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Supersonic Neutral Winds in an Outer Atmosphere. II* Effects of Variable Temperature

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Abstract

Using the concept of steady nozzle flow along a narrow tube, the effect a variable temperature has on achieving one or two critical points is considered for a neutral gas in a locally heated region of the outer atmosphere of a planet or star. An implicit finite-difference formula, which does not require restrictions on the temperature variation for convergence, is used to solve the general nozzle flow equation. The direct dependence on temperature variation of a single critical point at which the flow becomes supersonic is shown for a model in which the gravitational force component is almost constant. The mathematical possibility of achieving a solution that passes through two critical points, such that in the intermediate region a reversal occurs and the velocity is supersonic, is demonstrated and an example is given.

1. Introduction

In the preceding Part I (Gilbert and Cole 1974; present issue pp. 511–28) two models were considered in which lateral velocity components of neutral gas constituents in an intensely heated isothermal local region of the outer atmosphere of a planet or star may become supersonic. The concept of steady nozzle flow along a narrow tube, as developed for a number of types of wind (for a review, see Holzer and Axford 1970), was applied in the situation where an artificial nozzle throat could be achieved at a critical point. The purpose of the present paper is to consider the effect a variable temperature has on achieving one or two critical points.

The case when the contribution of the variation in the gravitational force component towards achieving a critical point is insignificant compared with the contribution of the temperature variation is first considered for a monotonically increasing velocity (Section 3). Corresponding to the approach in Part I, critical distance boundaries are introduced which result in corresponding temperature boundaries. The dependence of critical distance, temperature boundary values, and velocity and heating profiles on various parameters is also shown.

By carefully utilizing the effect of both gravitational and temperature variations, the possible existence of two critical points is then demonstrated (Section 4). This allows the mathematical possibility of a velocity reversal whereby the velocity decreases to subsonic having first increased from subsonic to supersonic. Such a reversal is usually explained in terms of a shock front at which the velocity drops sharply from supersonic to subsonic within a very short distance (e.g. Fahr 1971). Whether or not a reversal that relies on the occurrence of two critical points is at all physically likely, depends on the reliability of the many assumptions of the nozzle flow dynamics. To

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begin with, it is not improbable that the flow would become turbulent at some stage when supersonic, so that the walls of a narrow tube could no longer be defined in the assumed manner (Part I). Also, the deduced high degree of sensitivity to the various parameters in achieving such a solution suggests that the validity of the steady state assumption is very doubtful. Therefore, the solution is presented here more as a mathematical possibility that could be physically meaningful in very special circumstances, and a parameter study, as in the case of a single critical point, is not considered appropriate.

Given the same assumptions and notation as in Part I, the 'general nozzle flow equation' for a variable temperature T is given by (see equation (5) of Part I)

$$(v-c^2/v)\,\mathrm{d}v/\mathrm{d}r = F(r), \tag{1}$$

where

$$F(r) = nc^2/r - \mathrm{d}(c^2)/\mathrm{d}r - g_0 \cos\Phi\,(1+y/a)^{-2}, \tag{2}$$

or in terms of the dimensionless variables,

$$F(r) = F = \frac{1}{a}\left(\frac{nRT}{R_0+S} - R\frac{\mathrm{d}T}{\mathrm{d}S} - \frac{g_0\, a\cos\Phi}{(1+Y)^2}\right). \tag{3}$$

A monotonically increasing temperature function T is chosen such that

$$T = T_\infty - (T_\infty - T_0)\exp(-S/H_T), \tag{4}$$

where T_∞ is the 'exospheric' temperature approached asymptotically as S tends to infinity, T_0 the temperature at $S = 0$ (or $r = r_0$) and H_T the dimensionless temperature scale height. For a vertical streamline, such a profile is used frequently in describing reference atmospheres for the Earth above about 120 km altitude (CIRA 1965; Jacchia 1971). Although not generally applied to cases of intense heating, the profile (4) is not unrealistic under steady conditions.

When the temperature varies, equation (1) cannot be integrated exactly and a numerical technique must be used. An alternative method of solution to that used by Parker (1964) is considered here in which finite-difference formulae are used. The basic method, including some possible variations, has the advantage that no restrictions on the temperature variation are necessary for convergence.

The net accession of heat q (joules cm^{-3} s^{-1}) from all sources and sinks may be written as (Part I)

$$q = \frac{\rho_0 v_0}{v}\left(\frac{r_0}{r}\right)^n \left(c^2\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{v}{\gamma-1}\frac{\mathrm{d}(c^2)}{\mathrm{d}r}\right). \tag{5}$$

When the velocity decreases following the reversal in the case of a solution passing through two critical points, q will ultimately become negative for the temperature defined in equation (4), thus indicating an energy sink.

