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Drift of a Charged Particle in a Static Magnetic Field Having a Power Law Dependence

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Abstract

As a generalization of Seymour's (1959) exact solution for the drift velocity of a charged particle in a static magnetic field of constant gradient, exact solutions are obtained for charged particle drift in a static magnetic field represented by $B_z = \lambda x^\alpha$, where λ and α are constants. Four cases of bound orbits are analysed. Exact solutions in terms of hypergeometric, confluent hypergeometric and gamma functions are obtained for the displacement Δy per cycle, the periodic time T and the drift velocity v_d. The special solutions in terms of complete elliptic integrals obtained by Seymour (1959) are also recovered. Calculated exact drift velocity characteristics for representative conditions are presented, and the manner in which the exact curves merge into the Alfvén approximate drift velocity region is indicated.

1. Basic Equations for Bound Orbits

The basic equations for the motion of a charged particle in a field are conveniently written for an electron of charge $-e$ moving in a straight and parallel static magnetic field having the z direction of a cartesian coordinate system. For proton motion in such a field, the change of charge to $+e$ leads principally to a reversal of sign in the displacement and drift velocity expressions derived. Since $\nabla . \boldsymbol{B} = 0$, the field

$$\boldsymbol{B} = (0, 0, B_z) \tag{1}$$

in general has a coordinate dependence of the form

$$B_z = B_z(x, y). \tag{2}$$

In this treatment B_z is taken to depend on the x coordinate only, say, so that here

$$B_z = B_z(x),$$

with attendant simplicity of analysis.

In the absence of electric fields the nonrelativistic Lorentz force law becomes

$$m\,\mathrm{d}\boldsymbol{v}/\mathrm{d}t = -e\boldsymbol{v} \times \boldsymbol{B} \qquad \text{(e.m.u.)} \tag{3}$$

in the usual nomenclature. With $\boldsymbol{B}$ in the z direction, the z component of $\boldsymbol{v}$ is constant and does not require explicit consideration, while the x and y components of

v vary as

$$\mathrm{d}v_x/\mathrm{d}t = -(e/m)B_z v_y \qquad \text{and} \qquad \mathrm{d}v_y/\mathrm{d}t = +(e/m)B_z v_x . \tag{4a, b}$$

From equation (3),

$$\boldsymbol{v}\,.\,\mathrm{d}\boldsymbol{v}/\mathrm{d}t = \mathrm{d}(\tfrac{1}{2}v^2)/\mathrm{d}t = 0,$$

and so $v^2 = v_x^2 + v_y^2$ is a constant of the motion. In the notation of Fig. 1, which shows electron motion in the x–y plane, it is of physical significance in this treatment

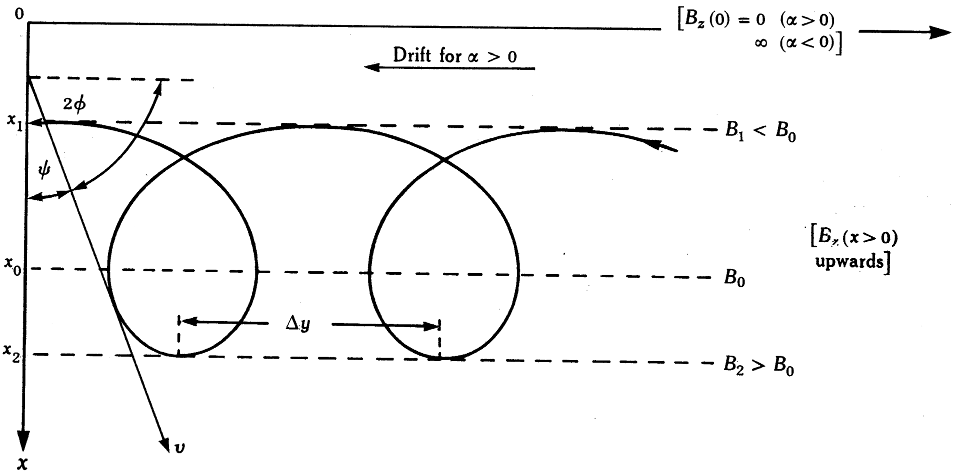

Fig. 1. Electron drift in $B_z = \lambda x^\alpha$. Case 1: $\alpha \neq -1$, motion does not cross the line $x = 0$. For $\alpha < 0$ the indicated drift motion reverses direction and has a pattern of the form shown in Fig. 2.

to introduce the angular variable ψ, so that v_x and v_y are represented parametrically as

$$v_x = \mathrm{d}x/\mathrm{d}t = v\cos\psi \qquad \text{and} \qquad v_y = \mathrm{d}y/\mathrm{d}t = v\sin\psi . \tag{5a, b}$$

Substitution of equations (5a) and (5b) into either equation (4a) or (4b) leads to

$$\cos\psi\,\mathrm{d}\psi = (e/mv)B_z(x)\,\mathrm{d}x,$$

so that

$$\sin\psi - \sin\psi_0 = (e/mv)\int_{x_0}^{x} B_z(x')\,\mathrm{d}x', \tag{6}$$

where $\psi = \psi_0$ when $x = x_0$.

Various forms of $B_z(x)$ lead to useful drift velocity results. The cases of $B_z(x)$ having sinusoidal, exponential and power law dependence on x have been found tractable. In upper atmosphere and laboratory plasma physics the power law is a particularly useful general form since, to give one example, it permits exact analysis of problems associated with the neutral sheet of the magnetosphere tail. Accordingly, in this paper the magnetic field is taken as

$$B_z(x) = \lambda x^\alpha, \tag{7}$$

where λ and α are constants. Then provided $\alpha \neq -1$ and $x_0 > 0$, equation (6) yields

$$x = x_0\{1 + \beta^{-1}(\rho_0/x_0)(\sin\psi - \sin\psi_0)\}^{\beta}, \tag{8}$$

where, with

$$B_0 = \lambda x_0^{\alpha}, \tag{9}$$

$$\rho_0 = mv/eB_0 \tag{10}$$

is the electron gyroradius for circular orbital motion in a uniform magnetic field of strength B_0, and

$$\beta = (1+\alpha)^{-1}. \tag{11}$$

The electron motion sketched in Fig. 1 is for $x_0 > 0$. For simplicity of mathematical formulation and without particular loss of generality, x_0 is chosen to correspond to $\psi_0 = 0$. Then equation (8) becomes

$$x = x_0\{1 + \beta^{-1}(\rho_0/x_0)\sin\psi\}^{\beta}. \tag{12}$$

Recalling that equation (12) is valid for all α except $\alpha = -1$, the electron orbits are bound between the limits

$$x_1 = x_0(1 - \beta^{-1}\rho_0/x_0)^{\beta} \qquad \text{for} \qquad \psi = \tfrac{3}{2}\pi, \tag{13}$$

and

$$x_2 = x_0(1 + \beta^{-1}\rho_0/x_0)^{\beta} \qquad \text{for} \qquad \psi = \tfrac{1}{2}\pi, \tag{14}$$

so that

$$0 \leqslant x_1 < x < x_2 < \infty. \tag{15}$$

For $\alpha > 0$ the magnetic field B_z vanishes on the line $x = 0$, and the electron motion is as shown in Fig. 1. For $\alpha < 0$ the field B_z becomes infinite on the line $x = 0$. Then the drift motion reverses direction and has the same general form as shown in Fig. 2 for $\alpha = -1$, which is considered in Case 2 of Section 2 below.

