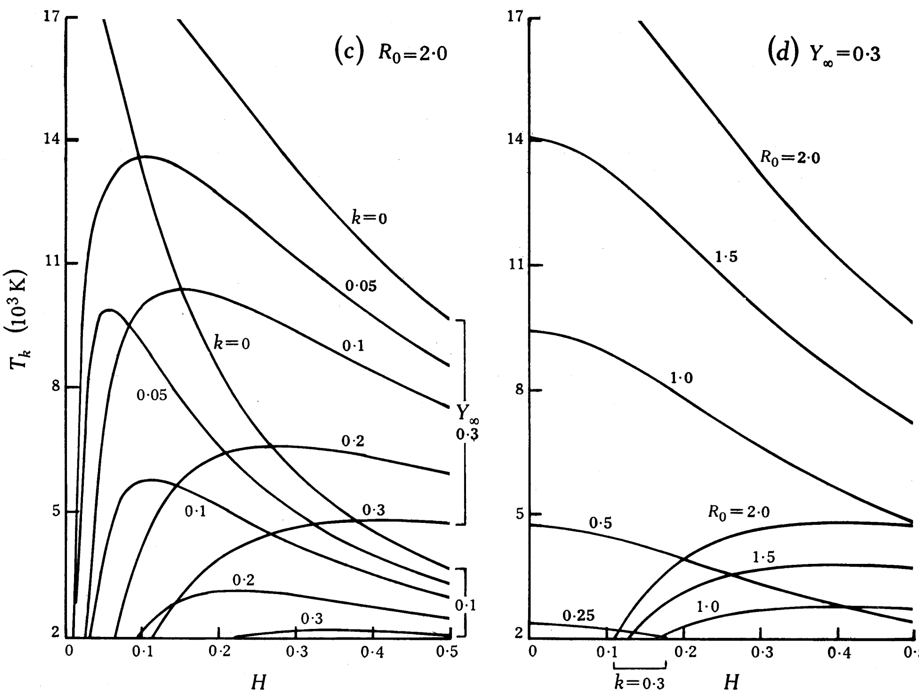
For solutions relating to the Earth's atmosphere, the four major neutral constituents N_2, O, He and H are considered for the parameter values given in Table 2. Hereafter, N_2 and O are referred to as the low altitude group and He and H as the high altitude group. A range of Y_∞ is introduced for each of these groups (Table 2) and values of R_0 up to $2\cdot0$ are considered. The parameter n may reasonably assume integral values from 1 to about 5, so that a central value of 3 is adopted.

The results below depend on c^2 ($= RT$) rather than on R and T separately, so that only one species of each of the low and high altitude groups need be considered, conversion to the other species being achieved by multiplying T by the ratio of the gas constants. Results are therefore presented for N_2 and He, and are equivalent to those of O and H when T is multiplied by $0\cdot5769$ and $0\cdot25$ respectively. Because in each case conversion reduces the temperature, suitably large temperatures are included for the cases of N_2 and He so that results for O and H at geophysically attainable temperatures may be inferred. The lower and upper limits for any gas are assumed to be 2000 and 10000 K respectively. Since T is proportional to $g_0\,a/nR$ (equation (14)), corresponding temperatures may also be obtained for alternative values of n, g_0 or a, the latter two enabling results to be readily related to other planets. The results serve as a reference in determining the conditions necessary for achieving supersonic flow in a locally heated region of an atmosphere.

Low altitude species group (N_2)



High altitude species group (He)



For the low altitude group, Figs 2*a* and 2*b* show the variation of the general temperature boundary value T_k with H for parameters Y_∞ and k, at $R_0 = 1\cdot0$, and R_0 for $k = 0$ and $0\cdot3$, at $Y_\infty = 0\cdot05$; the approximate curves obtained for $Y_\infty \leqslant 0\cdot05$ and $H \geqslant 2Y_\infty$ are also included. Figs 2*c* and 2*d* are the corresponding graphs for the high altitude group, where the approximations are inappropriate and are therefore omitted.