2. Solution of General Nozzle Flow Equation

Because c is a function of r, equation (1) cannot be integrated by separating the variables r and v. To numerically solve the equation, Parker (1964) has used an iterative approximation procedure whereby an initial approximation is derived by neglecting the term $v\,dv/dr$ when $v < c$, or the term $(c^2/v)\,dv/dr$ when $v > c$. A second approximation is obtained by replacing the previously neglected term by the value derived using the initial approximation, and the iterative cycle is then continued until convergence. Parker has proved that convergence is always achieved when the temperature is constant or decreases outwardly along the streamline as r increases. It is desired here, however, to obtain solutions for a temperature profile in which the temperature increases with r (see equation (4)). The initial approximation is most in error near $v = c$, where the terms $v\,dv/dr$ and $(c^2/v)\,dv/dr$ are comparable, so that solutions that are required to pass through two critical points may present problems in the intermediate region, when v could remain fairly close to c throughout. An alternative method is therefore presented which uses a finite-difference representation of dv/dr.

Given the velocity v_i at distance r_i, it is desired to calculate v_{i+1} at r_{i+1}, the next incremental value in r, where $\Delta r_i = r_{i+1} - r_i$. The initial values are at the critical point (r_c, v_c). An explicit 'forward-difference' formula, which is computationally simple, is given by

$$(v_{i+1} - v_i)/\Delta r_i = F(r_i)/(v_i - c_i^2/v_i)\,, \tag{6}$$

where $c_i = c$ at $r = r_i$. Unfortunately, at the first increment past the critical point the right-hand side is undefined, since $(r_i, v_i) = (r_c, v_c)$.

To overcome the problem with the explicit formula, an implicit 'backward-difference' formula is therefore introduced such that

$$(v_{i+1} - v_i)/\Delta r_i = F(r_{i+1})/(v_{i+1} - c_{i+1}^2/v_{i+1})\,. \tag{7}$$

This formula results in the following cubic equation in v_{i+1},

$$v_{i+1}^3 - [v_i]v_{i+1}^2 - [c_{i+1}^2 + \Delta r_i F(r_{i+1})]v_{i+1} + [v_i c_{i+1}^2] = 0\,, \tag{8}$$

which may be solved numerically, taking care to select the appropriate root (see Appendix). Having obtained a solution at the first increment, the forward-difference formula could then be used. Given two starting values, a 'central-difference' formula could also be used, where the left-hand side of equation (6) is replaced by an appropriate function of v_{i+1}, v_i, v_{i-1}, Δr_i and Δr_{i-1}, which reduces to $(v_{i+1} - v_{i-1})/2\Delta r_i$ when $\Delta r_i = \Delta r_{i-1}$.

Comparisons of accuracy between the above methods showed that for almost constant dv/dr there was very little difference, but that the larger the change in dv/dr, the smaller the increment required for the modified explicit methods to achieve the same accuracy as for the implicit method (in extreme cases, a quarter of the increment size was necessary). Almost no difference was observed between the two modified explicit methods. After first checking that the solution using the implicit method converged to the exact solution obtained for a special isothermal case (Part I), the convergence of the method was established experimentally by achieving convergence after successively reducing the increment size. Double precision was used in the computer program to minimize the truncation error, especially at the first increment when the difference of two almost equal numbers occurs in the denominator of the right-hand side of equation (7).

3. Single Critical Point

(a) Analysis

In order to demonstrate how a critical point may be achieved by a suitably varying temperature profile, a streamline function is chosen in which the contribution of the variation in the gravitational force component towards establishing an artificial nozzle throat is generally insignificant compared with the contribution of the temperature variation. Hence a 'constant-Φ' function is chosen where $\Phi = \phi$ (see Part I). Then the exact result

$$S = Y \sec\phi \tag{9}$$

is obtained and, when the condition $Y \ll 1$ is imposed, the exponential result given by equation (16) of Part I approximates to the linear form $Y = X\cot\phi$. Substitution of the temperature profile (4) into equation (3) then gives, with the help of equation (9),

$$F = \frac{1}{a}\left\{\frac{nRT_\infty}{R_0+S} - \frac{g_0\, a\cos\phi}{(1+S\cos\phi)^2} - R(T_\infty - T_0)\left(\frac{n}{R_0+S} + \frac{1}{H_T}\right)\exp\left(-\frac{S}{H_T}\right)\right\}. \tag{10}$$

The critical distance is found on solving this equation when $F = 0$, and the result may be written as

$$\frac{n}{R_0+S_c} - \frac{\Lambda}{(1+S_c\cos\phi)^2} = \left(1-\frac{T_0}{T_\infty}\right)\left(\frac{n}{R_0+S_c} + \frac{1}{H_T}\right)\exp\left(-\frac{S_c}{H_T}\right), \tag{11}$$

where $\Lambda = (g_0\, a/RT_\infty)\cos\phi$. In general, equation (11) cannot be solved analytically and a numerical procedure involving the method of false position is therefore employed.