Another class of motions of practical interest (Seymour 1959) occurs when $x_0 = 0$ and $\alpha > 0$, so that B_z is always zero on the line $x = 0$. Then with $\psi = \psi_0$ when $x_0 = 0$, equation (6) gives

$$x = \{\beta^{-1}(mv/e\lambda)(\sin\psi - \sin\psi_0)\}^{\beta}. \tag{16}$$

Under these conditions the electron enters a region of reversed magnetic field when it crosses the neutral plane $x = 0$ if α is an odd integer, and symmetrical electron motions of the type shown in Fig. 3 occur. From equation (16) x for any ψ is thus given by

$$x = \pm\{\beta^{-1}(mv/e\lambda)(\sin\psi - \sin\psi_0)\}^{\beta}, \tag{17}$$

with limits

$$x_2 = -x_1 = \{\beta^{-1}(mv/e\lambda)(1 - \sin\psi_0)\}^{\beta} \qquad \text{for} \qquad \psi = \tfrac{1}{2}\pi, \tag{18}$$

so that

$$x_1 < 0 \qquad \text{and} \qquad x_1 < x < x_2 < \infty. \tag{19}$$

For α an even integer, the magnetic field does not reverse direction when the electron crosses the neutral plane into the region $x < 0$. Particle motions of the type shown in Fig. 4 now occur, and from equation (16) the corresponding limits are given by

$$x_1 = -\{\beta^{-1}(mv/e\lambda)(1+\sin\psi_0)\}^\beta \qquad \text{for} \qquad \psi = \tfrac{3}{2}\pi, \tag{20}$$

$$x_2 = \{\beta^{-1}(mv/e\lambda)(1-\sin\psi_0)\}^\beta \qquad \text{for} \qquad \psi = \tfrac{1}{2}\pi, \tag{21}$$

so that the conditions (19) again apply.

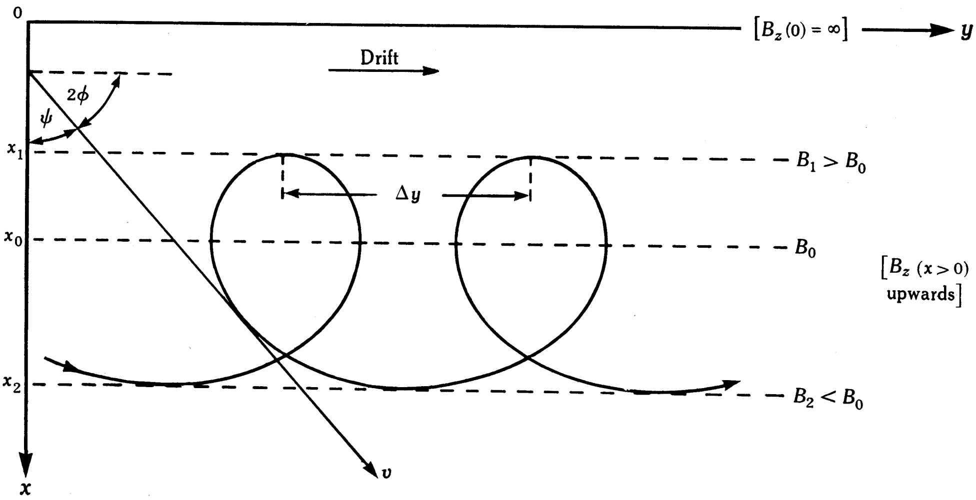

Fig. 2. Electron drift in $B_z = \lambda x^\alpha$. Case 2: $\alpha = -1$, motion does not cross the line $x = 0$.

In terms of $B_z(x)$ the magnitude of the angular gyrofrequency ω of the electron motion is given by (see e.g. Seymour 1970)

$$\omega = (e/m)B_z(x). \tag{22}$$

From Fig. 1 it is geometrically evident that $\mathrm{d}\psi/\mathrm{d}t = \omega$, and hence with the aid of equations (7) and (22)

$$\mathrm{d}t = (m/e\lambda)x^{-\alpha}\,\mathrm{d}\psi. \tag{23}$$

From equations (5b) and (23)

$$\mathrm{d}y = (mv/e\lambda)x^{-\alpha}\sin\psi\,\mathrm{d}\psi. \tag{24}$$

Results corresponding to those presented above for $\alpha = -1$ are given in Case 2 of Section 2 below.

2. Exact Solutions in Terms of Hypergeometric Functions

Case 1: $\alpha \neq -1$, *Electron Does Not Cross Line* $x = 0$

Consider the case in which $\alpha \neq -1$ and the electron does not cross the line $x = 0$, on which $B_z = 0$ when $\alpha > 0$ and $B_z = \infty$ when $\alpha < 0$. From equations (9),

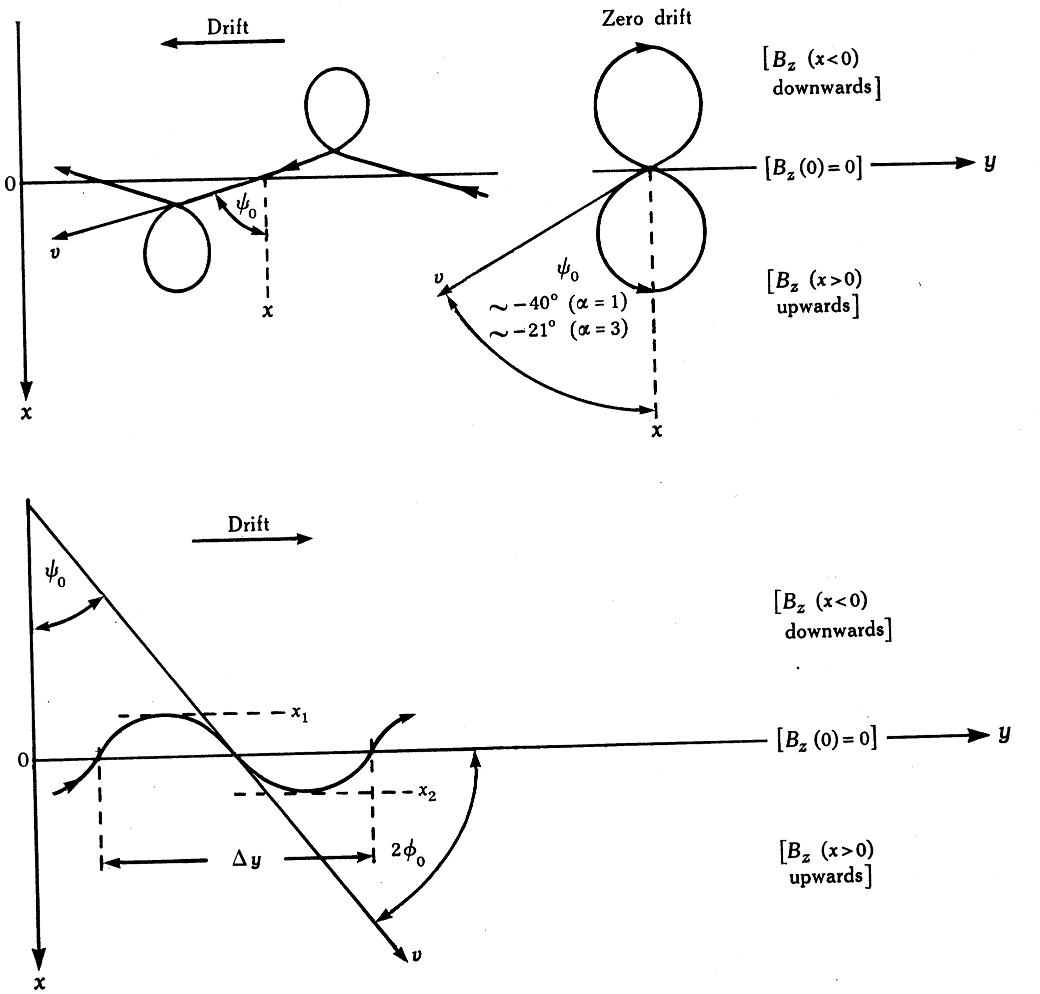

Fig. 3. Electron drift in $B_z = \lambda x^\alpha$. Case 3: α is a positive odd integer, motion crosses the line $x = 0$.

