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On a Class of Exact Spherically Symmetric Solutions to the Einstein Gravitational Field Equations

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Abstract

This paper presents a development of a well-known class of Einstein gravitational fields which describe, in comoving coordinates, spherically symmetric perfect fluids with spatially uniform density but with nonuniform pressure. It is noted that the sign of the spatial curvature for these distributions may vary with time, in contrast with the situation for the uniform model theory used in most cosmological applications. It is shown, however, that this entails physically unacceptable behaviour of the fluid. In conclusion there is given a preliminary investigation of isolated fluid spheres, and a description of their noncovariant 'spatial geometry' is discussed.

1. Introduction

Considerable interest has been shown over the past decade in classes of spherically symmetric Einstein gravitational fields which describe nonstatic fluids with spatially uniform density (see Nariai (1967) and Vaidya (1968), while Rao (1973) lists some recent work). The metric for such fields is known, in the sense that the field equations have been reduced (in general) to a system of ordinary differential equations for two functions of the time coordinate. It is intended to give in the present paper an alternative development of these solutions, which focuses attention on their close analogy with the standard Robertson–Walker field. This development provides simpler equations for classifying the dynamical behaviour of objects described by subclasses of solutions of this type (Bonnor and Faulks 1967; Thompson and Whitrow 1967, 1968; Bondi 1969; Banerjee 1972), although it is not our intention in this paper to duplicate the work of other authors in this way.

With some exceptions (e.g. that of Cahill and McVittie (1970) who discuss negative mass shells), global distributions have not been fully discussed in the literature. This is possibly because, while the solutions are *physical*, they do not satisfy any sort of *cosmological principle*. Nevertheless we consider here, and dispose of, an interesting possibility for the behaviour of a global distribution. Finally it is noted that the present approach provides a simple picture of the spatial geometry of isolated fluid spheres.

2. Solutions with Uniform Density

We begin with the metric

$$\mathrm{d}s^2 = \exp(\nu)\,\mathrm{d}t^2 - c^{-2}\exp(\lambda)\{\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\}, \tag{1}$$

where $\nu = \nu(r,t)$ and $\lambda = \lambda(r,t)$. (Thompson and Whitrow (1967, 1968) and others

initially choose a more general metric, but then immediately impose a dynamical condition which reduces it to the above form up to a transformation of the type $r = r(r^*)$.) If we require that the coordinates be comoving then the field equations (and appropriate Cauchy data) completely specify the functions ν and λ. This class of solutions contains a subclass for which $\rho_r = 0$, that is, the density is spatially uniform.

The appropriate conservation equations are:

$$\rho_t + \tfrac{3}{2}\lambda_t(\rho + p/c^2) = 0 \qquad \text{and} \qquad p_r + \tfrac{1}{2}\nu_r(\rho c^2 + p) = 0. \tag{2a, b}$$

The field equations $G_0^0 = -\kappa T_0^0$ and $G_1^1 = -\kappa T_1^1$ reduce respectively to (e.g. McVittie 1965, p. 74; the typographical error therein should be noted)

$$-2\kappa\rho = -\tfrac{3}{2}\exp(-\nu)\lambda_t^2 + c^2\exp(-\lambda)\{2\lambda_{rr} + 4\lambda_r/r + \tfrac{1}{2}\lambda_r^2\} \tag{3a}$$

and

$$-2\kappa p/c^2 = \exp(-\nu)\{2\lambda_{tt} + \tfrac{3}{2}\lambda_t^2 - \lambda_t\nu_t\} - c^2\exp(-\lambda)\{\lambda_{rr} + \nu_{rr} + (\lambda_r + \nu_r)/r + \tfrac{1}{2}\nu_r^2\}. \tag{3b}$$

The remaining field equations are

$$G_2^2 \equiv G_3^3 = -\kappa T_3^3 \equiv -\kappa T_2^2 \equiv -\kappa T_1^1 \qquad \text{and} \qquad G_0^1 = 0.$$

We thus have $G_1^1 = G_2^2$, which gives

$$\lambda_{rr} + \nu_{rr} - (\lambda_r + \nu_r)/r - \lambda_r\nu_r + \tfrac{1}{2}\nu_r^2 - \tfrac{1}{2}\lambda_r^2 = 0. \tag{4}$$

