Aust. J. Phys., 1975, **28**, 715–30

On the Classical Theory of Equilibrium Between Matter and Radiation

J. J. Monaghan

Department of Mathematics, Monash University, Clayton, Vic. 3168.

Abstract

The problem of establishing the Rayleigh–Jeans law for equilibrium electromagnetic radiation in a cavity is studied without making the customary simplifying assumptions. By using a Hamiltonian formalism analogous to that introduced by Fermi for quantum electrodynamics the analysis is simplified, general expressions for absorption and emission are obtained, and the correspondence with the quantum mechanical treatment is established. The model considered consists of a cavity which contains classical charged particles which move in an arbitrary potential while interacting with electromagnetic radiation. The work covers much the same ground as the fundamental but neglected work of McLaren, though the methods used are simpler and more direct. The applications are to those parts of radio astronomy where the wavelengths are sufficiently large to allow a classical description. In particular, Twiss's analysis of stimulated emission at radio wavelengths is incorporated in the analysis.

1. Introduction

The character of electromagnetic radiation in thermal equilibrium with matter may be determined in two ways. The first and simplest is to focus attention on the equilibrium state and pay no attention to whether, or in what manner, equilibrium occurs. The second method, which is more complicated but, correspondingly, is more informative, examines the processes of absorption and emission and establishes the conditions for equilibrium by balancing the two. The second method is clearly the more fundamental. In addition it has the advantage of giving a description of the non-equilibrium situation.

In both the quantum and the classical case the description of radiation by the first method is trivial. The electromagnetic energy in a cavity can be written as the energy of a set of independent simple harmonic oscillators. The classical equipartition theorem then gives the Rayleigh–Jeans law, while quantum statistics gives the Planck law.

In the literature the application of the second method is simplified by using thermodynamic arguments to establish that radiation in thermal equilibrium is independent of the matter with which it interacts. Indeed, if the investigation is theoretical, the matter need not even exist provided the equations of motion describing the matter are consistent with the laws of mechanics, classical or quantum, according to the approximation. The work of Planck (1912) and of Born (lectures reprinted 1969) assume that the matter is a collection of simple harmonic oscillators, each of which is linear and so massive that Doppler shifts may be ignored. Lorentz (1915) takes as his model a thin metal plate. A typical quantum calculation is that of Einstein

(1917) who established the Planck law by considering the emission and absorption of a hypothetical two-level quantum system.

None of the foregoing investigations can claim to be fundamental, for the thermodynamic laws should be a result of, not a tool for, the investigation of the equilibrium between matter and radiation. The first comprehensive attempt to place the classical model on a more fundamental basis was made by McLaren (1911, 1912) in two important, but neglected papers. McLaren discussed the general problem of the Hamiltonian description of the electron and its field, and showed that the Rayleigh–Jeans law resulted from the balance of emission and absorption. Much later Le Roux (1960), unaware of McLaren's work, considered the same problem, but allowed for relativistic motion. Le Roux also established the Rayleigh–Jeans law, but showed in addition that the Planck law could be obtained from the purely classical calculation provided the derivative of the particle distribution function with respect to energy was replaced by a difference expression in a plausible, if *ad hoc*, way.

Although the classical description is only approximate, it is a useful approximation in those areas where the quantum numbers are large. Thus, in radio astronomy the classical description is useful and considerations of the classical emission and absorption processes are relevant. For this reason, and also we believe because of its intrinsic interest we attempt, in this paper, to simplify and clarify the analysis of the problem. The plan is to follow Fermi (1932) and consider the particles and the radiation in a large cavity. The radiation field can then be represented as a superposition of standing waves satisfying periodic boundary conditions. The Hamiltonian of the system is easily constructed and, working to first order in the field–particle interaction, the emission and absorption can be found. By taking the limit as the volume goes to infinity, the interaction between particles and radiation in free space can be studied.

Because the formulation is similar to that used in quantum electrodynamics the method has the advantage that, at each stage, the classical expressions can be easily related to their quantum equivalents. In particular, the operator perturbation method employed has a simple analogue in quantum theory and the absorption and emission rates can be related to the Einstein coefficients for stimulated and spontaneous emission. The method used by Le Roux (1960) makes use of a special perturbation technique which does not easily allow suggestive comparisons with quantum theory and fails to clarify the nature of the approximations which result in Planck's law.

In order to make statistical arguments it is convenient to first consider a single electron in the radiation field and then average over phase space. Since the present analysis is concerned solely with the character of the radiation field, we take the distribution of the particles to be known. The equilibrium radiation field is then found when the particle distribution is governed by the Maxwell–Boltzmann distribution.