(10), (11), (12) and (24)

$$y = \rho_0 \int \frac{\sin\psi\, d\psi}{\{1+\beta^{-1}(\rho_0/x_0)\sin\psi\}^{1-\beta}}, \tag{25}$$

and thus from Fig. 1 the exact drift in the y direction is given by

$$\Delta y = \rho_0 \int_0^{2\pi} \frac{\sin\psi\, d\psi}{\{1+\beta^{-1}(\rho_0/x_0)\sin\psi\}^{1-\beta}}. \tag{26}$$

At $x = x_2$ (as defined by equation 14) equation (7) becomes $B_2 = \lambda x_2^\alpha$, and so

$$\rho_2 = mv/eB_2 = mv/e\lambda x_2^{(1-\beta)/\beta}. \tag{27}$$

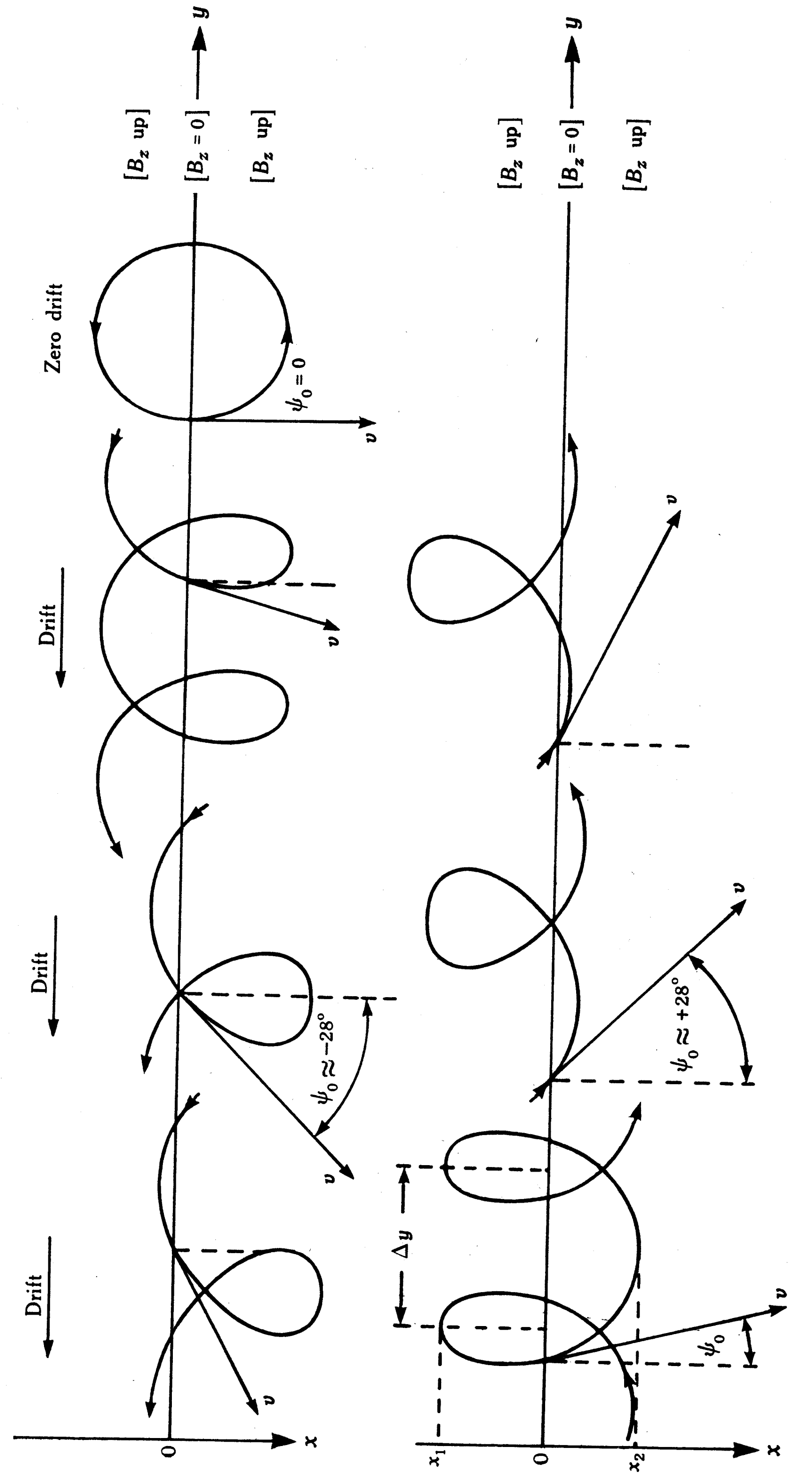

Fig. 4. Electron drift in $B_z = \lambda x^{\alpha}$. Case 4: α is a positive even integer, motion crosses the line $x = 0$.

Then use of a new variable $\phi = \frac{1}{4}\pi - \frac{1}{2}\psi$ in equation (26), with the help of equations (10), (14) and (27), leads to

$$\Delta y = 4\rho_2 \int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1-\sigma\sin^2\phi)^{1-\beta}} - 8\rho_2 \int_0^{\frac{1}{2}\pi} \frac{\sin^2\phi \, d\phi}{(1-\sigma\sin^2\phi)^{1-\beta}}, \tag{28}$$

where

$$\sigma = 2\beta^{-1}\rho_0 x_0^{(1-\beta)/\beta}/x_2^{1/\beta} = 2\beta^{-1}\rho_2/x_2. \tag{29}$$

Combination of equations (14) and (29) leads to the useful relationships

$$x_2/x_0 = (1-\tfrac{1}{2}\sigma)^{-\beta} \qquad \text{and} \qquad x_2/\rho_0 = 2(\sigma\beta)^{-1}(1-\tfrac{1}{2}\sigma)^{1-\beta}. \tag{30, 31}$$

For $x_1 \geqslant 0$ (cf. equation 15), equations (13), (14) and (29) yield

$$x_2 \geqslant x_0(2\rho_0/\beta x_0)^{\beta} \geqslant x_2\sigma^{\beta}, \tag{32}$$

so that the upper limits of $\beta^{-1}\rho_0/x_0$ and σ^{β} are unity.