The equation $G_0^1 = 0$ leads to

$$\lambda_{rt} = \tfrac{1}{2}\nu_r\lambda_t \tag{5a}$$

which, when integrated once with respect to r, becomes

$$\lambda_t = \sigma(t)\exp(\tfrac{1}{2}\nu), \tag{5b}$$

where σ is an arbitrary function of t. We use equation (5b) to replace the field equations (3a) and (3b) respectively by

$$\kappa\rho = \tfrac{3}{4}\sigma^2 - \tfrac{1}{2}c^2\exp(-\lambda)\{2\lambda_{rr} + 4\lambda_r/r + \tfrac{1}{2}\lambda_r^2\} \tag{6a}$$

and

$$\kappa p/c^2 = -\dot{\sigma}\exp(-\tfrac{1}{2}\nu) - \tfrac{3}{4}\sigma^2 + \tfrac{1}{2}c^2\exp(-\lambda)\{\lambda_{rr} + \nu_{rr} + (\lambda_r + \nu_r)/r + \tfrac{1}{2}\nu_r^2\}, \tag{6b}$$

where $\dot{\sigma} \equiv \mathrm{d}\sigma/\mathrm{d}t$.

The conservation equations (2), though not independent of the field equations, are sometimes more convenient to work with, however, and we shall need to refer to them later. If we now put $\rho = \rho(t)$ then we may integrate equation (2b) with respect to r to obtain

$$p = -\rho c^2 + h(t)\exp(-\tfrac{1}{2}\nu), \tag{7}$$

where $h(t)$ is an arbitrary function. This expression, once we have determined $\nu(r, t)$, gives $p(r, t)$, apart from arbitrary functions. We shall use the corresponding field equation (3b) to identify $h(t)$. We next use the condition $\rho = \rho(t)$ in equation (3a). If we define $-4c^2b = \tfrac{3}{4}\sigma^2 - \kappa\rho$, where $b = b(t)$, and if we put

$\lambda = 4\log\Lambda$ then equation (3a) becomes (for $\Lambda \not\equiv 0$)

$$\Lambda_{rr} + 2\Lambda_r/r + b\Lambda^5 = 0. \tag{8}$$

One family of solutions to equation (8) is

$$\Lambda^2 = S(t)\{1+\tfrac{1}{4}\alpha(t)r^2\}^{-1}. \tag{9}$$

This result is obtained by generalizing from the Robertson–Walker solutions which must also satisfy equation (8), which is an equation of the 'super-linear' type and is one form of the Emden–Fowler equation. This family of solutions (9) contains all solutions that are regular at $r = 0$ (J. A. Hempel, personal communication). The functions α and S are related by $bS^2 = 12\alpha$, but are otherwise arbitrary.

Now the function $\lambda(r,t)$ defined in this fashion also satisfies the equation

$$\lambda_{rr} - \lambda_r/r - \tfrac{1}{2}\lambda_r^2 = 0,$$

since equation (4) reduces to this in the case $p_r = 0$ (or by direct substitution). Thus equation (4) now becomes

$$\nu_{rr} - (\lambda_r + r^{-1})\nu_r + \tfrac{1}{2}\nu_r^2 = 0. \tag{10}$$

On integrating equation (10) twice with respect to r (using the integrating factor $r^{-1}\exp(-\lambda)$ to obtain ν_r), we obtain

$$\exp\nu = F^2(t)\{f(t)+\tfrac{1}{4}\alpha(t)r^2\}^2\{1+\tfrac{1}{4}\alpha(t)r^2\}^{-2}, \tag{11}$$

where F and f are arbitrary functions of t that are determined by substituting equation (11) into (5b). After some elementary calculation this substitution gives

$$\sigma F = 2(\dot{S}/S - \dot{\alpha}/\alpha) \qquad \text{and} \qquad f = (\dot{S}/S)(\dot{S}/S - \dot{\alpha}/\alpha)^{-1}.$$