2. Radiation Field

While adopting Fermi's method in principle it is preferable here to use Bethe's (1964) formulation. Thus for a cavity of volume V, the vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ is written

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_{\boldsymbol{k}}{}' \sum_{\lambda=1,2} \{q_{\boldsymbol{k}\lambda}(t)\,\boldsymbol{u}_{\boldsymbol{k}\lambda}(\boldsymbol{r}) + q^{*}_{\boldsymbol{k}\lambda}(t)\,\boldsymbol{u}^{*}_{\boldsymbol{k}\lambda}(\boldsymbol{r})\}\,, \tag{1}$$

where the asterisk denotes a complex conjugate;

$$\boldsymbol{u}_{k\lambda}(\boldsymbol{r}) = (4\pi c^2/V)^{\frac{1}{2}}\,\boldsymbol{\varepsilon}_{k\lambda}\exp(\mathrm{i}\,\boldsymbol{k}\,.\,\boldsymbol{r}),$$

$\boldsymbol{\varepsilon}_{k\lambda}$ being a unit polarization vector; the prime on the summation over $\boldsymbol{k}$ indicates that it is only over one hemisphere of k space; and $\boldsymbol{\varepsilon}_{k\lambda}\,.\,\boldsymbol{k} = 0$ since $\nabla\,.\,\boldsymbol{A} = 0$ for each mode, and the normalization is given by

$$\int \boldsymbol{u}_{k'\lambda'}\,.\,\boldsymbol{u}_{k\lambda}\,\mathrm{d}V = 0, \qquad \int \boldsymbol{u}_{k'\lambda'}\,.\,\boldsymbol{u}^*_{k\lambda}\,\mathrm{d}V = 4\pi c^2\,\delta_{kk'}\,\delta_{\lambda\lambda'}\,. \tag{2}$$

The two values of λ allow for the two independent polarization directions perpendicular to $\boldsymbol{k}$.

Using the foregoing results it is easy to verify that the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ become

$$\boldsymbol{E} = -c^{-1}\,\partial \boldsymbol{A}/\partial t = -c^{-1}\sum_{k}{}'\sum_{\lambda}\{\dot{q}_{k\lambda}(t)\,\boldsymbol{u}_{k\lambda} + \text{c.c.}\}, \tag{3a}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \sum_{k}{}'\sum_{\lambda}\{\mathrm{i}\,q_{k\lambda}(t)\,\boldsymbol{k}\times\boldsymbol{u}_{k\lambda} + \text{c.c.}\}, \tag{3b}$$

where c.c. denotes the complex conjugate of the other expression in the brackets. The electromagnetic field energy W_{em} is given by

$$W_{\text{em}} = \sum_{k}{}'\sum_{\lambda}\{\dot{q}_{k\lambda}\,\dot{q}^*_{k\lambda} + k^2c^2 q_{k\lambda}\,q^*_{k\lambda}\}. \tag{4}$$

The canonical momenta are then given by

$$\partial W_{\text{em}}/\partial \dot{q}_{k\lambda} = p_{k\lambda} = \dot{q}^*_{k\lambda}, \qquad \partial W_{\text{em}}/\partial \dot{q}^*_{k\lambda} = p^*_{k\lambda} = \dot{q}_{k\lambda}. \tag{5}$$

Therefore the free field Hamiltonian H is

$$H = \sum_{k}{}'\sum_{\lambda}\{p_{k\lambda}\,p^*_{k\lambda} + k^2c^2 q_{k\lambda}\,q^*_{k\lambda}\}. \tag{6}$$

3. Particle and Radiation

The Lagrangian of a charged particle with charge σ in an electromagnetic field described by a vector potential $\boldsymbol{A}'$ and a scalar potential U (which may include a non-electromagnetic part) is (see e.g. Jackson 1962)

$$L = -mc^2(1 - v^2/c^2)^{\frac{1}{2}} + (\sigma/c)\boldsymbol{v}\,.\,\boldsymbol{A}' - \sigma U. \tag{7}$$

Here

$$\boldsymbol{A}' = \boldsymbol{A}_0 + \boldsymbol{A}, \tag{8}$$

where $\boldsymbol{A}$ is the contribution from the radiation and $\boldsymbol{A}_0$ is due to a static background magnetic field. Following the normal rules, the canonical momentum is found to be

$$\boldsymbol{p} = m\boldsymbol{v}(1 - v^2/c^2)^{-\frac{1}{2}} + (\sigma/c)\boldsymbol{A}' \tag{9}$$

and the Hamiltonian is

$$W_p = \sigma U + \{m^2c^4 + c^2(\boldsymbol{p} - \sigma \boldsymbol{A}'/c)^2\}^{\frac{1}{2}}. \tag{10}$$

If the motion is nonrelativistic (the relativistic case is considered in Section 7)

$$W_p \approx \sigma U + mc^2 + (\boldsymbol{p} - \sigma \boldsymbol{A}'/c)^2/2m,$$

and, furthermore, if the radiation is a weak perturbation

$$W_p = \sigma U + mc^2 + p_0^2/2m - (\sigma/mc)\boldsymbol{p}_0 . \boldsymbol{A},$$

where

$$\boldsymbol{p}_0 = \boldsymbol{p} - \sigma \boldsymbol{A}_0/c.$$

The constant term mc^2 in W_p has no effect on the dynamics and can be left out. The final nonrelativistic Hamiltonian for a single particle in a cavity radiation field is then

$$H = p_0^2/2m + \sigma U + \sum_k{}' \sum_\lambda \{p_{k\lambda} p_{k\lambda}^* + k^2c^2 q_{k\lambda} q_{k\lambda}^*\}$$
$$-(\sigma/mc)\boldsymbol{p}_0 . \left(\sum_k{}' \sum_\lambda \{q_{k\lambda} \boldsymbol{u}_{k\lambda} + q_{k\lambda}^* \boldsymbol{u}_{k\lambda}^*\}\right). \tag{11}$$

