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Raman Processes in the Helium Atom in Intense Fields

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Abstract

Using the hydrogenic model for the helium atom, the amplitude of transition from states He(1s) to He(1s, *nl*) by absorption of N photons of an intense field together with the emission of one Raman photon is evaluated. From the general expression for the transition amplitude, the particular case of transition from He(1s) to He(1s, 2p) is considered. The 'reduced' transition amplitude is plotted against the number of photons N involved and against the intensity parameter y separately. It is found that the s wave contributes maximaly to the transition amplitude. An important feature of the calculations is the appearance of nonlinear behaviour at high intensity. The dominance of higher order processes over lower ones at high intensity is also found.

1. Introduction

With the advent of lasers a considerable amount of theoretical work has been done on multiphoton processes, such as multiphoton ionization, bound–bound excitation and Raman emission and absorption processes. Raman scattering of light is one of the most interesting interactions between electromagnetic radiation and matter and was observed for the first time by Daniltseva *et al.* (1963) in crystals using ruby and helium–neon lasers. The anti-Stokes Raman scattering of ruby laser light from Cr^{3+} ions in ruby has been observed by Ducuing *et al.* (1969). Braunlich and Lambropoulos (1970) and Braunlich *et al.* (1972) have observed anti-Stokes Raman scattering from metastable deuterium molecules. The results obtained are in good agreement with the theoretical calculations of Zernick (1963, 1964).

Saslow and Mills (1969) have calculated the cross section of Raman scattering of light from hydrogen-like atoms, while Rapoport and Zon (1968) have obtained the cross section in closed form in terms of hypergeometric functions. Many calculations have been made using the perturbation technique (Zernick 1964; Keldysh 1965; Gold and Bebb 1965; Bebb and Gold 1966; Gontier and Trahin 1968; Chang and Tang 1969; Karule 1971). The main difficulty found with this technique is in summing over intermediate states. The method of summation described by Dalgarno and Lewis (1956) has been used by Zernick (1964, 1968) in his calculations of the transition rate of the metastable 2s state, by Gontier and Trahin (1968, 1971*a*, 1971*b*) in their calculations of bound–bound transitions and spontaneous Raman emission in atomic hydrogen, and by S. N. Biswas (personal communication) to calculate the transition probability for two spontaneous Raman photons in bound–bound transitions in atomic hydrogen. However, the Dalgarno and Lewis method of summing over intermediate states is only useful where small numbers of photons take part in the transitions, because for many photons it becomes very difficult to solve the resulting

set of coupled differential equations. For multiphoton processes the method of Gold and Bebb (1965), which replaces the summation by a 'mean' frequency term, is quite simple. At high intensity, however, all calculations by perturbation techniques become very complicated, and also doubtful (Reiss 1970, 1971), and in this case the method of Reiss is very useful. Essentially his method consists of applying a unitary transformation to approximately remove the incident electromagnetic field from the problem. The accuracy of the method increases with the number of photons involved in the transition. Two of us (Thareja and Haque 1974) have used Reiss's method to calculate bound–bound transitions in the hydrogen atom, and the results obtained are in good agreement with those of Gontier and Trahin (1968). The method has also been applied by us to various problems of multiquantum transitions (Man Mohan and Thareja 1972, 1973*a*, 1973*b*, 1973*c*); in the case of Raman processes in atomic hydrogen, it has been found that the transition amplitude for two Raman photons is enhanced over that for a single photon (Man Mohan and Thareja 1972).

In Section 2 of the present paper we give the formalism for the helium atom using the momentum translation approximation. The main effect of this approximation is to reduce the influence of the incident electromagnetic field.