If we now transform t according to

$$\sigma(t)\,S(t)\,\mathrm{d}t^* = 2\dot{S}(t)\,\mathrm{d}t,$$

giving

$$\sigma = 2S^{-1}\,\mathrm{d}S/\mathrm{d}t^*,$$

and then relabel t^* by t, we obtain

$$\exp(\tfrac{1}{2}\nu) = \{1+\tfrac{1}{4}\alpha(t)\gamma(t)r^2\}\{1+\tfrac{1}{4}\alpha(t)r^2\}^{-1}, \tag{12}$$

where

$$\gamma(t) = 1 - \dot{\alpha}S/\alpha\dot{S}, \tag{13}$$

and

$$\exp(\tfrac{1}{2}\lambda) = S(t)\{1+\tfrac{1}{4}\alpha(t)r^2\}^{-1}. \tag{14}$$

We now substitute from equations (12) and (14) into equations (6a) and (6b) and obtain, ultimately,

$$\kappa\rho = 3S^{-2}(\dot{S}^2 + \alpha c^2) \tag{15}$$

and

$$\kappa p/c^2 = S^{-2}\{-2S\ddot{S} + 2\dot{S}^2 + \alpha c^2(1+\gamma)\}\exp(-\tfrac{1}{2}\nu) - 3S^{-2}(\dot{S}^2 + \alpha c^2). \tag{16}$$

Equations (12)–(16) complete the solution.

The functions α and S are arbitrary but may be determined by imposing extra conditions. One expects such conditions to be based on some physical assumptions about the distribution of matter–energy to be described. However, it is not necessary for α and S to be completely determined in order to investigate the properties of a general space–time described by equations (1), (12) and (14), and it is to this investigation that we devote the remaining sections.

3. Behaviour at Infinity for Global Distributions with Positive Curvature

In an investigation of this type we hope to obtain indications of what features are important by considering those that correspond to important quantities in the uniform distributions described by the Robertson–Walker metric. Undoubtedly the first such quantity is the spatial curvature $\hat{R}$ which determines the 'size', rate of expansion, finiteness or otherwise, sign of curvature, 'age', and most of the features of interest in observational cosmology. Usually the fundamental quantity is taken to be the so-called scale factor $|\hat{R}|^{-\frac{1}{2}}$, which satisfies the Friedmann differential equations (see e.g. McVittie 1965, p. 142, equations 8.209 and 8.210), but here we take $R = |\hat{R}|^{-\frac{1}{2}}$ and $k = \operatorname{sgn}\hat{R}$.

In general for a line element of the form (1), the term 'spatial curvature' denotes the gaussian curvature $\hat{R}$ of the three-parameter manifold described by the line element

$$\mathrm{d}l^2 = \exp(\lambda)\{\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\}, \tag{17}$$

i.e. the hypersurface in the space–time (1) defined by $\mathrm{d}t = 0$. For this line element, t is constant and (r, θ, ϕ) are comoving, since $\hat{R}$ is not a tensor but is invariantly defined for all comoving systems, i.e. for those related by transformations of the type $x^i = x^i(x^{j*})$ and $t = t(t^*)$.

For the space–time described by the equations (1), (12) and (14) we obtain

$$\hat{R} = \alpha(t)\,S^{-2}(t).$$

We thus have the rather surprising result that, simply by prescribing as α any function of time which changes sign, there are classes of solutions for which the sign of the spatial curvature (fixed in the Robertson–Walker models) may vary with time. As this has rather odd consequences for the geometry and for what we normally regard as the spatial volume (e.g. transition between finite and infinite values is now apparently possible), it seems worth while to examine this aspect more closely. (We note in passing that equation (6a) is $\kappa\rho = \frac{3}{4}\sigma^2 + 3c^2\hat{R}$, which is valid for all space–times described by the metric (1) and $u^1 = 0$, not just those with uniform density. We also note that the quantity σ is that introduced in the previous section, while $\frac{1}{2}\sigma$ corresponds directly to the Hubble parameter of uniform model theory.) Since not all such solutions are likely to provide realistic models of actual objects, we now consider what restrictions might follow from some reasonable physical assumptions.