It is easy to show that H gives the correct nonrelativistic equation of motion for the particle. It is not, however, evident that the interaction term correctly describes the electromagnetic field in the presence of a charged particle. Confirmation that H is the correct Hamiltonian is given in the next section. Finally we remark that $\boldsymbol{u}_{k\lambda}$ in the interaction term is to be evaluated at the particle.

4. Emission of Radiation

The equations for the radiation field are

$$\partial H/\partial p_{k\lambda} = \dot{q}_{k\lambda}, \qquad \partial H/\partial q_{k\lambda} = -\dot{p}_{k\lambda}$$

and their complex conjugates. Thus

$$\ddot{q}_{k\lambda}^* + k^2c^2 q_{k\lambda}^* = (\sigma/mc)\boldsymbol{p}_0 . \boldsymbol{u}_{k\lambda}, \tag{12a}$$

$$\ddot{q}_{k\lambda} + k^2c^2 q_{k\lambda} = (\sigma/mc)\boldsymbol{p}_0 . \boldsymbol{u}_{k\lambda}^*. \tag{12b}$$

These last two equations describe the pumping of energy into the radiation field by the nonuniform motion of the particle. Defining

$$f_{k\lambda}(t) = (\sigma/mc)\boldsymbol{p}_0 . \boldsymbol{u}_{k\lambda} = (\sigma/c)\boldsymbol{v} . \boldsymbol{u}_{k\lambda}$$
$$= \sigma(4\pi/V)^{\frac{1}{2}} \varepsilon_{k\lambda} . \boldsymbol{v} \exp(\mathrm{i}\boldsymbol{k} . \boldsymbol{r}), \tag{13}$$

where $\boldsymbol{r}(t)$ is the position vector of the particle, it is easily shown that if there is no radiation at time t_0 the solution of equation (12b) is

$$q_{k\lambda} = -(kc)^{-1} \int_{t_0}^{t} f_{k\lambda}^*(\xi) \sin\{kc(\xi - t)\}\, \mathrm{d}\xi \tag{14}$$

with

$$\dot{q}_{k\lambda} = \int_{t_0}^{t} f_{k\lambda}^*(\xi)\cos\{kc(\xi-t)\}\,\mathrm{d}\xi.$$

Taking the complex conjugate of equation (14) gives the solution of (12a). The rate of change of electromagnetic energy becomes

$$\mathrm{d}W_{\mathrm{em}}/\mathrm{d}t = \sum_k{}' \sum_\lambda \{\dot{q}_{k\lambda}^* f_{k\lambda}^*(t) + \dot{q}_{k\lambda} f_{k\lambda}(t)\}. \tag{15}$$

If we now average over many particles distributed in phase space according to the distribution function ρ, equation (15) gives the average emission rate $\mathscr{E}$, for a specific $\boldsymbol{k}, \lambda$, as

$$\mathscr{E} = \int \cdots \int \rho \left(f_{k\lambda}^*(t) \int_{t_0}^{t} f_{k\lambda}(\xi)\cos\{kc(\xi-t)\}\,\mathrm{d}\xi + \text{c.c.}\right) \mathrm{d}V_n, \tag{16}$$

where $\mathrm{d}V_n$ is an element of volume in phase space.

The foregoing expressions may be checked by comparing the results obtained when $V \to \infty$ with the known results for a particle of charge σ radiating in free space. In Appendix 2 the electrical field deduced from equations (3a) and (14) is shown to be correct. Here the total energy radiated by a particle at rest in the infinite past and at rest in the infinite future ($t = \pm\infty$) will be calculated.

The total energy radiated is

$$W_{\mathrm{T}} = \int_{-\infty}^{\infty} \left(\frac{\mathrm{d}W_{\mathrm{em}}}{\mathrm{d}t}\right) \mathrm{d}t = \sum_k{}' \sum_\lambda \int_{-\infty}^{\infty} \{\dot{q}_{k\lambda}^* f_{k\lambda}^*(t) + \dot{q}_{k\lambda} f_{k\lambda}(t)\}\,\mathrm{d}t$$

which, with the foregoing expressions for $q_{k\lambda}$ and $f_{k\lambda}$, becomes

$$W_{\mathrm{T}} = \left(\frac{\sigma}{mc}\right)^2 \frac{4\pi c^2}{V} \sum_k{}' \sum_\lambda \int_{-\infty}^{\infty} \int_{-\infty}^{t} \{\boldsymbol{p}_0(t)\,.\,\boldsymbol{\varepsilon}_{k\lambda}\}\{\boldsymbol{p}_0(\xi)\,.\,\boldsymbol{\varepsilon}_{k\lambda}\}$$
$$\times \cos\{kc(\xi-t)\}\cos(\boldsymbol{k}\,.\,\Delta\boldsymbol{r})\,\mathrm{d}\xi\,\mathrm{d}t,$$