In Section 3, using the hydrogenic model for the helium atom we evaluate the amplitude of transition from states He(1s) to He(1s, nl), accompanied by the emission of a Raman photon of frequency ω' and vector potential A', by absorption of N photons of an intense field of vector potential A and frequency ω. The value of N is such that the energy relation

$$N\omega \pm \omega' = E_{1s} - E_{nl}$$

is satisfied. An interesting result is that the transition probability is found to increase with the initial values of the intensity but at high intensity a nonlinear effect is observed. It is also found that the s wave contributes maximaly to the transition probability. The particular case of transition from He(1s) to He(1s, 2p) is calculated from the general expression for the transition amplitude.

In Section 4 we discuss the numerical procedure for obtaining a value of the transition amplitude. The expression for a transition from He(1s) to He(1s, 2p) involves an infinite sum which leads to divergent results for high intensity, and in order to sum the series in a closed form we make use of the Euler transformation method (Morse and Feshbach 1953). In the last section we discuss the physical significance of the numerical results obtained.

2. Formalism

We shall use the momentum translation approximation (Reiss 1970). This is an analytical method which is valid for transitions caused in bound systems by an external plane wave electromagnetic field subject to the conditions

$$(\omega/E) \ll 1, \qquad eaa_0(\omega/E) \ll 1, \tag{1}$$

where ω is the frequency of the electromagnetic field, E is a characteristic energy of the bound system, a is the amplitude of the vector potential A of the electromagnetic field and a_0 is the size of the bound system.

The s-matrix element for the transition can be evaluated by using the usual perturbation theory,

$$(s-1)_{fi} \equiv \tau_{fi} = -i \int dt \exp\{i(E_f - E_i)t\} (\Phi_f, H' \exp\{ieA.(x_1+x_2)\} \Phi_i) \quad (2)$$

$$= -2\pi i \sum_{N=-\infty}^{\infty} \delta(E_f - E_i + N\omega) T_{fi}^{(N)}, \quad (3)$$

where

$$T_{fi}^{(N)} \equiv \mathcal{T}(\Phi_f, H' \exp\{ieA.(x_1+x_2)\} \Phi_i) \propto \exp(\pm iN\omega t), \quad (4)$$

$\mathcal{T}(z)$ denoting the term in z that has the subsequent proportionality. After applying the commutator theorem we get

$$(\Phi_f, H' \exp\{ieA.(x_1+x_2)\} \Phi_i) = (E_i - E_f)M, \quad (5)$$

where

$$M = (\Phi_f, \exp\{ieA.(x_1+x_2)\} \Phi_i).$$

The transition probability per unit time ω_{fi} between an initial state Φ_i and a final state Φ_f is given by

$$\omega_{fi} = 2\pi \sum_N |T_{fi}^{(N)}|^2 \delta(E_f - E_i + N\omega). \quad (6)$$

Now when instead of one intense field, two fields, one arbitrarily intense and the other weak, are present the T-matrix becomes

$$T_{fi}^{(N,1)} \equiv \mathcal{T}(\Phi_f, \exp\{ieB.(x_1+x_2)\} \Phi_i) \propto \exp(\pm iN\omega t), \quad (7)$$

where $B = A + A'$, the vector potential A' being that of a weak Raman field of frequency ω'. The corresponding transition probability ω_{fi} is

$$\omega_{fi} = 2\pi \sum_N |T_{fi}^{(N,1)}|^2 \delta(E_f - E_i + N\omega \pm \omega'). \quad (8)$$

The contributions to ω_{fi} come only from those values of N which satisfy the relations

$$\begin{aligned} E_f - E_i &= N\omega - \omega' \quad (N \text{ odd}) \quad \text{for absorption from the initial state,} \\ &= (N-1)\omega + \omega' \quad (N \text{ even}) \quad \text{for emission from the final state,} \end{aligned}$$

ω' being the frequency of the Raman photon, as defined above.

3. Generalized Transition Matrix Element

Hydrogen atoms and hydrogenic ions are unique in a number of important respects. They are the only atomic systems for which exact wavefunctions are known and also the only ones that do not exhibit two-electron processes such as double excitation, double ionization and simultaneous ionization and excitation. In the hydrogenic model of the helium atom, it is assumed that on absorption of incident radiation one of the electrons remains in the 1s state while the other is excited to higher states or even to the continuum, depending on the energy of the incident radiation, i.e. on the number of photons absorbed. Since the energy difference between the initial and

final levels is much greater than the energy of a single photon, the process of excitation can only be effected by multiphoton transition.