Let us suppose that $\rho \geqslant 0$ which, from equation (15), is equivalent to $c^2|\alpha| \leqslant \dot{S}^2$. Since the visible region, to which we expect our results to ultimately apply, may be regarded as being of overall nonnegative density, a model possessing only a uniform distribution of negative mass–energy is unlikely to be satisfactory.

It is more difficult to reject the possibility that $p < 0$, and thereby to assume that $p \geqslant 0$ for all (r, t). Although negative pressure would be regarded as unusual in the current astronomical context (Synge 1960, p. 186) we should not, for that reason, preclude the possibility that a model which may be quite reasonable from the standpoint of present physics at some particular time, e.g. the current epoch, might evolve to display regions of negative pressure under some extreme conditions. We therefore adopt the following neutral position. We do not reject $p < 0$ out of hand, although we expect that $p \geqslant 0$ describes the observable universe during the current epoch. Furthermore, we hold reservations on those models which evolve to a negative pressure state, and we investigate the conditions under which this might occur. Some additional physical assumptions are discussed in the following section, but here we confine our attention to global distributions with $\alpha \geqslant 0$.

If we differentiate equation (7) with respect to r, we obtain

$$p_r = \tfrac{1}{2}h(t)\, r\dot{\alpha}\dot{S}^{-1}(1+\tfrac{1}{4}\alpha\gamma r^2)^{-2}.$$

For fixed t this expression does not change sign in the range $0 \leqslant r < \infty$ and it is therefore sufficient, when investigating the circumstances under which p may become negative, to consider only the behaviour of $p(0, t)$ and $p(\infty, t)$, always provided that p is not singular in this range. Our approach is to require $p(0, t) \geqslant 0$ and then investigate $p(\infty, t)$. We remark that $p(0, t) \geqslant 0$ implies that $h(t) \geqslant 0$ by equations (7) and (12), and thus we have $\operatorname{sgn}(p_r) = \operatorname{sgn}(\dot{\alpha}/\dot{S})$.

Returning now to models which may undergo transition, and thus taking $\dot{\alpha} < 0$, we firstly consider the case $\dot{S} < 0$. From the preceding remark we know that $p(\infty, t) > 0$ unless $\exp(-\frac{1}{2}\nu)$ is singular, and from equation (12) we have that $\exp(-\frac{1}{2}\nu)$ is nonsingular provided $\alpha\gamma > 0$, that is, we have $\alpha - \dot{\alpha}S/\dot{S} > 0$, or $\alpha\dot{S} - \dot{\alpha}S < 0$. Hence we find that $S < \text{const.}\alpha$ as $\alpha \to 0_+$. We infer that there are no nonsingular transition solutions having $\dot{S} < 0$ as $\alpha \to 0_+$.

For the case $\dot{S} > 0$ as $\alpha \to 0_+$, we may show by a more complicated argument (see Appendix) that there do not exist C^1 functions (required for metric smoothness) α and S for which $p(\infty, t) > 0$ or for which $p(\infty, t) \to 0_+$ as $\alpha \to 0_+$. We may therefore conclude that there are no global transition solutions which do not develop a region of negative pressure as the transition from $\alpha > 0$ to $\alpha < 0$ proceeds. Hence, in the absence of some known physical mechanism permitting $p < 0$, we must regard such models as being of mathematical interest only.

4. Preliminary Remarks on Isolated Matter with Negative Curvature

In this section we establish that the only distributions with negative spatial curvature which make sense physically are isolated ones. We note in passing that it is of course possible to have isolated distributions with positive curvature simply by introducing a boundary into the corresponding global distribution. Then the restrictions (considered in the previous section) on transitions of sign of curvature do not apply in all cases. In fact, for some subclasses of solutions to (15) and (16), this behaviour does occur, but without the interesting consequences noted above.

It will be convenient to introduce the terms 'open' and 'closed' to describe solutions with $\alpha < 0$ and $\alpha > 0$ respectively. The basis for these terms is the definition for the spatial volume (of a global distribution) which gives the volume V

associated with the line element (17) for fixed t. Thus

$$V = 2\pi^2 S^3(t)\alpha^{-3/2}(t) \qquad \text{for} \qquad \alpha > 0,$$
$$= \infty \qquad\qquad\qquad\qquad\qquad\qquad \alpha < 0.$$

We retain this terminology for isolated distributions although V is no longer infinite when $\alpha < 0$.