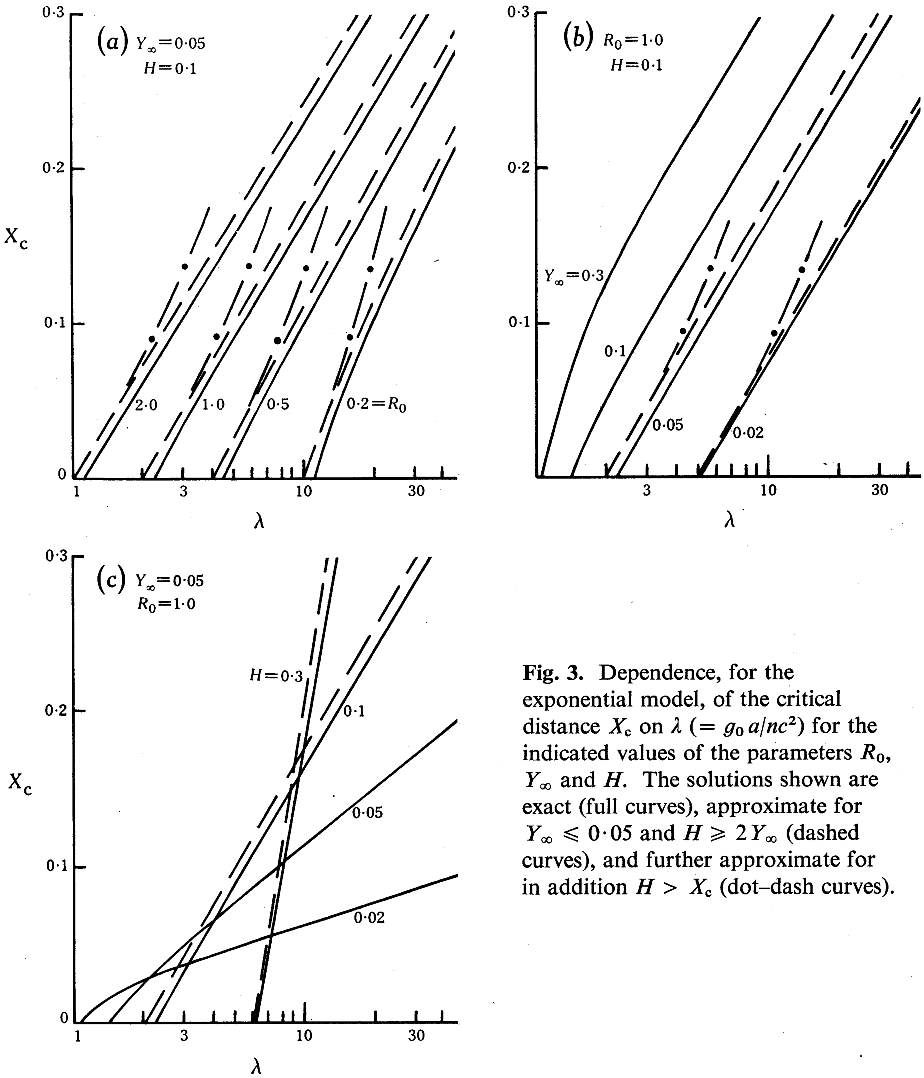


Fig. 3. Dependence, for the exponential model, of the critical distance X_c on λ ($= g_0\,a/nc^2$) for the indicated values of the parameters R_0, Y_∞ and H. The solutions shown are exact (full curves), approximate for $Y_\infty \leqslant 0\cdot05$ and $H \geqslant 2Y_\infty$ (dashed curves), and further approximate for in addition $H > X_c$ (dot–dash curves).

Fig. 2 (*opposite*). Variation, for the exponential model, of the temperature boundary values T_k with H for the low altitude species group (represented by N_2) and the high altitude group (He). The curves are for the indicated values of the parameters R_0, Y_∞ and k; only extreme upper and lower temperature boundaries, which correspond to $k = 0$ and $0\cdot3$ respectively, are given in (*b*) and (*d*). The solutions shown are exact (full curves) and approximate for $Y_\infty \leqslant 0\cdot05$ and $H \geqslant 2Y_\infty$ (dashed curves). The approximations are inappropriate for the high altitude group (*c* and *d*).

The variation of X_c with λ for parameters R_0, Y_∞ and H is shown in Figs 3*a*, 3*b* and 3*c* respectively. The approximate values derived for $Y_\infty \leqslant 0\cdot05$ and $H \geqslant 2Y_\infty$, and in addition for $H > X_c$, are compared where appropriate. The value of X_c is observed to always increase as λ, R_0 and Y_∞ each increases, but not necessarily as H increases.