As in Part I, a general critical distance boundary value k, corresponding to Y_c, is introduced such that $0 \leqslant k \leqslant K$, where K is a specified constant ($K \ll 1$ for consistency with the condition $Y \ll 1$). The value k represents the extreme lower and upper boundaries at $k = 0$ and K respectively and may be either a lower or upper boundary between these extremes.

An expression for the corresponding boundary value T_k of T_0 is required. For a monotonically increasing velocity, a lower boundary at k is equivalent to the condition $F < 0$ at $Y = k$ (Part I), which from equation (10) results in an upper boundary for T_0. It may similarly be shown that an upper boundary at k corresponds to a lower boundary for T_0. The general boundary value T_k is therefore found on substituting $S_c = k\sec\phi$ in equation (11), which gives

$$T_k = \frac{T_\infty}{B}\left(\frac{\Lambda}{(1+k)^2} + B - \frac{n}{R_0 + k\sec\phi}\right), \tag{12}$$

where

$$B = \left(\frac{n}{R_0 + k\sec\phi} + \frac{1}{H_T}\right)\exp\left(-\frac{k\sec\phi}{H_T}\right).$$

In order to obtain an analytical expression for the critical distance, the following approximations may be made. Since $S_c\cos\phi \leqslant K$ for $K \ll 1$, the second term on the left-hand side of equation (11) may be reasonably approximated by $-\Lambda$, which is equivalent to assuming a constant gravitational force component. S_c may also be

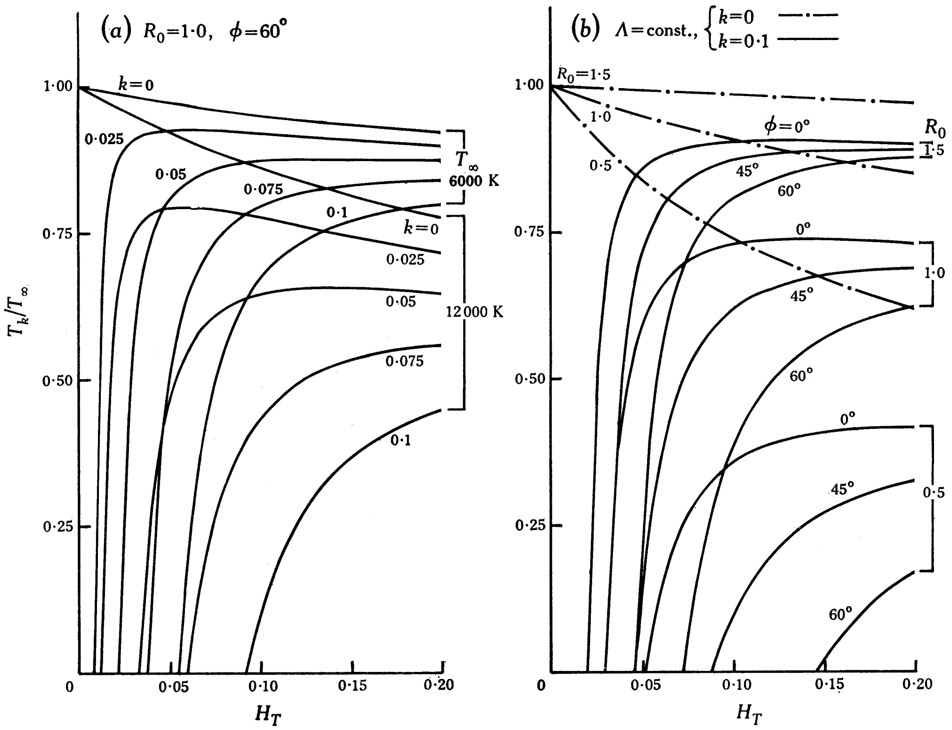

Fig. 1. Variation of the temperature boundary values with H_T for the case of a single critical point. In this figure and in Figs 2 and 3 the temperatures are for helium. The curves are for the indicated values of the parameters T_∞, k, R_0 and ϕ; only extreme upper and lower boundaries for T_k/T_∞, which correspond to $k = 0$ and $0 \cdot 1$ respectively, are shown in (*b*). Also, as ϕ is varied in (*b*) so is T_∞ such that $\Lambda = (g_0\, a/RT_\infty)\cos\phi$ remains constant (at $1 \cdot 76334$, which corresponds to $T_\infty = 8000$ K at $\phi = 60°$); thus T_k/T_∞ is independent of ϕ when $k = 0$.