The theory of hypergeometric functions gives the result (for its derivation, see Appendix 1),

$$\mathrm{F}(a,b;c;x) = \frac{2\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^{\frac{1}{2}\pi} \frac{(\sin\phi)^{2b-1}(\cos\phi)^{2c-2b-1}\, d\phi}{(1-x\sin^2\phi)^{a}}, \tag{33}$$

which for $b = \frac{1}{2}$ and $c = 1$ reduces to

$$\mathrm{F}(a,\tfrac{1}{2};1;x) = (2/\pi) \int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1-x\sin^2\phi)^{a}}. \tag{34}$$

Noting that the result (29) gives $\rho_2 = \frac{1}{2}\sigma\beta x_2$, equation (28) can be further manipulated to the form

$$\Delta y = 2\beta x_2\left(2\int_0^{\frac{1}{2}\pi} (1-\sigma\sin^2\phi)^{\beta}\, d\phi - (2-\sigma)\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1-\sigma\sin^2\phi)^{1-\beta}}\right). \tag{35}$$

Hence in terms of hypergeometric functions of the form (34), by putting $x = \sigma$ and $a = -\beta$ for the first integral of equation (35) and $a = 1-\beta$ for the second integral, Δy can be written as

$$\Delta y = \pi\beta x_2[2\{\mathrm{F}(-\beta,\tfrac{1}{2};1;\sigma) - \mathrm{F}(1-\beta,\tfrac{1}{2};1;\sigma)\} + \sigma\,\mathrm{F}(1-\beta,\tfrac{1}{2};1;\sigma)]. \tag{36}$$

From equations (9), (10), (11), (12) and (23)

$$t = (\rho_0/v)\int \frac{d\psi}{\{1+\beta^{-1}(\rho_0/x_0)\sin\psi\}^{1-\beta}} \tag{37}$$

and so, with reference to Fig. 1, the periodic time T corresponding to Δy is obtained with the help of equations (14) and (29) as

$$T = (2\sigma\beta x_2/v)\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1-\sigma\sin^2\phi)^{1-\beta}}. \tag{38}$$

Putting $a = 1-\beta$ and $x = \sigma$ in the result (34), equation (38) becomes

$$T = (\pi\sigma\beta x_2/v)\,\mathrm{F}(1-\beta,\tfrac{1}{2};1;\sigma)\,. \tag{39}$$

Elimination of x_2 from equation (39) by means of equation (31) and use of the zero-order circular orbit result

$$T_0 = 2\pi\rho_0/v\,, \tag{40}$$

gives the interesting form

$$T = T_0(1-\tfrac{1}{2}\sigma)^{1-\beta}\,\mathrm{F}(1-\beta,\tfrac{1}{2};1;\sigma)\,. \tag{41}$$

Equation (41) expresses the periodic time T for noncircular orbital motion in terms of the zero-order periodic time T_0 corresponding to electron circular orbital motion in a constant magnetic field $B_0 = \lambda x_0^{\alpha}$.

From equations (36) and (39) the exact drift velocity for electron motion in the magnetic field $B_z = \lambda x^{(1-\beta)/\beta}$ is given by

$$v_{\mathrm{d}} = \Delta y/T = -v[2\sigma^{-1}\{1-\mathrm{F}(-\beta,\tfrac{1}{2};1;\sigma)/\mathrm{F}(1-\beta,\tfrac{1}{2};1;\sigma)\}-1]\,. \tag{42}$$

When $\alpha = 1$ equation (7) reduces to a magnetic field of constant gradient λ. The results (36), (39) and (42) above then reduce to those obtained by Seymour (1959, Section III, Case 1) since $\alpha = 1$ gives $\beta = \frac{1}{2}$, and thus from equation (34), with $x = \sigma = k_1^2$,

$$\tfrac{1}{2}\pi\,\mathrm{F}(\tfrac{1}{2},\tfrac{1}{2};1;k_1^2) = \int_0^{\frac{1}{2}\pi}(1-k_1^2\sin^2\phi)^{-\frac{1}{2}}\,\mathrm{d}\phi = K\,, \tag{43a}$$

$$\tfrac{1}{2}\pi\,\mathrm{F}(-\tfrac{1}{2},\tfrac{1}{2};1;k_1^2) = \int_0^{\frac{1}{2}\pi}(1-k_1^2\sin^2\phi)^{\frac{1}{2}}\,\mathrm{d}\phi = E\,, \tag{43b}$$

the complete elliptic integrals of the first and second kinds respectively. Both elliptic integrals are of modulus k_1, where k_1 is bounded by the inequality

$$0 \leqslant k_1 \leqslant 1\,. \tag{44}$$

It has been seen in Section 1 above that for $\alpha > 0$ the electron drift has the pattern and direction shown in Fig. 1, and that for $\alpha < 0$ the electron drift is in the positive direction of the y axis and has a pattern like that shown in Fig. 2, which relates to Case 2 below. When $\alpha = 0$ equation (7) gives the magnetic field as $B_z = \lambda = \text{const.}$ Then, with $\beta = 1$, $\mathrm{F}(0,\frac{1}{2};1;\sigma) = 1$, $\mathrm{F}(-1,\frac{1}{2};1;\sigma) = 1-\frac{1}{2}\sigma$ and equations (13), (14), (36), (41) and (42) respectively reduce to

$$x_1 = x_0-\rho_0\,,\qquad x_2 = x_0+\rho_0\,,\qquad \Delta y = 0\,,\qquad T = T_0\,,\qquad v_{\mathrm{d}} = 0\,, \tag{45}$$

as is correct for zero-order circular electron orbital motion in a constant magnetic field.

Case 2: $\alpha = -1$, *Electron Does Not Cross Line* $x = 0$

Consider the case in which $\alpha = -1$ and the electron does not cross the line $x = 0$ on which $B_z = \infty$. When $\alpha = -1$ equation (7) becomes $B_z = \lambda/x$, and β

becomes infinite. For electron motion in this magnetic field, the results (36), (39) and (42) for Δy, T and v_d respectively are not valid. The infinite value of β here suggests solutions for these quantities in terms of confluent hypergeometric functions. Again choosing x_0 to correspond to $\psi_0 = 0$, use of equations (7) and (10) in (6) yields in this case

$$x = x_0 \exp\{(\rho_0/x_0)\sin\psi\}. \tag{46}$$

From this result the electron motion of Fig. 2 in the field $B_z = \lambda/x$ is bounded by

$$x_1 = x_0 \exp(-\rho_0/x_0) \qquad \text{for} \qquad \psi = \tfrac{3}{2}\pi, \tag{47}$$

$$x_2 = x_0 \exp(\rho_0/x_0) \qquad \text{for} \qquad \psi = \tfrac{1}{2}\pi. \tag{48}$$

From equations (24) and (46),

$$y = \rho_0 \int \exp\{(\rho_0/x_0)\sin\psi\} \sin\psi \, \mathrm{d}\psi, \tag{49}$$

and similarly equation (23) yields

$$t = \rho_0 v^{-1} \int \exp\{(\rho_0/x_0)\sin\psi\} \, \mathrm{d}\psi. \tag{50}$$

Thus, with reference to Fig. 2, equation (49) gives

$$\Delta y = 2\rho_0 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \exp\{(\rho_0/x_0)\sin\psi\} \sin\psi \, \mathrm{d}\psi. \tag{51}$$

From equations (7) and (27) it is found in this case that

$$mv/e\lambda = \rho_0/x_0 = \rho_2/x_2. \tag{52}$$

With the substitutions $\nu = 2\rho_0/x_0 = 2\rho_2/x_2$ and $\tau = \sin\psi$, Δy becomes

$$\Delta y = 2\rho_0 \int_{-1}^{1} \tau(1-\tau^2)^{-\frac{1}{2}} \exp(\tfrac{1}{2}\nu\tau) \, \mathrm{d}\tau. \tag{53}$$