where $\Delta\boldsymbol{r} \equiv \boldsymbol{r}(\xi) - \boldsymbol{r}(t)$ and $t_0 = -\infty$. The sum over polarization can be simplified by recalling that there are two independent polarization vectors perpendicular to $\boldsymbol{k}$ and perpendicular to each other. Apart from this last constraint they are distributed uniformly in the plane perpendicular to $\boldsymbol{k}$. From Appendix 1 we find

$$\mathrm{Av}\left(\sum_\lambda \{\boldsymbol{p}_0(t)\,.\,\boldsymbol{\varepsilon}_{k\lambda}\}\{\boldsymbol{p}_0(\xi)\,.\,\boldsymbol{\varepsilon}_{k\lambda}\}\right) = \boldsymbol{p}_0(t)\times\hat{\boldsymbol{k}}\,.\,\boldsymbol{p}_0(\xi)\times\hat{\boldsymbol{k}}, \tag{17}$$

where $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/k$. Finally, writing the cosines in complex exponential form and converting the ξ integration to the range $(-\infty, \infty)$ by using the symmetry, we find

$$W_{\mathrm{T}} = \left(\frac{\sigma}{mc}\right)^2 \frac{2\pi c^2}{V} \sum_k{}' \left[\left| \int_{-\infty}^{\infty} \boldsymbol{p}_0(t)\times\hat{\boldsymbol{k}} \exp\{-\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{r}(t) + \mathrm{i}kct\}\,\mathrm{d}t \right|^2 \right.$$
$$\left. + \left| \int_{-\infty}^{\infty} \boldsymbol{p}_0(t)\times\hat{\boldsymbol{k}} \exp\{\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{r}(t) + \mathrm{i}kct\}\,\mathrm{d}t \right|^2 \right]. \tag{18}$$

When V is large the sum over $\boldsymbol{k}$ can be written as an integral. Recalling that the number of modes for a given polarization with k in the range k to $k+\mathrm{d}k$ and in the element of solid angle $\mathrm{d}\Omega$ is

$$Vk^2\,\mathrm{d}k\,\mathrm{d}\Omega/8\pi^3\,, \tag{19}$$

we find

$$\frac{\Delta W_\mathrm{T}}{\Delta k\,\Delta\Omega} = \frac{\sigma^2 k^2}{4m^2\pi^2}\left[\left|\int_{-\infty}^{\infty} \boldsymbol{p}_0 \times \hat{\boldsymbol{k}} \exp\{-\mathrm{i}\boldsymbol{k}.\boldsymbol{r} + \mathrm{i}kct\}\,\mathrm{d}t\right|^2 + \left|\int_{-\infty}^{\infty} \boldsymbol{p}_0 \times \hat{\boldsymbol{k}} \exp\{\mathrm{i}\boldsymbol{k}.\boldsymbol{r} + \mathrm{i}kct\}\,\mathrm{d}t\right|^2\right], \tag{20}$$

where, if an integration over Ω is later performed, it is only over a hemisphere. Indeed if the lower term in the square brackets is omitted the integration can be taken over a sphere. If a conversion to frequency $\omega = kc$ is made and $I(\omega)$ is defined by

$$\Delta W_\mathrm{T}/\Delta k\,\Delta\Omega = c\,\Delta I(\omega)/\Delta\omega\,\Delta\Omega\,,$$

the result (20) is equivalent to that given by Jackson (1962), provided it is recalled that in his formula the solid angle ranges over a sphere. We can therefore take equation (11) as the correct Hamiltonian, for it describes both the motion of the particle and the radiation field correctly.

Finally, for later reference it is convenient to establish the relationship between the classical emission (15) and its quantum equivalent. If the particle motion is periodic, with period $T = 2\pi/\omega_0$, the average rate of radiation of energy is

$$\langle \dot{W} \rangle = T^{-1} \int_0^T \dot{W}_\mathrm{em}\,\mathrm{d}t\,.$$

Performing manipulations analogous to those already performed to get W_T, we find, on replacing the summation by an integration with $k = \omega/c$, that

$$\left\langle \frac{\Delta\dot{W}}{\Delta\Omega} \right\rangle = \frac{1}{T\pi^2 c^3} \int_0^\infty \omega^2 \left(\left| \int_0^T (\sigma/m)\boldsymbol{p}(t) \times \hat{\boldsymbol{k}} \exp(-\mathrm{i}\boldsymbol{k}.\boldsymbol{r} + \mathrm{i}\omega t)\,\mathrm{d}t \right|^2 + (\boldsymbol{k} \to -\boldsymbol{k}) \right) \mathrm{d}\omega\,,$$

where $(\boldsymbol{k} \to -\boldsymbol{k})$ means the addition of a term similar to the preceding one with $\boldsymbol{k}$ replaced by $-\boldsymbol{k}$. If there is a continuous distribution of charge and therefore current density $\boldsymbol{J}(\boldsymbol{r}, t)$ this last expression generalizes to

$$\left\langle \frac{\Delta\dot{W}}{\Delta\Omega} \right\rangle = \frac{1}{T\pi^2 c^3} \int_0^\infty \omega^2 \left(\left| \iint_0^T \boldsymbol{J}(\boldsymbol{r}, t) \times \hat{\boldsymbol{k}} \exp(-\mathrm{i}\boldsymbol{k}.\boldsymbol{r} + \mathrm{i}\omega t)\,\mathrm{d}t\,\mathrm{d}\boldsymbol{r} \right|^2 + (\boldsymbol{k} \to -\boldsymbol{k}) \right) \mathrm{d}\omega\,.$$