When the helium atom is present in an intense external electromagnetic field of long wavelength, $\omega a_0 \ll 1$, where ω is the frequency of the intense field and a_0 the Bohr radius, the matrix element governing the transition from the initial 1s state to any final state (nl) on absorption of N photons of vector potential A, accompanied by the emission of a single Raman photon of frequency ω' and vector potential A', is given by

$$M = \left(\Phi_f(r_1, r_2), \exp\{ie(A+A').r\}\, \Phi_i(r_1, r_2)\right) + (1 \leftrightarrow 2). \tag{9}$$

The ground state wavefunction is symmetric with respect to interchange of r_1 and r_2, as is the wavefunction of the final states. Thus

$$M = 2\left(\Phi_f(r_1, r_2), \exp\{ie(A+A').r\}\, \Phi_i(r_1, r_2)\right). \tag{10}$$

The degree of complexity of the ensuing calculations depends on the choice of wavefunctions. The simplest ground state wavefunction for helium is the one-parameter variational form obtained by Hylleraas (1929),

$$\Phi_i(r_1, r_2) = \phi_{1s}(\alpha \mid r_1)\, \phi_{1s}(\alpha \mid r_2), \qquad \alpha = 1{\cdot}6875, \tag{11}$$

where $\phi_{1s}(\alpha \mid r)$ is the ground state wavefunction of a hydrogenic system with nuclear charge α.

Eckart (1930) has shown that an approximate wavefunction for the excited state of the helium atom can be taken as the symmetrized product of the hydrogenic wavefunctions, one representing the ground state of an electron in the field of an effective charge $\lambda = 2$ and the other the excited state of an electron in the field of a charge $\gamma = 1$. We thus let the excited wavefunction of helium be

$$\Phi_f(r_1, r_2) = 2^{-\frac{1}{2}}\{\phi_{1s}(\lambda \mid r_1)\, \phi_{nl}(\gamma \mid r_2) + \phi_{nl}(\gamma \mid r_1)\, \phi_{1s}(\lambda \mid r_2)\}. \tag{12}$$

With this form and equation (11) in (10), the matrix element is given by

$$M = 2^{\frac{1}{2}} \int \phi^*_{nl}(\gamma \mid r_1) \exp\{ie(A+A').r_1\}\, \phi_{1s}(\alpha \mid r_1)\, dr_1 \int \phi^*_{1s}(\lambda \mid r_2)\, \phi_{1s}(\alpha \mid r_2)\, dr_2. \tag{13}$$

In deriving the expression for M, use has been made of the condition

$$\int \phi^*_{nl}(\gamma \mid r_2)\, \phi_{1s}(\alpha \mid r_2)\, dr_2 = 0 \qquad \text{for} \qquad l \neq 0,$$

since the angular parts of $\phi_{nl}(\gamma \mid r_2)$ and $\phi_{1s}(\alpha \mid r_2)$ are then orthogonal.

Now we have

$$\int \phi_{1s}(\alpha \mid r_2)\, \phi^*_{1s}(\lambda \mid r_2)\, dr_2 = 8(\alpha\lambda)^{3/2}/(\alpha+\lambda)^3. \tag{14}$$