neglected in the terms $n/(R_0+S_c)$ provided that R_0 is sufficiently large; the condition $S_c \leqslant 2R_0 K$ ensures that the relative error so introduced is no larger than that in the term already approximated. Using these approximations, the critical distance S_c is given by

$$S_c = H_T \ln\left(\frac{(1-T_0/T_\infty)(n/R_0+1/H_T)}{n/R_0-\Lambda}\right), \tag{13}$$

and the corresponding approximation for T_k is

$$T_k = \frac{T_\infty}{D}\left(\frac{\Lambda}{(1+k)^2}+D-\frac{n}{R_0}\right), \tag{14}$$

where

$$D = \left(\frac{n}{R_0}+\frac{1}{H_T}\right)\exp\left(-\frac{k\sec\phi}{H_T}\right).$$

As H_T tends to zero, the approximation for S_c is inappropriate; this may be observed from equation (11) where the right-hand side tends to zero, so that the two terms on the left-hand side cannot be approximated as above if a solution for S_c is required.

(b) Results

Consistent with the presentation of results in Part I, solutions are given here only for helium in the Earth's atmosphere (i.e. from Table 2 of Part I, $R = 2{\cdot}08 \times 10^7$ $\mathrm{cm^2\,s^{-2}\,K^{-1}}$, $a = 6800$ km, $g_0 = 863\ \mathrm{cm\,s^{-2}}$ and $n = 3$). The results may be readily related to other species and to other planets by suitably scaling the temperature (e.g. for hydrogen the temperature should be multiplied by $0{\cdot}25$).