The theory of confluent hypergeometric functions yields the result (for its derivation, see Appendix 2)

$$\mathrm{M}(\tfrac{1}{2}; 1; -\nu) = \pi^{-1} \exp(-\tfrac{1}{2}\nu) \int_{-1}^{1} (1-\tau^2)^{-\frac{1}{2}} \exp(\tfrac{1}{2}\nu\tau) \, \mathrm{d}\tau. \tag{54}$$

By means of the standard result (e.g. Hochstadt 1971)

$$\mathrm{dM}(a; b; -\nu)/\mathrm{d}\nu = -ab^{-1} \mathrm{M}(a+1; b+1; -\nu) \tag{55}$$

(which can be derived directly from the series (A6) of Appendix 2) and the result (54), equation (53) becomes

$$\Delta y = 2\pi\rho_0 \exp(\tfrac{1}{2}\nu)\{\mathrm{M}(\tfrac{1}{2}; 1; -\nu) - \mathrm{M}(\tfrac{3}{2}; 2; -\nu)\}. \tag{56}$$

Similarly equations (40), (50) and (54) yield for the periodic time

$$T = T_0 \exp(\tfrac{1}{2}v)\,\mathrm{M}(\tfrac{1}{2};1;-v), \tag{57}$$

and so the drift velocity for an electron in a magnetic field $B_z = \lambda/x$ becomes from equations (40), (56) and (57)

$$v_{\mathrm{d}} = \Delta y/T = v\{1-\mathrm{M}(\tfrac{3}{2};2;-v)/\mathrm{M}(\tfrac{1}{2};1;-v)\}. \tag{58}$$

Case 3: α a Positive Odd Integer, Electron Crosses Line $x = 0$ and Enters Reversed Field

Consider the case in which α is a positive odd integer and the electron crosses the line $x = 0$ on which $B_z = 0$ and enters a region of reversed magnetic field. When α is a positive odd integer, equation (7) shows that the magnetic field reverses for $x < 0$, and the symmetrical motions of Fig. 3 occur. Here equations (17) and (24) give for $x > 0$

$$y = (mv/e\lambda)^\beta \beta^{1-\beta} \int (\sin\psi - \sin\psi_0)^{\beta-1} \sin\psi \,\mathrm{d}\psi, \tag{59}$$

and similarly equation (23) yields

$$t = v^{-1}(mv/e\lambda)^\beta \beta^{1-\beta} \int (\sin\psi - \sin\psi_0)^{\beta-1}\,\mathrm{d}\psi. \tag{60}$$

Using $\psi = \frac{1}{2}\pi - 2\phi$, then $\psi_0 = \frac{1}{2}\pi - 2\phi_0$ where, from Fig. 3, $-\frac{1}{2}\pi \leqslant \psi_0 \leqslant \frac{1}{2}\pi$ and $0 \leqslant \phi_0 \leqslant \frac{1}{2}\pi$. In terms of ϕ_0 the limits given by equation (18) reduce to

$$x_2 = -x_1 = \{2mv\sin^2(\phi_0)/\beta e\lambda\}^\beta. \tag{61}$$

Utilizing the symmetry of the drift patterns of Fig. 3, equation (59) gives

$$\begin{aligned}\Delta y &= 4(mv/e\lambda)^\beta \beta^{1-\beta} \int_{\psi_0}^{\frac{1}{2}\pi} \frac{\sin\psi\,\mathrm{d}\psi}{(\sin\psi - \sin\psi_0)^{1-\beta}} \\ &= 2^{3+\beta}(mv/e\lambda)^\beta \beta^{1-\beta}\left(\int_0^{\phi_0} \frac{\cos^2\phi\,\mathrm{d}\phi}{(\sin^2\phi_0 - \sin^2\phi)^{1-\beta}} - \tfrac{1}{2}\int_0^{\phi_0} \frac{\mathrm{d}\phi}{(\sin^2\phi_0 - \sin^2\phi)^{1-\beta}}\right).\end{aligned} \tag{62}$$

If another variable of integration is defined by $\sin\phi = \sin\phi_0 \sin\theta$, equations (27) (61) and (62) give

$$\begin{aligned}\Delta y &= 2^{3+\beta}(mv/e\lambda)^\beta \beta^{1-\beta} \gamma^{2\beta-1} \\ &\times\left(\int_0^{\frac{1}{2}\pi} \frac{(1-\gamma^2\sin^2\theta)^{\frac{1}{2}}\,\mathrm{d}\theta}{(\cos\theta)^{1-2\beta}} - \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}\theta}{(1-\gamma^2\sin^2\theta)^{\frac{1}{2}}(\cos\theta)^{1-2\beta}}\right),\end{aligned} \tag{63}$$

where

$$\gamma = \sin\phi_0 = (\tfrac{1}{2}\beta x_2/\rho_2)^{\frac{1}{2}}. \tag{64}$$

The result (33) enables equation (63) to be written as

$$\Delta y = 2\Sigma\gamma^{2\beta-1}(2F_1 - F_2), \tag{65}$$

where

$$\Sigma = \pi^{\frac{1}{2}}2^{\beta}(mv/e\lambda)^{\beta}\beta^{1-\beta}\,\Gamma(\beta)/\Gamma(\beta+\tfrac{1}{2}), \tag{66}$$

$$F_1 = \mathrm{F}(-\tfrac{1}{2},\tfrac{1}{2};\beta+\tfrac{1}{2};\gamma^2) \qquad \text{and} \qquad F_2 = \mathrm{F}(\tfrac{1}{2},\tfrac{1}{2};\beta+\tfrac{1}{2};\gamma^2). \tag{67, 68}$$

Similarly equation (60) yields for the periodic time

$$T = 2v^{-1}\Sigma\gamma^{2\beta-1}F_2. \tag{69}$$

From the results (65) and (69) the drift velocity for α a positive odd integer may be written

$$v_{\mathrm{d}} = \Delta y/T = v(2F_1/F_2-1). \tag{70}$$

Recalling the forms (43), when $\alpha = 1$ and thus $\beta = \frac{1}{2}$, the results (65), (69) and (70) reduce to those obtained by Seymour (1959, Section III, Case 2).

Case 4: α a Positive Even Integer, Electron Crosses Line $x = 0$ and Enters Non-reversed Field

Consider the case in which α is a positive even integer and the electron crosses the line $x = 0$ on which $B_z = 0$ and enters a region of non-reversed magnetic field. When α is a positive even integer, equation (7) shows that the magnetic field does not reverse direction for $x < 0$. In this case the magnetic field itself has symmetry about the neutral plane defined by $x = 0$, and typically the electron drift patterns are as shown in Fig. 4. From equations (16) and (24) the form (59) is again obtained for y, while use of equation (23) gives the form (60) for t.