If $J(r,t) = j(r)\exp(-i\omega_0 t)$, the integral over frequency is dominated by the contribution near $\omega = \omega_0$, and we find

$$\left\langle\frac{\Delta\dot{W}}{\Delta\Omega}\right\rangle \approx \frac{1}{T\pi^2c^3}\left(\left|\int j(r)\times\hat{k}\exp(-ik.r)\,dr\right|^2 + (k\to -k)\right)$$
$$\times\,\omega_0^2\int_{-\infty}^{\infty}\frac{\sin^2\{\frac{1}{2}(\omega-\omega_0)T\}}{(\omega-\omega_0)^2}\,d(\omega-\omega_0)$$
$$= \frac{\omega_0^2}{2\pi c^3}\left(\left|\int j(r)\times\hat{k}\exp(-ik.r)\,dr\right|^2 + (k\to -k)\right).$$

This result provides one method of establishing the quantum transition probability A_{ω_0} for spontaneous emission (Bethe 1964, p. 142). We find

$$A_{\omega_0} = (k\omega_0)^{-1}\langle\Delta\dot{W}/\Delta\Omega\rangle,$$

or alternatively with k confined within $\Delta\Omega$

$$A_{\omega_0} = (k\omega_0\,\Delta\Omega)^{-1}\sum_k{}'\sum_\lambda\left\langle f_{k\lambda}^*(t)\int_0^t f_{k\lambda}(\xi)\cos\{kc(\xi-t)\}\,d\xi + \text{c.c.}\right\rangle.$$

5. Absorption of Energy

The absorption of electromagnetic energy per unit time is given by $\sigma\boldsymbol{E}.\boldsymbol{v}$. However, the only part of $\boldsymbol{v}$ which contributes to the net absorption is that part $\delta\boldsymbol{v}$ which is produced by $\boldsymbol{E}$. Then the absorption rate $\mathscr{A}$ is $\sigma\boldsymbol{E}.\delta\boldsymbol{v}$. For the unperturbed motion $\boldsymbol{p}_0 = m\boldsymbol{v}$ while, after the motion is perturbed,

$$\boldsymbol{p}_0+\delta\boldsymbol{p}_0 = m(\boldsymbol{v}+\delta\boldsymbol{v})+\sigma\boldsymbol{A}/c,$$

so that

$$\delta\boldsymbol{p}_0 = m\,\delta\boldsymbol{v}+\sigma\boldsymbol{A}/c.$$

The absorption rate is therefore

$$\mathscr{A} = \sigma\boldsymbol{E}.\{m^{-1}\delta\boldsymbol{p}_0-(\sigma/mc)\boldsymbol{A}\}. \tag{21}$$

However, the term involving $\boldsymbol{E}.\boldsymbol{A}$ does not contribute to the average absorption rate since $\boldsymbol{E}$ is proportional to $\partial\boldsymbol{A}/\partial t$ and the two are uncorrelated, i.e. the long-time average of $\boldsymbol{A}.(\partial\boldsymbol{A}/\partial t)$ tends to zero. Therefore $\mathscr{A}$ may be taken as

$$\mathscr{A} = (\sigma/m)\boldsymbol{E}.\delta\boldsymbol{p}_0 = -(\sigma/mc)\delta\boldsymbol{p}_0.\sum_k{}'\sum_\lambda\{\dot{q}_{k\lambda}(t)\,\boldsymbol{u}_{k\lambda}+\text{c.c.}\}$$
$$= -\sum_k{}'\sum_\lambda\{\dot{q}_{k\lambda}(t)\,\delta f_{k\lambda}+\dot{q}_{k\lambda}^*(t)\,\delta f_{k\lambda}^*\}, \tag{22}$$

and, for a particular k and λ,

$$\mathscr{A} = \dot{q}_{k\lambda}(t)\,\delta f_{k\lambda}+\dot{q}_{k\lambda}^*(t)\,\delta f_{k\lambda}^*. \tag{23}$$

In the absence of the interaction term $-(\sigma/mc)\boldsymbol{p}_0.\boldsymbol{A}$ in the Hamiltonian, the motion of the particle is unperturbed. This interaction term results in a perturbation to the motion and therefore causes the absorption of radiation. The perturbation is easily determined using an operator method (Kilmister and Reeves 1966), the essential details of which are the following.