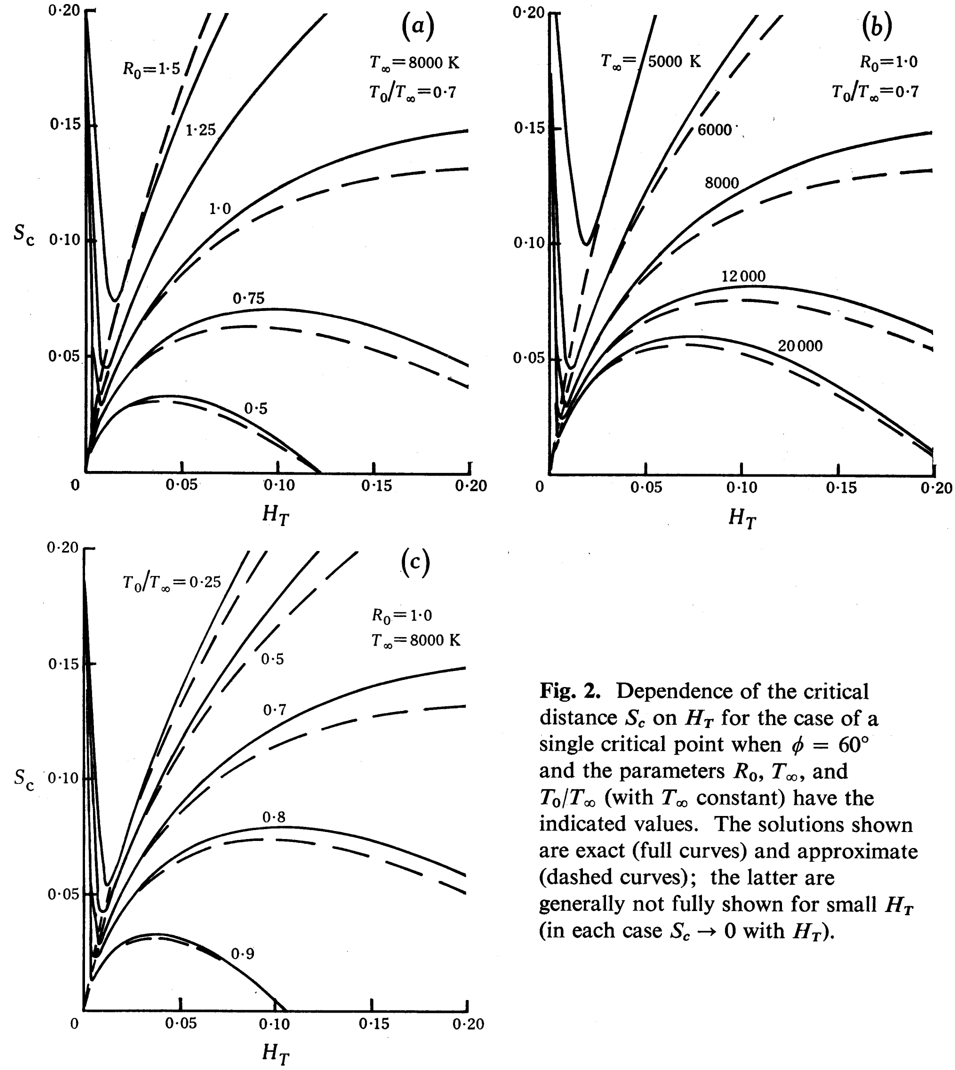

Fig. 2. Dependence of the critical distance S_c on H_T for the case of a single critical point when $\phi = 60°$ and the parameters R_0, T_∞, and T_0/T_∞ (with T_∞ constant) have the indicated values. The solutions shown are exact (full curves) and approximate (dashed curves); the latter are generally not fully shown for small H_T (in each case $S_c \to 0$ with H_T).

For $K = 0{\cdot}1$, Figs 1*a* and 1*b* show the variation of T_k/T_∞ with H_T for parameters T_∞ and k, at $R_0 = 1{\cdot}0$ and $\phi = 60°$, and R_0 and ϕ for $k = 0$ and $0{\cdot}1$. When $k = 0$, the only dependence of T_k/T_∞ on ϕ is through the inclusion of $\cos\phi$ in Λ, and hence as ϕ is varied in Fig. 1*b* so is T_∞ such that Λ remains constant.

At $\phi = 60°$, the variation of S_c with H_T for parameters R_0, T_∞ and T_0/T_∞ (with T_∞ constant) is shown in Figs 2*a*, 2*b* and 2*c* respectively. The approximate values given by equation (13) are compared with the exact values obtained by numerically solving equation (11), and are observed to be in good agreement except, as predicted, when H_T tends to zero.

The effect of parameter variations on the velocity and heating profiles has been shown in Fig. 4 of Part I when the temperature is constant, so that only the effects of the additional parameters T_0, T_∞ and H_T when the temperature varies are illustrated in Fig. 3 here. As discussed in Part I, comparisons with any experimental data are inappropriate at present. The values derived using the approximations, which are not shown, are in error of the true values by no more than about 3% for the parameter values assumed in Fig. 3.

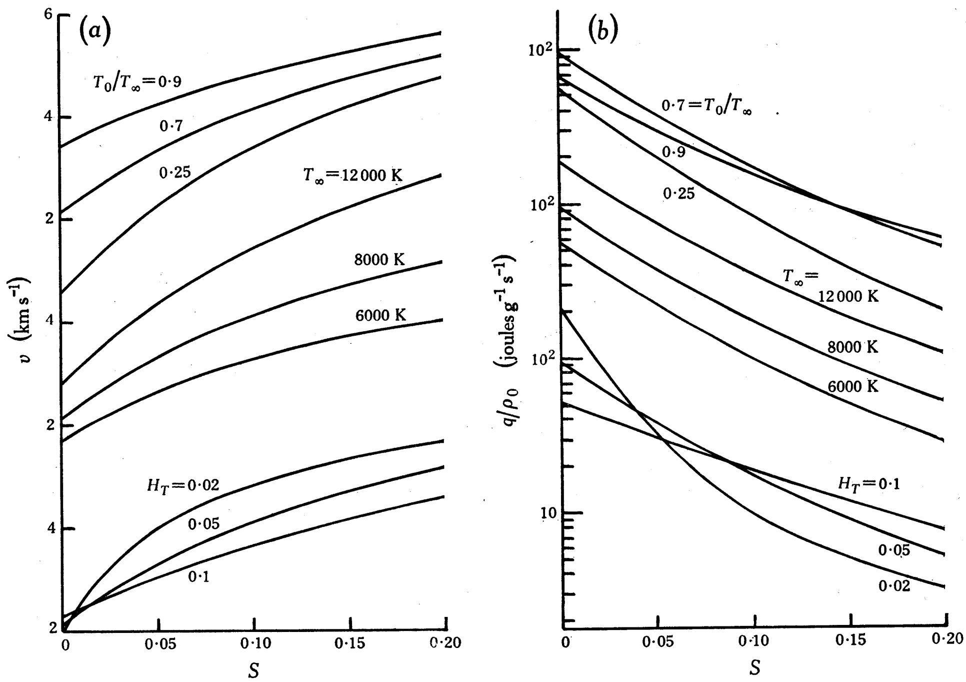

Fig. 3. Effect of variations in the parameters T_0/T_∞ (with T_∞ constant), T_∞ and H_T on the (*a*) velocity and (*b*) heating profiles for the case of a single critical point when $\phi = 60°$ and $R_0 = 1$. While each parameter is varied, the other two assume constant values appropriately from $T_0/T_\infty = 0\cdot7$, $T_\infty = 8000$ K or $H_T = 0\cdot05$. In (*b*) the specific heat ratio γ is taken to be $1\cdot67$.