In terms of ϕ_0 the limits given by equations (20) and (21) are

$$x_1 = -\{2mv\cos^2(\phi_0)/\beta e\lambda\}^{\beta} \qquad \text{and} \qquad x_2 = \{2mv\sin^2(\phi_0)/\beta e\lambda\}^{\beta}. \tag{71, 72}$$

From the drift pattern of Fig. 4, Δy can be expressed by means of equation (59) as

$$\Delta y = 2(mv/e\lambda)^{\beta}\,\beta^{1-\beta}\left(\int_{\psi_0}^{\frac{1}{2}\pi}\frac{\sin\psi\,\mathrm{d}\psi}{(\sin\psi-\sin\psi_0)^{1-\beta}}+\int_{3\pi/2}^{\pi-\psi_0}\frac{\sin\psi\,\mathrm{d}\psi}{(\sin\psi-\sin\psi_0)^{1-\beta}}\right). \tag{73}$$

Again using $\psi = \frac{1}{2}\pi-2\phi$, and $\psi_0 = \frac{1}{2}\pi-2\phi_0$, equation (73) becomes

$$\Delta y = 2^{1+\beta}(mv/e\lambda)^{\beta}\,\beta^{1-\beta}\left(\int_0^{\phi_0}\frac{(2\cos^2\phi-1)\,\mathrm{d}\phi}{(\sin^2\phi_0-\sin^2\phi)^{1-\beta}}+\int_{-\phi_0}^{-\frac{1}{4}\pi}\frac{(1-2\sin^2\phi)\,\mathrm{d}\phi}{(\cos^2\phi-\cos^2\phi_0)^{1-\beta}}\right). \tag{74}$$

Changing the variable ϕ to θ in the first integral of equation (74) by means of the relationship $\sin\phi = \sin\phi_0\sin\theta$, and changing ϕ to μ in the second integral by use of $\cos\phi = \cos\phi_0\sin\mu$, equation (74) can be written

$$\Delta y = 2^{2+\beta}(mv/e\lambda)^{\beta}\,\beta^{1-\beta}\left(\gamma^{2\beta-1}\int_0^{\frac{1}{2}\pi}\frac{\cos^{2\beta-1}\theta\,\mathrm{d}\theta}{(1-\gamma^2\sin^2\theta)^{-\frac{1}{2}}}-\tfrac{1}{2}\gamma^{2\beta-1}\int_0^{\frac{1}{2}\pi}\frac{\cos^{2\beta-1}\theta\,\mathrm{d}\theta}{(1-\gamma^2\sin^2\theta)^{\frac{1}{2}}}\right.$$

$$\left.+\tfrac{1}{2}\xi^{2\beta-1}\int_0^{\frac{1}{2}\pi}\frac{\cos^{2\beta-1}\mu\,\mathrm{d}\mu}{(1-\xi^2\sin^2\mu)^{\frac{1}{2}}}-\xi^{2\beta-1}\int_0^{\frac{1}{2}\pi}\frac{\cos^{2\beta-1}\mu\,\mathrm{d}\mu}{(1-\xi^2\sin^2\mu)^{-\frac{1}{2}}}\right), \tag{75}$$

where, from equations (27) and (72),

$$\gamma = \sin\phi_0 = (\tfrac{1}{2}\beta x_2/\rho_2)^{\frac{1}{2}} \qquad \text{and} \qquad \xi = \cos\phi_0 = (1-\gamma^2)^{\frac{1}{2}}.$$

In terms of the result (33), equation (75) finally becomes

$$\Delta y = \Sigma\{\gamma^{2\beta-1}(2F_1-F_2)-\xi^{2\beta-1}(2F_3-F_4)\}, \tag{76}$$

where Σ is given by equation (66), F_1 and F_2 are given by equations (67) and (68) respectively (but with β determined for α a positive even integer), and F_3 and F_4 are given by

$$F_3 = \mathrm{F}(-\tfrac{1}{2},\tfrac{1}{2};\beta+\tfrac{1}{2};\xi^2) \qquad \text{and} \qquad F_4 = \mathrm{F}(\tfrac{1}{2},\tfrac{1}{2};\beta+\tfrac{1}{2};\xi^2). \tag{77,78}$$

Similarly, reference to Fig. 4 and equation (60) enables the periodic time T in this case to be expressed as

$$T = v^{-1}\Sigma(\gamma^{2\beta-1}F_2+\xi^{2\beta-1}F_4). \tag{79}$$

From the results (76) and (79) the drift velocity for α a positive even integer becomes

$$v_{\mathrm{d}} = \Delta y/T = v\{\gamma^{2\beta-1}(2F_1-F_2)-\xi^{2\beta-1}(2F_3-F_4)\}/(\gamma^{2\beta-1}F_2+\xi^{2\beta-1}F_4). \tag{80}$$

Since α cannot be assigned the value unity in this case, there is no solution here which can be expressed in terms of the complete elliptic integrals given by the forms (43).

3. Alfvén's Approximate Drift Velocity

In Cases 1 and 2 of the previous section, the electron does not cross the neutral plane, and Alfvén's drift velocity is readily obtained from the exact results (42) and (58) as follows. Considering first the ratio of hypergeometric functions appearing in the result (42), use of the series (A1) of Appendix 1 and the binomial expansion $(1+x)^{-1} = 1-x+x^2-x^3+\ldots$ leads, for $\sigma \ll 1$, to the approximate result

$$\mathrm{F}(-\beta,\tfrac{1}{2};1;\sigma)/\mathrm{F}(1-\beta,\tfrac{1}{2};1;\sigma) \approx 1-\tfrac{1}{2}\sigma-\tfrac{1}{8}(1-\beta)\sigma^2. \tag{81}$$

It can be seen from equation (29) that the smallness of σ relative to unity implies orbital motion far from the neutral plane. Insertion of the result (81) into equation (42) yields the first-order β-dependent Alfvén drift velocity expression

$$v_{\mathrm{d}}/v = -\tfrac{1}{4}(1-\beta)\sigma. \tag{82}$$

When $\alpha = 1$ and $\beta = \frac{1}{2}$, equation (29) gives $\sigma = 4\rho_2/x_2$ and equation (82) reduces to

$$v_{\mathrm{d}}/v = -\tfrac{1}{8}\sigma, \tag{83}$$

as obtained by Seymour (1959, Section III, Case 1).

Considering now the ratio of confluent hypergeometric functions appearing in the result (58) for $\alpha = -1$, use of the series (A6) of Appendix 2 similarly leads, for $\nu \ll 1$, to the approximate result

$$\mathrm{M}(\tfrac{3}{2};2;-\nu)/\mathrm{M}(\tfrac{1}{2};1;-\nu) \approx 1-\tfrac{1}{4}\nu. \tag{84}$$

Substitution of this result into equation (58) gives the Alfvén drift velocity expression

$$v_d/v = \tfrac{1}{4}\nu \qquad \text{for} \qquad \alpha = -1. \tag{85}$$

From equation (29)

$$\rho_0/\rho_2 = (x_2/x_0)^{(1-\beta)/\beta}. \tag{86}$$

Elimination of σ from equation (82) by means of equation (29) leads to

$$\frac{v_d}{v} = -\tfrac{1}{2}\frac{1-\beta}{\beta}\frac{\rho_2}{x_2}. \tag{87}$$

When α approaches -1, β approaches infinity, and the limiting forms of equations (86) and (87) lead to the conclusion that

$$v_d/v = \tfrac{1}{4}\nu, \tag{88}$$

which is consistent with the result (85) in the small-perturbation limit.