We now consider open distributions. The metric coefficient $\exp \nu$ may, depending on the sign of $\alpha\gamma$, have one or two singularities. One of these, at $r_\infty(t)$, defined by $1+\frac{1}{4}\alpha r_\infty^2 = 0$ for $\alpha < 0$, defines the limits of the distribution since the volume enclosed in the region $0 \leqslant r \leqslant r_\infty$ is infinite. The other is at $r_s(t)$ defined by $1+\frac{1}{4}\alpha\gamma r_s^2 = 0$ for $\alpha\gamma < 0$, for which $p(r_s, t) = \infty$ by equation (16). If we are not to permit infinite pressures within the distribution then we must assume that $r_s > r_\infty$, which is equivalent to $\alpha\gamma > \alpha$. Hence, by equation (13), this implies that $\dot{\alpha}/\dot{S} < 0$, that is, $\dot{\alpha}$ and $\dot{S}$ are of opposite sign.

From equations (7) and (12) we have $p(r_\infty, t) = -\rho c^2$. Thus any open global distribution possesses a region of negative pressure at all times. In view of the remarks in the preceding section, and since we lack the physical knowledge of the behaviour of such regions that is needed in order to apply extra conditions to determine α and S, we confine our attention to isolated distributions. We therefore place a boundary at $r = r^*$ (a constant, since the coordinates are comoving). We note that we might now allow $r^* < r_s < r_\infty$ and still meet the requirement for finite pressure, and we consider this case in Section 6. However, the result for global distributions remains true, namely that to avoid a singularity $\dot{\alpha}$ and $\dot{S}$ must have opposite signs.

To conclude this section, we briefly consider the possible choices of boundary conditions to impose at r^*. We assume that the metric components are always at least continuous across hypersurfaces with space-like normals (cf. Lichnerowicz 1955). This avoids a discontinuity in the velocity of a material particle crossing the boundary (with arbitrary velocity). However, this requirement provides in general no additional information to determine α and S. Further conditions, which we will use in the determination of α and S, include $p(r^*, t) = 0$ and continuity in the first derivatives of the metric at r^*. These conditions are not independent if the distribution is located in vacuum, in which case the exterior field is Schwarzschildian. This result for a general spherically symmetric isolated fluid has been obtained by Misner and Sharp (1964). For the field defined by equations (1), (12) and (14), it is possible to give a quite elementary and direct proof of this result (Cook 1973). Of course we need not impose the boundary condition $p(r^*, t) = 0$ and still require metric smoothness. In such a case we need to assume that the distribution is surrounded by something other than vacuum, for which there may be difficulty in determining the exterior field, although this is by no means necessary for some applications.

5. Geometric Picture

The geometry of the class of space–times corresponding to the models considered in this paper is completely determined by the line element (1) considered in Section 2 and is independent of the coordinate system used. It is illuminating, however, to construct a picture of the geometry of 'space' (which does depend on

the coordinate system) and to consider how this evolves with time, in comparison with the corresponding picture for the Robertson–Walker solutions (Robertson and Noonan 1968, p. 340).

If we define the restriction of the line element (1) to the hypersurface t = const. by $\mathrm{d}l^2 = -c^2\,\mathrm{d}s^2$ for $\mathrm{d}t = 0$ then we have

$$\mathrm{d}l^2 = S^2(t)\{1+\tfrac{1}{4}\alpha(t)r^2\}^{-2}\{\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\}. \tag{18}$$

This is also the 'spatial distance' element $\mathrm{d}l'^2$ measured by light signals. It is well known that for a line element of the form

$$\mathrm{d}s^2 = g_{00}(\mathrm{d}x^0)^2 + 2c^{-1}g_{0j}\,\mathrm{d}x^0\mathrm{d}x^j - c^{-2}g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j,$$

we have

$$\mathrm{d}l'^2 = (g_{ij} + g_{0i}g_{0j}/g_{00})\,\mathrm{d}x^i\mathrm{d}x^j. \tag{19}$$

Since the line element (1) is orthogonal in the chosen coordinate system, we have, for this choice of coordinates, $\mathrm{d}l^2 = \mathrm{d}l'^2$. It was remarked in Section 3 that equation (18) is the line element for a 3-space of (spatially) uniform curvature αS^{-2}. For $\alpha > 0$, we apply the term 'spherical', following Robertson and Noonan (1968), and define its 'radius' by $S\alpha^{-\frac{1}{2}}$ in the usual way.