4. Two Critical Points

It is mathematically possible to achieve a large number of critical points by suitably choosing F and the characteristic thermal velocity c. Generally, only one is considered in physical cases, so that an investigation into the conditions necessary for more than an additional critical point, though mathematically interesting, is not discussed here. For a solution which is supersonic only between the two critical points, the case of a velocity reversal while supersonic is considered. If c were to increase sufficiently rapidly in this region, it would be possible for v to still be increasing monotonically while becoming subsonic at the second critical point, a reversal then being achieved while subsonic.

(*a*) *Existence of Two Critical Points*

To achieve two critical points satisfying the above condition then, it is necessary that the equation $F(r) = 0$ have three roots, r_1, r_2 and r_3 say, such that $r_0 < r_1 < r_2 < r_3$ and $F'(r_1) > 0$, $F'(r_2) < 0$ and $F'(r_3) > 0$ (Part I), where the prime denotes differentiation with respect to r. The critical points are at r_1 and r_3, while r_2 is a stationary point where the reversal occurs. A function $F(r)$ possessing these properties is not possible for the models considered in Part I, where the temperature is constant, or for the model in Section 3 above, in which the temperature varies while the gravitational force component remains almost constant. However, by considering the case when both the temperature and the gravitational force component vary significantly, it is possible to obtain such a function. Other ways of doing so include the use of a more complicated temperature function or the addition of other forces such as friction (Banks and Holzer 1968). In practice, it was found to be very difficult to choose a set of parameter values that would produce such a function, so that the following semi-analytical method, which could be applied only in special cases, was adopted. It is emphasized here that the purpose in this paper is to illustrate by an example how two critical points may be achieved, rather than to present a general parameter study.

The method for obtaining a suitable function $F(r)$ consists of first choosing values of r_2 and r_3 and then determining analytically the necessary parameters for a constant temperature such that $F'(r_2) < 0$ and $F'(r_3) > 0$. In place of the conditions on the derivatives, it is sufficient to show that $F(r_{23}) < 0$, where $r_{23} = \frac{1}{2}(r_2+r_3)$. By then setting T_∞ equal to the calculated temperature and adjusting T_0 and H_T, the additional root r_1 may be created. The roots r_2 and r_3 need not be affected, because T_0 and H_T may be chosen such that as r approaches r_2 the temperature becomes almost constant. The analytical part of the method is described below for the constant-Φ or 'linear' model when $Y \ll 1$ and for the 'exponential' model when $Y_\infty \leqslant 0{\cdot}05$ and $H \geqslant 2Y_\infty$ (see Part I).

For the linear model, using the exact result for S in equation (9) when the temperature is constant, equation (10) may be written as

$$F = \frac{1}{a}\left(\frac{nRT}{R_0 + Y\sec\phi} - \frac{g_0\, a\cos\phi}{(1+Y)^2}\right). \tag{15}$$

Let subscripts 2, 3 and 23 for Y (for X in the exponential model) correspond to the value at r_2, r_3 and r_{23} respectively. For roots at r_2 and r_3 (i.e. when $F(r_2) = F(r_3) = 0$),

$$\lambda(R_0\cos\phi + Y_2) = (1+Y_2)^2 \tag{16a}$$

and

$$\lambda(R_0\cos\phi + Y_3) = (1+Y_3)^2, \tag{16b}$$

where $\lambda = g_0\, a/nRT$. The solution of equations (16) and the condition $F(r_{23}) < 0$ result in

$$R_0 = \frac{\{Y_3(1+Y_2)^2 - Y_2(1+Y_3)^2\}\sec\phi}{(1+Y_3)^2-(1+Y_2)^2}, \qquad \lambda = \frac{(1+Y_2)^2}{R_0\cos\phi + Y_2}$$

and

$$(1+Y_{23})^2 < \lambda(R_0\cos\phi + Y_{23}).$$

The exponential function, which 'bends over' asymptotically towards the horizontal in an exponentially decaying manner, is defined in Section 3*b* of Part I as

$$Y = Y_\infty\{1-\exp(-X/H)\}.$$

The assumptions $Y_\infty \leqslant 0\cdot05$ and $H \geqslant 2Y_\infty$ discussed in Part I allow the approximations

$$S \approx X \qquad \text{and} \qquad \cos\Phi \approx (Y_\infty/H)\exp(-X/H).$$