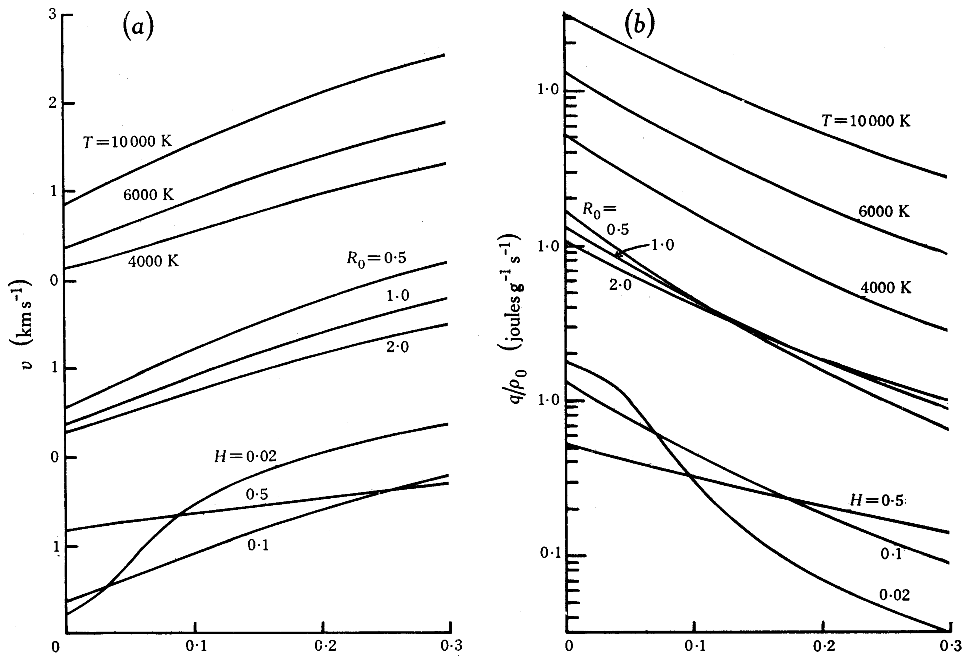


Fig. 4. Effect of variations in the parameters T, R_0 and H on the (*a*) velocity and (*b*) heating profiles for the exponential model when $Y_\infty = 0\cdot05$ (which is applicable to the low altitude species group, and thus the temperatures are for N_2). While each parameter is varied, the other two assume constant values appropriately from $T = 6000$ K, $R_0 = 1\cdot0$ or $H = 0\cdot1$.

The velocity and heating profiles in Fig. 4 are presented in order to illustrate the effect of variations in the parameters T, R_0 and H rather than as a comparison with any experimental data. Even if the temperature and streamline path were accurately known, possible variations in n and R_0 would produce a large range of possible profiles. An increase in n produces a similar effect to an increase in T. The parameter values are limited by the boundary conditions necessary for a critical point to exist within the limits already assumed. Values derived using the approximation for $Y_\infty \leqslant 0\cdot05$ and $H \geqslant 2Y_\infty$, which are not shown, differ from the true values by no more than about 3% for the parameter values assumed in Fig. 4. Generally, results could not be derived when the additional approximation $H > X_c$ was made, because the values of X_c obtained were only an approximation to the true numerical solution of equation (23) consistent with the assumptions $Y_\infty \leqslant 0\cdot05$ and $H \geqslant 2Y_\infty$. In practice, a sign change will occur in $\mathrm{d}v/\mathrm{d}r$ at the critical point if the error in X_c is greater than the computational increment in X there.

6. Conclusions

Two models in which lateral velocity components of neutral gas constituents in an intensely locally heated isothermal region of the atmosphere of a planet or star may become supersonic have been presented. The models, defined by different streamline functions in a vertical plane, each achieve an artificial nozzle throat at which the flow becomes supersonic. The 'linear' function, which spirals above the surface of the planet at constant elevation, achieves the throat by the decrease in gravitational force with distance from the centre of the planet, while the 'exponential' function, which bends over asymptotically towards the horizontal, does so by the decrease in gravitational force component along the streamline as the horizontal is approached.

Conditions under which the effects of viscosity and frictional interactions in the Earth's atmosphere may be neglected from the hydrodynamic equations have also been considered. It has been shown that through the inseparable link between viscosity and the mean free path, viscosity may be neglected provided the hydrodynamic description applies, and that this approximation becomes increasingly less justifiable for altitudes above 1200 km.

The dependence of the critical distance, temperature boundary values, and velocity and heating profiles on variations in the parameters has been fully explored. No comparison of velocity and heating profiles with any experimental data has been made because, even if the temperature and streamline path were accurately known, possible variations in other more uncertain parameters would produce a large range of possible profiles. Results for the critical distance and temperature boundaries serve as a reference in determining the conditions necessary for transition from subsonic to supersonic speeds. The analysis and results have shown that the exponential model approximations are best at low altitudes and for either a sufficiently small initial angle of elevation or when the critical distance is large enough. By suitably scaling the temperature, the results presented for the Earth's atmosphere may be readily related to other species and to other planets.

For the isothermal case considered, it has been shown that a velocity reversal, by which the flow may subsequently become subsonic having passed from the subsonic to supersonic states, is not possible unless a shock front is introduced. It is likely though that the flow would become turbulent before such a reversal.

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