The radial contribution to the matrix element M is

$$I = \int_0^\infty r_1^2\, dr_1\, R^*_f(\gamma \mid r_1)\, R_{1s}(\alpha \mid r_1)\, j_{l_A}(eAr_1)\, j_{l_{A'}}(eA'r_1), \tag{15}$$

where $R_{1s}(\alpha\,|\,r_1)$ and $R_f(\gamma\,|\,r_1)$ are the radial wavefunctions of the hydrogen atom with nuclear charge α and γ respectively. These functions occur in the partial wave expansions of $\exp(ie\boldsymbol{A}.\boldsymbol{r}_1)$ and $\exp(ie\boldsymbol{A}'.\boldsymbol{r}_1)$ and are defined as

$$R_f(\gamma\,|\,r) = \beta(n,l)\sum_{s=0}^{n-l-1}\alpha(n,l,s)\,r^{l+s}\exp(-\gamma r/na_0), \tag{16}$$

with

$$\beta(n,l) = -\left\{\left(\frac{2\gamma}{na_0}\right)^3\frac{(n-l-1)!}{2n\{(n+l)!\}^3}\right\}^{\frac{1}{2}}\left(\frac{2\gamma}{na_0}\right)^l,$$

$$\alpha(n,l,s) = (-)^{s+2l+1}\frac{\{(n+l)!\}^2}{(n-l-1-s)!(2l+1+s)!\,s!}\left(\frac{2\gamma}{na_0}\right)^s.$$

The spherical Bessel functions $j_{l_A}(eAr_1)$ and $j_{l_{A'}}(eA'r_1)$ in equation (15) are defined as

$$j_{l_A}(eAr) = \tfrac{1}{2}\pi^{\frac{1}{2}}\sum_{k=0}^{\infty}(-)^k\{k!\,\Gamma(l_A+k+\tfrac{3}{2})\}^{-1}(\tfrac{1}{2}eA)^{l_A+2k}r^{l_A+2k}. \tag{17}$$

Since A' is a weak field, we can take

$$j_{l_{A'}}(eA'r) = (eA'r)^{l_{A'}}/(2l_{A'}+1)!!. \tag{18}$$

Now for the absorption of N photons of frequency ω in the initial state, accompanied by the simultaneous emission of a single Raman photon of frequency ω', we should find the coefficient of $\exp(-\mathrm{i}N\omega t)$ in the expression for I in order to obtain the T-matrix. Using the usual dipole approximation for eA, namely

$$eA = \tfrac{1}{2}ea\beta,$$

$$\beta = \exp(\mathrm{i}\omega t)+\exp(-\mathrm{i}\omega t),$$

we have

$$\sum_{k=0}^{\infty}(\tfrac{1}{2}eA)^{l_A+2k} = \sum_{k=0}^{\infty}(b\beta)^{l_A+2k}(\tfrac{3}{4}a_0^{-1})^{l_A+2k},$$

where

$$b = \tfrac{1}{3}eaa_0$$

and

$$\beta^{l_A+2k} = \sum_{j=0}^{l_A+2k}\binom{l_A+2k}{j}\exp(-\mathrm{i}N\omega t),$$

with

$$l_A+2k-2j = -N.$$

It follows that

$$\sum_{k=0}^{\infty}(b\beta)^{l_A+2k}(\tfrac{3}{4}a_0^{-1})^{l_A+2k} = \sum_{k=\frac{1}{2}(N-l_A)}^{\infty}(\tfrac{3}{4}ba_0^{-1})^{l_A+2k}\binom{l_A+2k}{\frac{1}{2}l_A+\frac{1}{2}N+k}. \tag{19}$$