The rate of change of any function of the coordinates q and canonical momenta p is given by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \sum \left(\frac{\partial H}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial F}{\partial p}\right), \tag{24}$$

which in Poisson bracket notation is

$$\mathrm{d}F/\mathrm{d}t = [H, F], \tag{25}$$

for which there is an exact quantum equivalent (Dirac 1958, p. 112). By using equation (24) all the derivatives of F can be found. If the initial values are known then Taylor's expansion can be used to find F at some later time t. It is convenient to introduce an operator $\hat{\Omega}$ defined by

$$\mathrm{d}F/\mathrm{d}t = \hat{\Omega}F, \tag{26}$$

so that Taylor's expansion becomes

$$F_t(p,q) = \exp(t\hat{\Omega}_0)\, F_{t_0}(p,q). \tag{27}$$

Here subscripts denote the time, and in $\hat{\Omega}_0$ the derivatives are with respect to the initial ($t = t_0$) values of p and q. For two systems with the same generalized coordinates and momenta and the same initial values, but with slightly different Hamiltonians H and H', we find, assuming $t_0=0$, that

$$F_t(p',q') = \exp(t\hat{\Omega}'_0)\, F_{t_0}(p,q) = \exp(t\hat{\Omega}'_0)\exp(-t\hat{\Omega}_0)\, F_t(p,q).$$

The operator

$$\hat{S} = \exp(t\hat{\Omega}'_0)\exp(-t\hat{\Omega}_0) \tag{28}$$

converts the unperturbed motion derived from H to the perturbed motion derived from H'. By forming $\mathrm{d}/\mathrm{d}t$ of equation (28) we find

$$\hat{S} = 1 + \int_{t_0}^{t} \exp(s\hat{\Omega}'_0)(\hat{\Omega}'_0 - \hat{\Omega}_0)\exp(-s\hat{\Omega}_0)\, \mathrm{d}s. \tag{29}$$

The reader familiar with quantum mechanics will recognize this expression as similar to the perturbation expression for the time evolution operator in quantum mechanics. The perturbation to F may then be written

$$\begin{aligned}\delta F &= F_t(p',q') - F_t(p,q)\\ &\approx \int_{t_0}^{t} \exp(s\hat{\Omega}_0)(\hat{\Omega}'_0 - \hat{\Omega}_0)\exp(-s\hat{\Omega}_0)\, \mathrm{d}s \,.\, F_t(p,q).\end{aligned} \tag{30}$$

For the problem considered here

$$\hat{\Omega}'_0 - \hat{\Omega}_0 = \sum \left(\frac{\partial(H'-H)}{\partial p_0}\frac{\partial}{\partial q_0} - \frac{\partial(H'-H)}{\partial q_0}\frac{\partial}{\partial p_0}\right) \tag{31}$$

and

$$H' - H = -\sum_{k}{}' \sum_{\lambda} (q_{k\lambda} f_{k\lambda} + q_{k\lambda}^* f_{k\lambda}^*). \tag{32}$$

It is convenient to deal with the contribution from a given $\boldsymbol{k}$ and λ separately. We shall also use the fact that terms with different values of $\boldsymbol{k}$ and λ are uncorrelated, so that to determine $\delta f_{k\lambda}$ we need only consider the terms in $H'-H$ which have the same $\boldsymbol{k}$ and λ as $\delta f_{k\lambda}$. From equation (30)

$$\delta f_{k\lambda} = \int_{t_0}^{t} \exp(s\hat{\Omega}_0)(\hat{\Omega}_0' - \hat{\Omega}_0)\exp(-s\hat{\Omega}_0) f_{k\lambda}(t)\, \mathrm{d}s$$

and, since $\exp(-s\hat{\Omega}_0)$ produces a time shift of $-s$,

$$\begin{aligned} \delta f_{k\lambda} &= \int_{t_0}^{t} \exp(s\hat{\Omega}_0)(\hat{\Omega}_0' - \hat{\Omega}_0) f_{k\lambda}(t-s)\, \mathrm{d}s \\ &= \int_{t_0}^{t} \exp(s\hat{\Omega}_0)\, [H'-H, f_{k\lambda}(t-s)]\, \mathrm{d}s. \end{aligned}$$

Retaining now only those terms in $H-H'$ which have the same $\boldsymbol{k}, \lambda$ as in $f_{k\lambda}$, we find

$$\delta f_{k\lambda} = -\int_{t_0}^{t} [q_{k\lambda}(s) f_{k\lambda}(s) + q_{k\lambda}^*(s) f_{k\lambda}^*(s),\ f_{k\lambda}(t)]\, \mathrm{d}s, \tag{33}$$

where in the last two equations the Poisson bracket notation has been used. Since $f_{k\lambda}$ only contains the coordinates and momentum of the particle, the derivatives with respect to p, q of the field in the Poisson bracket vanish. Therefore

$$\delta f_{k\lambda} = -\int_{t_0}^{t} [f_{k\lambda}(s), f_{k\lambda}(t)]\, q_{k\lambda}(s)\, \mathrm{d}s - \int_{t_0}^{t} [f_{k\lambda}^*(s), f_{k\lambda}(t)] q_{k\lambda}^*(s)\, \mathrm{d}s. \tag{34}$$