4. Discussion

For Cases 1 and 3 of Section 2 above, which yielded Δy, T and v_d/v in terms of hypergeometric functions, it was noted that when $\alpha = 1$ these results assumed forms containing the well-tabulated complete elliptic integrals E and K (see e.g. Byrd and Friedman 1971). For $\alpha = -2$ in Case 1, equation (42) simplifies by means of linear transformations of hypergeometric functions (Abramowitz and Stegun 1965) to

$$v_d/v = \sigma/(\sigma-2), \tag{89}$$

where, from equation (29),

$$\sigma = -2\rho_2/x_2, \qquad \text{with} \qquad -\infty \leqslant \sigma \leqslant 0. \tag{90}$$

Similarly, when $\alpha = -3$ in Case 1, equation (42) becomes

$$v_d/v = -\{2\sigma^{-1}(1-K/E)-1\}, \tag{91}$$

where the complete elliptic integrals K and E have modulus

$$k = \{\sigma/(\sigma-1)\}^{\frac{1}{2}}, \qquad \text{with} \qquad \sigma = -4\rho_2/x_2 \qquad \text{and} \qquad 0 \leqslant k \leqslant 1. \tag{92}$$

The results (89) and (91) were used respectively to obtain the curves for $\alpha = -2$ and -3 shown in Fig. 5. However, in general, when particular values of α are chosen in Cases 1, 3 and 4 the hypergeometric functions so determined are not related to well-tabulated functions of mathematical physics. Since $\mathrm{F}(a,b;c;x)$ is not extensively tabulated for the great range and variety of combinations of a, b, c and x that may be encountered in practical situations, the most effective way of utilizing the principal

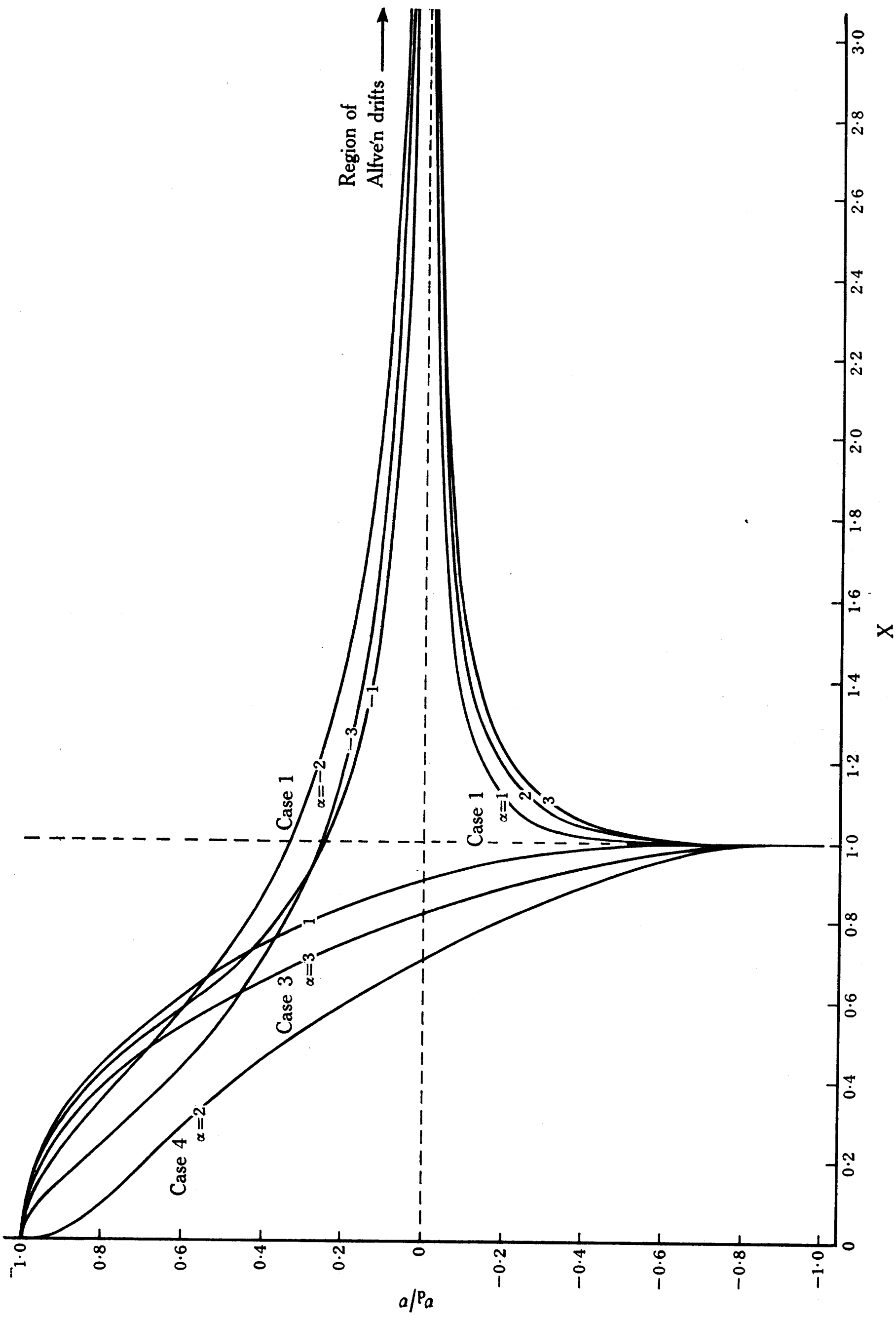

Fig. 5. Electron drift velocity curves v_d/v plotted against the parameter X as defined in Table 1 for the indicated cases and values of α.

results for v_d/v given by equations (42), (70) and (80) of Cases 1, 3 and 4 of Section 2 is to first calculate the specific $F(a,b;c;x)$ that are required.

Although the confluent hypergeometric function appearing in Case 2 is better tabulated (see e.g. Slater 1960; Abramowitz and Stegun 1965) the above conclusion appeared also to apply to the numerical evaluation of equation (58), which expresses v_d/v in terms of $M(a;b;x)$. Accordingly, one of us (P.W.S.) developed programs for use with the Monroe 1656 desk top computer–calculator (having 256 program step capacity) to obtain $F(a,b;c;x)$ and $M(a;b;x)$, and hence normalized drift velocity characteristics for selected values of α in Cases 1, 2, 3 and 4, as shown in Fig. 5. The parameter X used in this figure is defined for the different cases in Table 1. While the calculations leading to Fig. 5 were tedious, it is a simple matter to extend the Alfvén drift velocity region to smaller values of v_d/v for selected values of α by employing the small-perturbation results (82) and (85).

Table 1. Definition of parameter X for different cases

Case	$0 \leqslant X \leqslant 1$	$X \geqslant 1$
$1,\ \beta > 0$	Not applicable	$X = \sigma^{-\frac{1}{2}} = (\frac{1}{2}\beta x_2/\rho_2)^{\frac{1}{2}}$
$1,\ \beta < 0$	$X = \lvert\sigma\rvert^{-\frac{1}{2}} = (\frac{1}{2}\lvert\beta\rvert x_2/\rho_2)^{\frac{1}{2}}$	$X = \lvert\sigma\rvert^{-\frac{1}{2}} = (\frac{1}{2}\lvert\beta\rvert x_2/\rho_2)^{\frac{1}{2}}$
2	$X = \nu^{-\frac{1}{2}} = (\frac{1}{2}x_2/\rho_2)^{\frac{1}{2}}$	$X = \nu^{-\frac{1}{2}} = (\frac{1}{2}x_2/\rho_2)^{\frac{1}{2}}$
3	$X = \gamma = (\frac{1}{2}\beta x_2/\rho_2)^{\frac{1}{2}}$	Not applicable
4	$X = \gamma = (\frac{1}{2}\beta x_2/\rho_2)^{\frac{1}{2}}$	Not applicable