Using equation (19), the T-matrix for the required transition is found to be

$$
\begin{aligned}
T &= (E_{1s}-E_{nl})M \\
&= (E_{1s}-E_{nl})\frac{8\sqrt{2}(\alpha\lambda)^{3/2}}{(\alpha+\lambda)^3}\exp(\pm i\eta)\,P_l(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}') \\
&\quad\times 4\pi \sum_{l_A,m_A,l_{A'},m_{A'}} (i)^{l_A+l_{A'}}\{(2l+1)(2l_A+1)(2l_{A'}+1)\}^{\frac{1}{2}}\begin{pmatrix} l_A & l_{A'} & l \\ m_A & m_{A'} & -m\end{pmatrix}\begin{pmatrix} l_A & l_{A'} & l \\ 0 & 0 & 0\end{pmatrix} \\
&\quad\times \beta(n,l)\,\tfrac{1}{2}\pi^{\frac{1}{2}}(-)^{\frac{1}{2}(N-l_A)}\frac{(eA')^{l_{A'}}}{(2l_{A'}+1)!!}\sum_{s=0}^{n-l-1}\alpha(n,l,s) \\
&\quad\times\left(\frac{2\gamma}{na_0}\right)^l\sum_{k=0}^{\infty}(-)^k\binom{N+2k}{k}\left(\frac{3b}{4a_0}\right)^{N+2k} \\
&\quad\times\frac{1}{\Gamma\left(\frac{1}{2}(N-l_A)+k+1\right)\Gamma\left(\frac{1}{2}(N+l_A)+k+\frac{3}{2}\right)}\frac{\Gamma(N+2k+s+5)}{(\alpha a_0^{-1}+\frac{1}{2}\gamma a_0^{-1})^{N+2k+s+5}},
\end{aligned}
\tag{20}
$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ are the polarization vectors of A and A', and η is the phase angle displacement between the A and A' fields. The Clebsch–Gordon coefficient $\left(\begin{smallmatrix} l_A & l_{A'} & l \\ 0 & 0 & 0\end{smallmatrix}\right)$ vanishes unless

$$
|l_A-l_{A'}| \leqslant l \leqslant |l_A+l_{A'}| \qquad \text{and} \qquad l_A+l_{A'}+l = 2\mathscr{I},
$$

where $\mathscr{I}$ is an integer.

From the general relation (20) for the T-matrix, we easily obtain the expression for the amplitude of transition from states He(1s) to He(1s, 2p) as

$$
\begin{aligned}
\tau_{1s\to 2p} &= \left(\frac{3y}{8}\right)^N\sum_{k=0}^{\infty}(-)^k\binom{N+2k}{k}\left(\frac{3y}{8}\right)^{2k} \\
&\quad\times\frac{1}{\Gamma\left(\frac{1}{2}(N-l_A)+k+1\right)\Gamma\left(\frac{1}{2}(N+l_A)+k+\frac{3}{2}\right)}\frac{\Gamma(N+2k+5)}{(\alpha+\frac{1}{2}\gamma)^{N+2k+5}},
\end{aligned}
\tag{21}
$$

where τ_{fi} is the 'reduced' transition amplitude which is defined in such a way that it involves all the intensity-dependent terms in the T-matrix. The quantity $y = 2b$ is the intensity parameter and the $3j$-symbol implies the constraints $l_A = 0$ and 2.

4. Numerical Calculations

We consider here the Euler transformation method which has been used to obtain the final results in closed form. The series in equation (21) is divergent for large values of y. Now this equation can be written as

$$
\tau = \left(\tfrac{3}{8}y\right)^N f(z), \tag{22}
$$

where

$$
f(z) = \sum_{k=0}^{\infty} a_k c_k z^k, \tag{23}
$$

with

$$
a_k = \binom{N+2k}{k}, \qquad z = 9y^2/64
$$

and

$$c_k = \frac{1}{\Gamma\left(\frac{1}{2}(N-l_A)+k+1\right)\Gamma\left(\frac{1}{2}(N+l_A)+k+\frac{3}{2}\right)} \frac{\Gamma(N+2k+5)}{(\alpha+\frac{1}{2}\gamma)^{N+2k+5}}.$$

The series (22) is also divergent for large values of y, but it can be summed if a function of the form

$$g(z) = \sum_{k=0}^{\infty} a_k z^k \tag{24}$$

is known. Expressing the parameter a_k in terms of the derivatives $g', g'', \ldots$ of g, we have

$$f = C_0 g - (\delta C_0) z g' + (2!)^{-1} (\delta^2 C_0) z^2 g'' - \ldots, \tag{25}$$

where

$$\delta^q C_0 = \sum_{p=0}^{\infty} (-)^p \binom{q}{p} C_p,$$

$\binom{q}{p}$ being a binomial coefficient, and the value of g is given by

$$g = (1+y^2)^{-\frac{1}{2}} \{1+(1+y^2)^{\frac{1}{2}}\}^{-N} = \sum_{k=0}^{\infty} \binom{N+2k}{k} (-z)^k.$$

The numerical values of the reduced transition amplitude τ_{fi} were found by directly feeding the values of C into an IBM 360 computer. The results obtained are plotted in Figs 1*a*–1*c* below.