The contribution to the absorption from $\delta f_{k\lambda}$ is $\dot{q}_{k\lambda}(t)\,\delta f_{k\lambda}$, and this can now be simplified in the following way. First note that since the field hardly changes in the absorption process we can take $q_{k\lambda}(t)$ as being given by its value in the absence of matter. Thus

$$q_{k\lambda}(t) = a\exp(-\mathrm{i}\omega t + \mathrm{i}\alpha) + b\exp(+\mathrm{i}\omega t + \mathrm{i}\beta), \tag{35}$$

where, since in the standing wave representation waves of equal amplitude move in opposite directions,

$$aa^* = bb^*.$$

The phase constants α and β may be taken as random since ultimately we sum over a large number of particles. Taking the average over phase we find

$$\mathrm{Av}\{\dot{q}_{k\lambda}(t)\, q_{k\lambda}^*(s)\} = -2aa^*kc\sin\{kc(t-s)\}, \qquad \mathrm{Av}\{\dot{q}_{k\lambda}(t)\, q_{k\lambda}(s)\} = 0.$$

Therefore

$$\mathrm{Av}(\dot{q}_{k\lambda}\,\delta f_{k\lambda}) = +2aa^*kc \int_{t_0}^{t} [f_{k\lambda}^*(s), f_{k\lambda}(t)] \sin\{kc(t-s)\}\, \mathrm{d}s.$$

The total absorption for the given $\boldsymbol{k}$ and λ is then

$$+2aa^*kc\int_{t_0}^{t}\{[f_{k\lambda}^*(s),f_{k\lambda}(t)]+[f_{k\lambda}(s),f_{k\lambda}^*(t)]\}\sin\{kc(t-s)\}\,\mathrm{d}s\,. \tag{36}$$

The contribution to the energy in the electromagnetic field from the mode with the given $\boldsymbol{k}$ and λ is found by substituting equation (35) into the relation (4). It is

$$W_{k\lambda} = 2(kc)^2aa^*\,.$$

Therefore, for a fixed $\boldsymbol{k}$ and λ, the absorption rate $\mathscr{A}$ is given by

$$\mathscr{A} = +\frac{W_{k\lambda}}{kc}\int_{t_0}^{t}\{[f_{k\lambda}^*(s),f_{k\lambda}(t)]+[f_{k\lambda}(s),f_{k\lambda}^*(t)]\}\sin\{kc(t-s)\}\,\mathrm{d}s\,. \tag{37}$$

If the cavity contains a large number of similar independent particles which are distributed in phase space according to some distribution ρ, the average absorption is found by multiplying equation (37) by ρ and integrating over phase space. Liouville's theorem tells us that the phase space can be specified either by using the current values of p, q for the particles or by using the initial values p_0, q_0. For the present case, since the Poisson brackets in (37) imply differentiation with respect to p_0, q_0, we choose to use a volume element $\mathrm{d}V$ of phase space defined by the initial values. Furthermore we can integrate by parts if ρ vanishes at the end points of the integration. Therefore

$$\int\cdots\int\rho[f_{k\lambda}^*(s),f_{k\lambda}(t)]\,\mathrm{d}V_n = -\int\cdots\int[\rho,f_{k\lambda}(t)]\,f_{k\lambda}^*(s)\,\mathrm{d}V_n\,, \tag{38}$$

where, if each particle has n degrees of freedom,

$$\mathrm{d}V_n \equiv \mathrm{d}q_{01}\,\mathrm{d}q_{02}\ldots\mathrm{d}q_{0n}\,\mathrm{d}p_{01}\ldots\mathrm{d}p_{0n}\,.$$

If we now assume that ρ is a function of the particle energy W_p alone, then

$$[\rho,f_{k\lambda}(t)] = \frac{\mathrm{d}\rho}{\mathrm{d}W_\mathrm{p}}\frac{\mathrm{d}f_{k\lambda}}{\mathrm{d}t}\,. \tag{39}$$

The condition for equilibrium between emission and absorption is not necessarily the strong condition that at each instant there should be equality of emission and absorption. It is entirely sufficient to use the weaker condition, that the time average of the emission and absorption of a single particle should balance. Since we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f_{k\lambda}(t)\int_{t_0}^{t}f_{k\lambda}^*(s)\sin\{kc(t-s)\}\,\mathrm{d}s\right)$$
$$= \frac{\mathrm{d}f_{k\lambda}(t)}{\mathrm{d}t}\int_{t_0}^{t}f_{k\lambda}^*(s)\sin\{kc(t-s)\}\,\mathrm{d}s + kc\,f_{k\lambda}(t)\int_{t_0}^{t}f_{k\lambda}^*\cos\{kc(t-s)\}\,\mathrm{d}s\,,$$

and recalling that

$$\theta^{-1}\int_{t}^{t+\theta}(\mathrm{d}G/\mathrm{d}t)\,\mathrm{d}t \approx 0\,,$$