When the temperature is constant, equation (10) may be written as

$$F = a^{-1}\{nRT/(R_0+X) - g_0\, a(Y_\infty/H)\exp(-X/H)\}.$$

For roots at r_2 and r_3,

$$\lambda(Y_\infty/H)(R_0+X_2) = \exp(X_2/H) \tag{17a}$$

and

$$\lambda(Y_\infty/H)(R_0+X_3) = \exp(X_3/H). \tag{17b}$$

Given R_0, the solution of equations (17) and the condition $F(r_{23}) < 0$ result in

$$H = \frac{X_3-X_2}{\ln\{(R_0+X_3)/(R_0+X_2)\}}, \qquad \lambda = \frac{(H/Y_\infty)\exp(X_2/H)}{R_0+X_2}$$

and

$$\lambda(Y_\infty/H)(R_0+X_{23}) > \exp(X_{23}/H).$$

An example of the use of this method is illustrated in Fig. 4*a* for an exponential model. The values $X_2 = 0\cdot1$, $X_3 = 0\cdot2$, $R_0 = 1$, and $Y_\infty = 0\cdot05$ are specified. When the temperature is constant, the analysis yields $H = 1\cdot1493$, $\lambda = 22\cdot7955$, and $F(r_{23}) = -0\cdot000823$. For N_2 in the Earth's atmosphere (i.e. from Table 2 of Part I, $R = 3\times10^6\ \mathrm{cm^2\,s^{-2}\,K^{-1}}$, $a = 6500$ km, $g_0 = 945\ \mathrm{cm\,s^{-2}}$ and $n = 3$), the temperature is 2994 K. By putting $T_\infty = 2994$ K, $T_0 = 2990$ K and $H_T = 0\cdot01$, the root X_1 $(= 0\cdot021)$ is created.

(*b*) *Solutions Passing Through Two Critical Points*

In general, a solution does not pass through both critical points. To demonstrate this, consider the solution for v at r_2 (that is, v_2). Since there are two known critical values, v_2 may be derived in two ways by integrating from each of the critical points. Ignoring the variation in c on the left-hand side of equation (1), so that the equation may be integrated exactly as in the isothermal case, the solution is given by

$$v^2 - \langle c^2\rangle \ln(v^2) = \langle c^2\rangle(1-\ln\langle c^2\rangle) + 2\int_{r_c}^{r} F(r)\,\mathrm{d}r,$$

where $\langle c^2\rangle$ is the mean value of c^2. Equating the solution at r_2 by integrating from both r_1 and r_3,

$$\int_{r_1}^{r_3} F(r)\,\mathrm{d}r = 0, \tag{18}$$

which in Fig. 4*a* requires the shaded areas to be equal. In general, this condition would have to be specified, thus losing an additional degree of freedom in the choice of the parameters. Because of its arbitrariness, r_0 is the most convenient parameter to alter. The limits of integration r_1 and r_3 in equation (18) are themselves functions of most parameters, including r_0, so that an analytical solution is not possible even for the various approximations presented in Part I and for an assumed mean value of c in the definition of $F(r)$. For a solution to pass through both critical points therefore, r_0 must be determined numerically using an iterative method, such as the following.