5. Discussion

In the presence of intense electromagnetic radiation of frequency ω the metastable state decays by absorbing an incident quantum and emitting a Raman photon of frequency ω' such that $N\omega \pm \omega'$ is equal to the energy difference between the initial and final states, where the plus and minus signs correspond to pure absorption and stimulated emission processes respectively. Calculations for the absorption of two or more quanta and stimulated Raman scattering in plasmas have been made by Reinheimer (1964) using the second-order time-dependent perturbation theory of the Stark effect for high frequencies. Burrell and Kunze (1972) have shown that high frequency electric fields in plasmas can be measured by simultaneously focusing microwave radiation and the beam from a tunable laser into a helium plasma and observing two-photon absorption and stimulated Raman scattering from the excited helium atoms.

The processes of photoionization and photoexcitation in the helium atom are old problems in physics (Magnaron and Levinger 1965). The accuracy of the calculations depends on the choice of wavefunction. Dalgarno and Stewart (1960) and Salpeter and Zaidi (1962) have calculated the oscillator strength for one particular transition involving simultaneous photoexcitation and photoionization of helium, namely $1s^2 \rightarrow (2s, \varepsilon p)^1P$, using an 18-parameter Hylleraas wavefunction for the bound state of helium and the Hartree approximation for the free state. Brown (1970) has evaluated the single photoionization cross section using the Eckart wavefunction, and his results for energies greater than threshold agree well with those obtained by Salpeter and Zaidi, who used a much more accurate ground state representation.

The matrix element given by equation (9) requires only a knowledge of radial wavefunctions. If these wavefunctions can be found analytically or numerically, all that is needed to obtain the final results is a single integration and the performance of an infinite sum. The latter can be easily reduced to a closed form by using the Euler transformation method discussed in Section 4. The results so obtained are applicable at very high intensities where perturbation theory predictions are doubtful. Thus this method provides an opportunity to examine nontrivial general features of high order processes.