For the Robertson–Walker solutions we may take $\alpha = 1$, and in this case the transformation

$$r = 2\tan(\tfrac{1}{2}\psi),$$

which corresponds to stereographic projection of the surface of a 4-sphere onto a 'flat' 3-space, transforms the line element (18) with $\alpha = 1$ to

$$\mathrm{d}l^2 = S^2(t)(\mathrm{d}\psi^2 + \sin^2\psi\,\mathrm{d}\omega^2),$$

where

$$\mathrm{d}\omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2.$$

This is the line element for a 4-sphere in 4-spherical polar coordinates (ψ, θ, ϕ), and so we have for the case $\alpha = 1$ a picture of a 4-spherical surface with a time-dependent radius $S(t)$.

For the case $\dot{\alpha} \neq 0$, the corresponding transformation for equation (18) is

$$\alpha^{\frac{1}{2}}r = 2\tan(\tfrac{1}{2}\psi), \tag{20}$$

which tranforms the line element into

$$\mathrm{d}l^2 = S^2(t)\alpha^{-1}(t)(\mathrm{d}\psi^2 + \sin^2\psi\,\mathrm{d}\omega^2).$$

However, ψ is no longer a comoving coordinate and $\mathrm{d}l^2$ in this system is no longer equivalent to $\mathrm{d}l'^2$ (as it was for $\alpha = 1$) since the term corresponding to g_{01} in equation (19) is not zero after the transformation (20).

If we differentiate equation (20) with respect to t then, for any particular fluid particle, we have after some slight simplification

$$\mathrm{d}\psi/\mathrm{d}t = \tfrac{1}{2}\dot{\alpha}\sin(\psi/\alpha).$$

Now for Robertson–Walker solutions which describe global distributions of matter, the range of ψ is $(0, \pi)$. Since we are here concerned with isolated distributions, the

range of ψ becomes $(0, \psi^*)$ where from equation (20) ψ^* is defined by

$$\psi^* = 2\arctan(\alpha^{\frac{1}{2}}K), \qquad \text{where} \qquad K = 2/r^*.$$

We note that ψ^* is thus a function of t and that $\psi^* \to \pi$ as $\alpha \to \infty$. In conclusion then, we have the picture of a uniform 4-spherical surface with a time-dependent radius $S\alpha^{-\frac{1}{2}}$, with a 'cap' (that is, $\pi \geqslant \psi > \psi^*$) removed, and with motion of the particles on the surface given by $(\frac{1}{2}\dot{\alpha}\sin\psi/\alpha, 0, 0)$, such that for a contracting model ($\dot{\alpha} > 0$ and $\alpha \to \infty$) the 'cap' shrinks ($\psi^* \to \pi$) and the model approaches a global distribution in this limit.

6. Conclusions

In order to justify the remark in Section 4 concerning the sign of $\dot{\alpha}/\dot{S}$ we need only substitute from equations (12) and (15) into (16) to obtain

$$p(r,t) = \frac{p(0,t) + \frac{1}{4}r^2\{p(0,t) + c^2\rho\dot{\alpha}S/\dot{S}\}}{1 + \frac{1}{4}\alpha\gamma r^2}$$

and observe that the value of r for which the numerator vanishes will be less than that for which the denominator vanishes provided $\dot{\alpha}$ and $\dot{S}$ have opposite signs. Applications of the models of isolated fluid spheres studied here have been clearly documented in the works quoted. For global distributions the requirement for spherical symmetry must be seen as being physically implausible, although it is significant that there is no nonzero lower limit to the extent of departure from uniformity necessary for the behaviour described here to ensue. To this point we have considered fields described by equations (1), (12) and (14) for α and S any admissible functions of time. Various subclasses which permit further integration of equations (15) and (16) have been extensively studied in the literature. We consider one particular case here because this integration is particularly simple and because of an interesting physical interpretation.