The motion of a charged particle in the magnetic field of a straight current-carrying conductor of infinite length has been investigated by Hertweck (1959). (One of us (M.H.) recently translated this paper from German into English.) In terms of a parameter λ (the reduced angular momentum of the particle), Hertweck considers four special cases of particle motion in the magnetic field external to the conductor. The only case which is analytically tractable is that of $\lambda = 0$, in which the particle motion is confined to a meridian plane. Detailed examination of Hertweck's case of $\lambda = 0$ shows that it corresponds in effect to the present Case 2 of $\alpha = -1$. Expressing our ratio v_d/v for an electron in terms of Hertweck's analysis parameters,

$$\frac{v_d}{v} = \frac{1}{(2\varepsilon)^{\frac{1}{2}}}\frac{\Delta\zeta}{\Delta\tau} = \frac{-\mathrm{i}\,J_1\big(\mathrm{i}(2\varepsilon)^{\frac{1}{2}}\big)}{J_0\big(\mathrm{i}(2\varepsilon)^{\frac{1}{2}}\big)}, \tag{93}$$

where ε is a dimensionless parameter formed from the ratio of two energies. From standard Bessel function theory (e.g. Bell 1968), equation (93) can be expressed in terms of modified Bessel functions as

$$\frac{v_d}{v} = \frac{1}{(2\varepsilon)^{\frac{1}{2}}}\frac{\Delta\zeta}{\Delta\tau} = \frac{I_1\big((2\varepsilon)^{\frac{1}{2}}\big)}{I_0\big((2\varepsilon)^{\frac{1}{2}}\big)}. \tag{94}$$

It is readily shown that $(2\varepsilon)^{\frac{1}{2}} = \frac{1}{2}\nu$, and so

$$v_d/v = I_1(\tfrac{1}{2}\nu)/I_0(\tfrac{1}{2}\nu), \tag{95}$$

an alternative form of our result (58). In direct confirmation of this result one can show that equations (56) and (57) may be further expressed as

$$\Delta y = 2\pi\rho_0\,\mathrm{I}_1(\tfrac{1}{2}\nu) \qquad \text{and} \qquad T = (2\pi\rho_0/v)\mathrm{I}_0(\tfrac{1}{2}\nu)\,, \tag{96,97}$$

after a number of transformations, whereupon, with $v_d = \Delta y/T$, the result (95) is immediately obtained. Calculations of v_d/v from equation (95), using suitable mathematical tables of modified Bessel functions (e.g. British Association for the Advancement of Science Mathematical Tables 1937, 1952), lead to results agreeing generally to four decimal places with those calculated from equation (58) by means of the Monroe 1656 computer–calculator approach described above. The advantage of the latter approach is, of course, that for all characteristics appearing in Fig. 5, the plotted points can be obtained as required for any selected value of the variable X, as defined in Table 1. On the other hand, for the particular characteristics which may be plotted using mathematical tables of special functions, one does not have this freedom of choice, and this may prove to be inconvenient.

With reference to Table 1 for the region $0 \leqslant X \leqslant 1$, the angle ψ_0 corresponding to zero drift velocity is found from the numerical work to be approximately $-40°$ for $\alpha = 1$, precisely zero for $\alpha = 2$ and approximately $-21°$ for $\alpha = 3$. From Fig. 4, for $\alpha = 2$ and electron drift in the negative direction of the y axis, the left-hand part of the drift cycle has zero displacement when $\psi_0 \approx -28°$. For electron drift in the positive direction of the y axis, the right-hand part of the cycle has zero displacement when $\psi_0 \approx +28°$.

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References

Abramowitz, M., and Stegun, I. A. (Eds) (1965). 'Handbook of Mathematical Functions', p. 559 (Dover: New York).
Bell, W. W. (1968). 'Special Functions for Scientists and Engineers', p. 92. (Van Nostrand: London).
British Association for the Advancement of Science Mathematical Tables, Vol. 6 (1937). 'Bessel Functions, Part 1' (Cambridge Univ. Press).
British Association for the Advancement of Science Mathematical Tables, Vol. 10 (1952). 'Bessel Functions, Part 2' (Cambridge Univ. Press).
Byrd, P. F., and Friedman, M. D. (1971). 'Handbook of Elliptic Integrals for Engineers and Scientists', 2nd Ed., p. 323 (Springer: Berlin).
Erdélyi, A. (Ed.) (1953). 'Higher Transcendental Functions', Vol. 1, p. 115 (McGraw–Hill: New York).
Hertweck, F. (1959). *Z. Naturforsch.* A **14**, 47.
Hochstadt, H. (1971). 'The Functions of Mathematical Physics', p. 195 (Wiley–Interscience: New York).
Seymour, P. W. (1959). *Aust. J. Phys.* **12**, 309.
Seymour, P. W. (1970). *Aust. J. Phys.* **23**, 133.
Slater, L. J. (1960). 'Confluent Hypergeometric Functions' (Cambridge Univ. Press).

Appendix 1. Derivation of Equation (33)

The hypergeometric differential equation has a series solution of the form

$$\mathrm{F}(a, b; c; x) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r r!} x^r , \tag{A1}$$

where the Pochhammer symbol $(a)_r$ is defined by

$$(a)_r = a(a+1)(a+2)\dots(a+r-1) . \tag{A2}$$

In terms of gamma functions,

$$(a)_r = \Gamma(a+)/\Gamma(a) \qquad \text{for} \qquad r \text{ a positive integer} . \tag{A3}$$

Hence, using the beta function

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \tag{A4}$$

the solution (A1) may be written as

$$\mathrm{F}(a, b; c; x) = \{\Gamma(c)/\Gamma(b)\,\Gamma(c-b)\} \int_0^1 t^{b-1}(1-t)^{c-b-1}\,\mathrm{d}t \sum_{r=0}^{\infty} (a)_r (tx)^r/r! .$$

Since by expansion

$$\sum_{r=0}^{\infty} (a)_r (tx)^r/r! = (1-tx)^{-a} ,$$

it follows that

$$\mathrm{F}(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}\,\mathrm{d}t}{(1-tx)^a} . \tag{A5}$$

The trigonometrical substitution $t = \sin^2 \phi$ reduces the form (A5) to the result (33) of Section 2, as quoted by Erdélyi (1953).

Appendix 2. Derivation of Equation (54)

The confluent hypergeometric differential equation has a series solution of the form

$$\mathrm{M}(a; b; x) = \sum_{r=0}^{\infty} \{(a)_r/(b)_r r!\} x^r . \tag{A6}$$

Following the procedure adopted in Appendix 1, the series (A6) may be written

$$\mathrm{M}(a; b; x) = \{\Gamma(b)/\Gamma(a)\,\Gamma(b-a)\} \int_0^1 \exp(xt)\, t^{a-1}(1-t)^{b-a-1}\,\mathrm{d}t \tag{A7}$$

since

$$\sum_{r=0}^{\infty} (xt)^r/r! = \exp(xt).$$

Changing the variable in equation (A7) by means of the relationship $2t = 1-\tau$, the solution (A7) becomes

$$\mathrm{M}(a;b;x) = \frac{2^{1-b}\Gamma(b)\exp(\frac{1}{2}x)}{\Gamma(a)\Gamma(b-a)} \int_{-1}^{1} \exp(-\tfrac{1}{2}x\tau)(1-\tau)^{a-1}(1+\tau)^{b-a-1}\,\mathrm{d}\tau, \tag{A8}$$

and in particular, when $a = \frac{1}{2}$, $b = 1$ and $v = -x$, the result (54) of Section 2 is obtained.

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