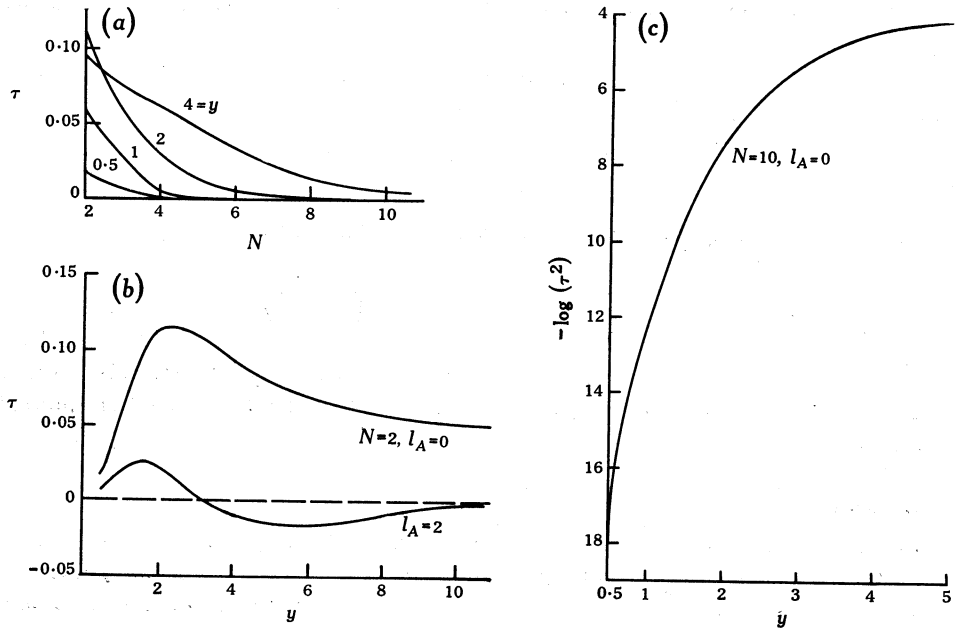

Fig. 1. Reduced transition amplitude τ plotted (*a*) against the number N of photons involved for four values of the intensity parameter y, (*b*) against y for $N = 2$ and $l_A = 0, 2$ and (*c*) against y for $N = 10$ and $l_A = 0$.

Fig. 1*a* shows a plot of the reduced transition amplitude τ against the photon number N for four values of the intensity parameter y. It is seen that as the intensity increases the minimum value of τ is reached at larger values of N. This implies that higher order processes are dominant at very high intensities. The plot of τ against y for $N = 2$ shown in Fig. 1*b* indicates that the initial linear increase of τ with y does not apply at high intensity. This nonlinear behaviour may be attributed to a deviation from power law dependence of the transition probability on intensity. With $x = b^2$ $(= \frac{1}{4}y^2)$, the power law

$$\partial(\log|T|^2)/\partial(\log x) = N, \tag{26}$$

is the usual perturbation result for a transition of order N. From equation (9) we find

$$\partial(\log|T|^2)/\partial(\log x) = N - \Delta x, \tag{27}$$

where Δx involves the constants and N-dependent terms. Thus at high intensities the

contributions from higher order terms have the effect of reducing the power law dependence of the transition probability to something less than the lowest order transition. This effect has already been observed experimentally in ionization processes by Voronov *et al.* (1965), Voronov and Delone (1966), Bystrova *et al.* (1967) and Delone and Delone (1968). Arutyun Yan *et al.* (1970) have theoretically analysed the multiquantum effect in the focus of a laser beam and have shown that this behaviour may be due to a number of reasons: e.g. broadening of the upper level in strong fields, transition through a resonance level, etc. They have also shown that, at large radiation densities, saturation and expansion of the interactive region occurs which substantially changes the form of the dependence of the transition probability on intensity. Barashev (1972) has demonstrated theoretically that this effect could also be observed in an unfocused laser beam and in the region far from saturation.

As is obvious from Fig. 1*b*, with $N = 2$ the contribution to the transition probability for $l_A = 0$ is much greater than that for $l_A = 2$, showing that the s wave contributions are dominant. After attaining a maximum at $y \sim 2$, the $l_A = 0$ curve is seen to decrease for further increase in intensity. However, for high values of N (see Fig. 1*c* with $N = 10$) the transition amplitude increases with intensity and the plot resembles an ionization curve. This effect may be due to peculiar behaviour of the atom in the presence of an intense field. Voronov (1967) has shown that at high intensity there is a smearing of the upper energy levels of the atom, resulting in an overlapping and merging of the levels into a quasi-continuum spectrum. Thus for the fields under consideration the 2p state could act as a quasi-continuum and lead to a higher transition amplitude with increase in intensity. However, for transitions to the 2s state, it is expected that τ will decrease with y, since this state in the helium atom is much further from the continuum than any other level.