Let us choose to regard the nonzero pressure gradient in a fluid sphere as describing a varying proportion of radiation to particle matter throughout the distribution, and put

$$p(0,t) = c^2\rho/n \qquad \text{and} \qquad p(r^*,t) = mp(0,t),$$

where m and n are dimensionless constants and $n \geqslant 3$ and $0 \leqslant m \leqslant 1$. The case $n = 3$ describes a sphere with a pure radiation core. The case $m = 0$ describes a pressureless boundary and $m = 1$ a uniform fluid. These conditions now completely determine α and S, and integration yields

$$S = A(\alpha + K^2)^\beta, \qquad \text{where} \qquad \beta = (m+n)/(1-m),$$

A is a positive constant and $K = 2/r^*$. This condition is similar to that chosen by Banerjee (1972) to define a subclass of the general system. We are left with a first-order ordinary differential equation for α which is autonomous and, while it has no closed form solution, it may be readily integrated numerically. This equation is

$$\beta^2\dot{\alpha}^2 + c^2\alpha A^{-2}(\alpha + K^2)^{2+2\beta} = B^2 A^{-3(1-1/n)}(\alpha + K^2)^{2+3(1+1/n)\beta},$$

where B is a constant.

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References

Banerjee, A. (1972). *J. Phys.* A **5**, 1305.
Bondi, H. (1969). *Mon. Notic. Roy. Astron. Soc.* **142**, 333.
Bonnor, W. B., and Faulks, M. C. (1967). *Mon. Notic. Roy. Astron. Soc.* **137**, 239.
Cahill, M. E., and McVittie, G. C. (1970). *J. Math. Phys.* **11**, 1382.
Cook, M. W. (1973). Ph.D. Thesis, University of New England.
Lichnerowicz, A. (1955). 'Théories Relativistes de la Gravitation et de L'électromagnétisme', p. 5 (Masson: Paris).
McVittie, G. C. (1965). 'General Relativity and Cosmology', 2nd Ed. (Chapman & Hall: London).
Misner, C. W., and Sharp, D. H. (1964). *Phys. Rev.* B **136**, 571.
Nariai, H. (1967). *Progr. Theor. Phys.* **38**, 92.
Rao, J. Krishna (1973). *Gen. Relativ. Gravitat.* **4**, 351.
Robertson, H. P., and Noonan, T. W. (1968). 'Relativity and Cosmology' (Saunders: New York).
Synge, J. L. (1960). 'Relativity: The General Theory' (North-Holland: Amsterdam).
Thompson, I. H., and Whitrow, G. J. (1967). *Mon. Notic. Roy. Astron. Soc.* **136**, 207.
Thompson, I. H., and Whitrow, G. J. (1968). *Mon. Notic. Roy. Astron. Soc.* **139**, 499.
Vaidya, P. C. (1968). *Phys. Rev.* **147**, 1615.

Appendix

This appendix gives in detail the argument which establishes the result mentioned in Section 3: that there do not exist 'transition solutions' with nonnegative pressure. Our first step is to rewrite equation (2a) as

$$\partial\{\rho\exp(\tfrac{3}{2}\lambda)\}/\partial t = -pc^{-2}\,\partial\{\exp(\tfrac{3}{2}\lambda)\}/\partial t,$$

that is, from equation (14),

$$\frac{\partial}{\partial t}\left(\frac{\rho S^3}{(1+\frac{1}{4}\alpha r^2)^3}\right) = -pc^{-2}\,\frac{\partial}{\partial t}\left(\frac{S^3}{(1+\frac{1}{4}\alpha r^2)^3}\right). \tag{A1}$$