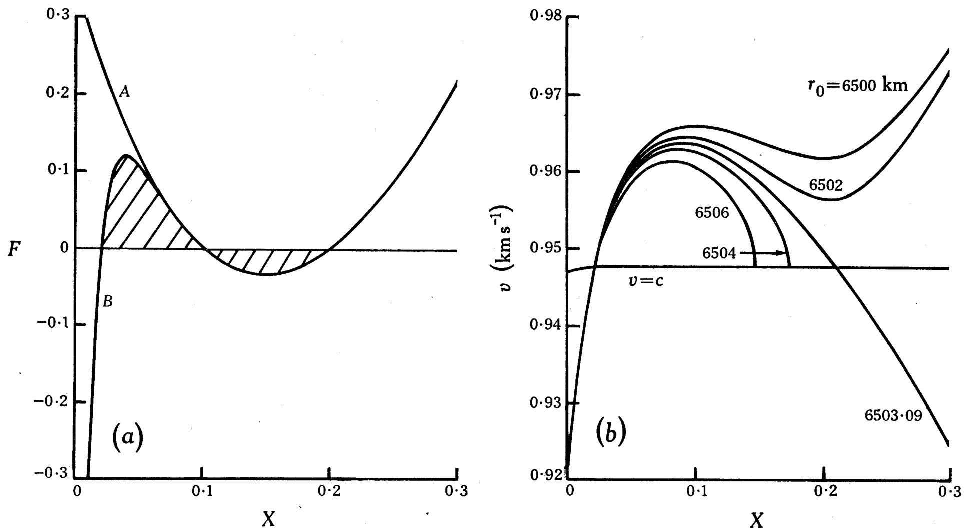

Fig. 4. Example of the method for the case of two critical points using an approximation of the exponential model, showing the variation with X of (*a*) the function F and (*b*) the velocity v. The temperatures are for N_2. In (*a*): Curve A corresponds to the function F given by the analysis for roots at $X = 0\cdot1$ and $0\cdot2$ when $R_0 = 1$, $Y_\infty = 0\cdot05$ and the temperature is constant. This analysis results in the values $H = 1\cdot1493$ and $T = 2994$ K. Curve B is obtained by then putting $T_\infty = 2994$ K, $T_0 = 2990$ K and $H_T = 0\cdot01$ in a variable temperature atmosphere. The shaded areas represent the integral in equation (18) and should be equal (allowing for an error introduced by assuming a mean value for c^2) for a solution to pass through both critical points. In (*b*): Four values of r_0 that give solutions either side of the second critical point and also the value for the solution to pass through both critical points are specified on the curves.

Two values of r_0 are found that 'straddle' the second critical point (e.g. $r_0 = 6502$ and 6504 km in Fig. 4*b*). The mean value is calculated and replaces one of the original values such that the critical point is still straddled. The process is continued until a solution is found that is sufficiently close to the second critical point. For $r \geqslant r_3$, where $v \leqslant c$, the solution is continued by integrating separately from the second critical point. Fig. 4*b* shows four values of r_0 that give solutions either side of the second critical point, and the value (6503·09 km) such that the solution passes through both critical points.

5. Conclusions

The effect a variable temperature has on achieving one or two critical points has been considered. The direct dependence of a single critical point on temperature variation has been shown for the case of a 'linear' model when the gravitational force component is almost constant. By making use of variations in both the temperature and gravitational force component, the possible mathematical existence of two critical points has been demonstrated. To achieve two critical points such that in the intermediate region a reversal occurs and the velocity is supersonic, a semi-analytical method has been proposed. For a solution to then pass through both critical points, an iterative procedure has been suggested to determine the necessary adjustment to one of the parameters. Because of the doubt on the validity of the steady nozzle flow assumptions, the solution has been presented more as a mathematical possibility that could, under special circumstances, be physically meaningful.

A new numerical method of solving the general nozzle flow equation for any temperature profile has been developed. Unlike Parker's (1964) iterative method, it does not require restrictions on the temperature variation for convergence. The basic method uses an implicit finite-difference formula, but a computationally simple explicit formula may be used after the first one or two steps.

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Appendix. Solution of Implicit Formula

In order to determine the velocity v_{i+1} at the next incremental value of r, namely r_{i+1}, using equation (8), which is a cubic in v_{i+1}, the appropriate root must first be isolated. The equation may be written as

$$f(v_{i+1}) = 0,$$

where

$$f(v_{i+1}) = v_{i+1}^3 - [v_i]v_{i+1}^2 - [c_{i+1}^2 + \Delta r_i F(r_{i+1})]v_{i+1} + [v_i c_{i+1}^2].$$

Since $f(0)$ is positive and $f(\infty)$ tends to infinity, there are only two possible positive real roots, say v_l and v_u, where $v_l \leqslant v_u$. These two roots are equal when $v_{i+1} = c_{i+1}$, and then

$$f(c_{i+1}) = -\Delta r_i F(r_{i+1}) c_{i+1}^2.$$

Omitting the subscript $i+1$ now, in the region where v is increasing or decreasing monotonically through the critical point, $f(c) \leqslant 0$ (equal when $r = r_c$), since, for

$r < r_c$, $F(r)$ is negative (Part I) and Δr_i is also negative, while, for $r > r_c$, $F(r)$ and Δr_i are positive. Hence $v_l < c < v_u$, so that v_l is the required root when $v < c$ and v_u the required root when $v > c$. However, following a velocity reversal $F(r)$ changes sign while Δr_i does not, so that $f(c) \geqslant 0$. Hence $v_l < v_u < c$ when $v < c$, and $c < v_l < v_u$ when $v > c$; for a continuous velocity profile, the same root must be selected as before the reversal.

The solution is found by using the method of false position when $v = c$ is one fixed point, and the other, such that $f(v)$ is opposite in sign to $f(c)$, is located by successively incrementing v in the appropriate direction.

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