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References

Arutyun Yan, I. N., Askar'yan, G. A., and Pogosyan, V. A. (1970). *Sov. Phys. JETP* **31**, 548–50.
Barashev, P. P. (1972). *Sov. Phys. JETP* **34**, 1225–9.
Bebb, H. B., and Gold, A. (1966). *Phys. Rev.* **143**, 1–24.
Braunlich, P., Hall, R., and Lambropoulos, P. (1972). *Phys. Rev.* A **5**, 1013.
Braunlich, P., and Lambropoulos, P. (1970). *Phys. Rev. Lett.* **25**, 135–8, 980–8.
Brown, R. L. (1970). *Phys. Rev.* A **1**, 341–7.
Burrell, C. F., and Kunze, H. J. (1972). *Phys. Rev. Lett.* **29**, 1445–8.
Bystrova, T. B., Voronov, G. S., Delone, G. A., and Delone, N. B. (1967). *Sov. Phys. JETP Lett.* **5**, 178.
Chang, F. T., and Tang, C. L. (1969). *Phys. Rev.* **185**, 42–3.
Dalgarno, A., and Lewis, J. T. (1956). *Proc. Roy. Soc.* A **233**, 70.
Dalgarno, A., and Stewart, A. L. (1960). *Proc. Phys. Soc. London* **76**, 49–55.
Daniltseva, P. E., Zubov, Z. A., Sushchinskii, M., and Shuvalav, I. L. (1963). *Sov. Phys. JETP* **17**, 1473–8.
Delone, G. A., and Delone, N. B. (1968). *Sov. Phys. JETP* **27**, 570–1.
Ducuing, J., Hauchercorne, G., Nysyrowicz, A., and Padere, F. (1969). *Phys. Lett.* A **28**, 746–7.

Eckart, C. (1930). *Phys. Rev.* **36**, 878.
Gold, A., and Bebb, H. B. (1965). *Phys. Rev. Lett.* **14**, 80–2.
Gontier, Y., and Trahin, M. (1968). *Phys. Rev.* **172**, 83–7.
Gontier, Y., and Trahin, M. (1971*a*). *Phys. Rev.* A **4**, 1896–1906.
Gontier, Y., and Trahin, M. (1971*b*). *Phys. Lett.* A **36**, 463–4.
Hylleraas, E. A. (1929). *Z. Phys.* **54**, 347.
Karule, E. (1971). *J. Phys.* B **4**, 67–70.
Keldysh, L. V. (1965). *Sov. Phys. JETP* **20**, 1307–14.
Magnaron, R., and Levinger, J. S. (1965). *Phys. Rev.* **139**, 646–8.
Man Mohan and Thareja, R. K. (1972). *J. Phys.* B **5**, L134–6.
Man Mohan and Thareja, R. K. (1973*a*). *Phys. Rev.* A **7**, 34.
Man Mohan and Thareja, R. K. (1973*b*). *J. Phys.* B **6**, 809–15.
Man Mohan and Thareja, R. K. (1973*c*). *J. Phys.* B **6**, 2660–7.
Morse, P. M., and Feshbach, H. (1953). 'Methods of Theoretical Physics', Vol. 1 (McGraw-Hill: New York).
Rapoport, L. P., and Zon, B. A. (1968). *Phys. Lett.* A **36**, 564–5.
Reinheimer, J. (1964). *J. Quant. Spectrosc. Radiat. Transfer* **4**, 671.
Reiss, H. B. (1970). *Phys. Rev.* A **1**, 803–18.
Reiss, H. B. (1971). *Phys. Rev. Lett.* **25**, 1149–51.
Salpeter, E. E., and Zaidi, M. H. (1962). *Phys. Rev.* **125**, 248–55.
Saslow, W. M., and Mills, D. L. (1969). *Phys. Rev.* **187**, 1025.
Thareja, R. K., and Haque, S. N. (1974). *Phys. Lett.* A **48**, 231–3.
Voronov, G. S. (1967). *Sov. Phys. JETP* **24**, 1009–11.
Voronov, G. S., Delone, G. A., Delone, N. B., and Kudrevatova, O. V. (1965). *Sov. Phys. JETP Lett.* **2**, 237–9.
Voronov, G. S., and Delone, N. B. (1966). *Sov. Phys. JETP* **23**, 54–8.
Zernick, W. (1963). *Phys. Rev.* **133**, A117–20.
Zernick, W. (1964). *Phys. Rev.* **135**, A51–7.
Zernick, W. (1968). *Phys. Rev.* **176**, 420.

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