Designating $p(0,t)$ and $p(\infty,t)$ as p_0 and p_∞ respectively, we have for, $r = 0$,

$$\mathrm{d}(\rho S^3)/\mathrm{d}t = -p_0\,c^{-2}\,\mathrm{d}S^3/\mathrm{d}t, \tag{A2}$$

and, for $r = \infty$, after multiplying equation (A1) by r^6 and taking limits as $r \to \infty$,

$$\mathrm{d}(\rho S^3\alpha^{-3})/\mathrm{d}t = -p_\infty\,c^{-2}\,\mathrm{d}(S^3\alpha^{-3})/\mathrm{d}t. \tag{A3}$$

Expansion of equation (A3) gives

$$\alpha^{-3}\,\mathrm{d}(\rho S^3)/\mathrm{d}t + \rho S^3\,\mathrm{d}\alpha^{-3}/\mathrm{d}t = -p_\infty\,c^{-2}\{\alpha^{-3}\,\mathrm{d}S^3/\mathrm{d}t + S^3\,\mathrm{d}\alpha^{-3}/\mathrm{d}t\}.$$

Following substitution into the first term on the left-hand side of this equation from (A2), we obtain after some simplification

$$p_\infty = (p_0\,\alpha\dot{S} + \rho c^2\dot{\alpha}S)/(\alpha\dot{S} - \dot{\alpha}S). \tag{A4}$$

This indicates that as $\alpha \to 0_+$ we have $p_\infty < 0$ unless $\dot{\alpha} \to 0$ also. We require $S \nrightarrow 0$ to avoid a global singularity at $\alpha = 0$ and, since $\dot{\alpha}/\alpha \nrightarrow 0$ as $\alpha \to 0$, the right-hand term in the denominator of equation (A4) is always dominant as $\alpha \to 0$. Hence to maintain $p_\infty > 0$ as $\alpha \to 0_+$, we need firstly that $\rho \to 0$ (and thus $\dot{S} \to 0$ by equation 15) and secondly that $p_0 \alpha \dot{S} \succ \rho c^2 \dot{\alpha} S$, where the symbol $\succ$ denotes 'dominates'.

If we put $\rho \approx (3/\kappa) g(\alpha)$, where $g \succ \alpha$ (since $\dot{S}^2 \to 0_+$), we obtain

$$\dot{S} \approx g^{\frac{1}{2}} S_0, \qquad \text{where} \qquad S_0 = \lim_{\alpha \to 0_+} S.$$

From equation (A2) we have

$$p_0 = -c^2(\dot{\rho} S + 3\rho \dot{S})/3\dot{S},$$

and hence

$$\kappa c^{-2} p_0 \approx -(\dot{\alpha} g' S_0 + 3g^{3/2} S_0)/g^{\frac{1}{2}} S_0$$

$$\approx -(\dot{\alpha} g^{-\frac{1}{2}} g' + 3g), \qquad \text{where} \qquad g' \equiv \mathrm{d}g/\mathrm{d}\alpha.$$

Therefore, from equation (A4) we have

$$\kappa c^{-2} p_\infty \approx \{-(\dot{\alpha} g^{-\frac{1}{2}} g' + 3g)\alpha g^{\frac{1}{2}} S_0 + 3g S_0 \dot{\alpha}\}/(\alpha g^{\frac{1}{2}} S_0 - \dot{\alpha} S_0).$$

Now, we have $\alpha g^{\frac{1}{2}} \prec \dot{\alpha}$ since $\dot{\alpha} \succ \alpha$ and $g^{\frac{1}{2}} \to 0$, and hence it follows that

$$\kappa c^{-2} p_\infty \approx \alpha g' + 3(\alpha/\dot{\alpha}) g^{3/2} - 3g.$$

However, as $\alpha g^{3/2}/\dot{\alpha} \prec g$ (since $\alpha g^{\frac{1}{2}} \prec \dot{\alpha}$), we have

$$\kappa c^{-2} p_\infty \approx \alpha g' - 3g.$$

Now $\alpha g' - 3g > 0$ holds if and only if $g \prec \alpha^3$, so that we therefore have: if $p_\infty \geqslant 0$ then

$$g \prec \alpha^3 \prec \alpha \prec g.$$

This statement is, of course, self-contradictory, thus completing our argument.

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