if θ is sufficiently large and G is bounded, we find

$$\left\langle \frac{\mathrm{d}f_{k\lambda}(t)}{\mathrm{d}t} \int_{t_0}^{t} f_{k\lambda}^*(s) \sin\{kc(t-s)\}\, \mathrm{d}s \right\rangle = -kc\left\langle f_{k\lambda}(t) \int_{t_0}^{t} f_{k\lambda}^* \cos\{kc(t-s)\}\, \mathrm{d}s \right\rangle, \quad (40)$$

where the angular brackets denote time averages. The average of equation (37) over phase space and over time can now be found using the relations (38), (39) and (40). Thus the average absorption rate $\mathscr{A}$ is, for a fixed $\boldsymbol{k}, \lambda$,

$$\mathrm{Av}(\mathscr{A}) = -W_{k\lambda} \int \dots \int \frac{\mathrm{d}\rho}{\mathrm{d}W_{\mathrm{p}}} \left(\left\langle f_{k\lambda}(t) \int_{t_0}^{t} f_{k\lambda}^*(s) \cos\{kc(t-s)\}\, \mathrm{d}s \right\rangle + \mathrm{c.c.} \right) \mathrm{d}V_n. \quad (41)$$

From equation (16) the time average of $\mathscr{E}$ is

$$\mathrm{Av}(\mathscr{E}) = \int \dots \int \rho \left(\left\langle f_{k\lambda}(t) \int_{t_0}^{t} f_{k\lambda}^*(s) \cos\{kc(t-s)\}\, \mathrm{d}s \right\rangle + \mathrm{c.c.} \right) \mathrm{d}V_n. \quad (42)$$

We now consider various applications of equations (41) and (42).

6. Applications

(*a*) *Classical Equilibrium*

This is the simplest case to consider. If the particles are distributed according to the canonical distribution

$$\rho \propto \exp(-W_{\mathrm{p}}/KT), \quad (43)$$

where K is Boltzmann's constant and T the temperature, we find on equating equations (41) and (42) that

$$W_{k\lambda} = KT, \quad (44)$$

which shows that each mode has the same energy. Recalling that the number of modes with k in the range k to $k+\mathrm{d}k$ is, on taking account of the two polarization directions,

$$Vk^2\, \mathrm{d}k/\pi^2, \quad (45)$$

the energy density per unit wave number is

$$KTk^2/\pi^2, \quad (46)$$

the usual Rayleigh–Jeans law. This last result shows that, if similar charged particles have the canonical distribution, the condition that they are in equilibrium with electromagnetic radiation in a cavity is that the energy distribution of the radiation is given by the Rayleigh–Jeans law. This result is *independent* of the character of other conservative time-independent fields with which the particles interact.

(*b*) *Relation to Quantum Mechanics*

If the particle motion is periodic with period $2\pi/\omega_0$ the analysis of Section 4 shows, for a given volume element of phase space, after summing over $\boldsymbol{k}, \lambda$, recognizing that the major contribution comes from $k_0 \sim \omega_0/c$ and introducing the spontaneous emission coefficient $A_{\omega 0}$, that the absorption rate $\mathscr{A}$ is given by

$$\mathscr{A} = -W_{k_0}(\mathrm{d}\rho/\mathrm{d}W_{\mathrm{p}})(h\omega_0 A_{\omega 0}), \tag{47}$$

where $k_0 = \omega_0/c$, and the emission rate $\mathscr{E}$ is

$$\mathscr{E} = \rho(h\omega_0 A_{\omega 0}). \tag{48}$$

In terms of the radiation intensity per unit frequency $I(\omega_0)$, defined by c times the product of the energy density per mode by the number of modes in $\mathrm{d}\omega$, that is,

$$I(\omega_0) = c(W_{k_0}/V)(V\omega_0^2/\pi^2c^3) = W_{k_0}\omega_0^2/c^2\pi^2, \tag{49}$$

the absorption rate becomes

$$\mathscr{A} = -I(\omega_0)(c^2\pi^2/\omega_0^2)(\mathrm{d}\rho/\mathrm{d}W_{\mathrm{p}})(h\omega_0 A_{\omega 0}). \tag{50}$$

Therefore at equilibrium we find

$$-I(\omega_0)(c^2\pi^2/\omega_0^2)\,\mathrm{d}\rho/\mathrm{d}W_{\mathrm{p}} = \rho. \tag{51}$$

The quantum equivalent of equation (51) was derived by Einstein (see Bethe 1964, p. 144), by balancing absorption and emission between two levels f, n and introducing a coefficient of stimulated emission B_ω. Einstein found that

$$I(\omega_0)\,c\,B_{\omega 0}\,\rho(E_f) + A_{\omega 0}\,\rho(E_f) = B_{\omega 0}\,c\,\rho(E_n)\,I(\omega_0), \tag{52}$$

where

$$E_f = E_n + h\omega_0.$$

Equation (52) may be written

$$-I(\omega_0)\,c\,B_{\omega 0}\,h\omega_0\{\rho(E_n+h\omega_0)-\rho(E_n)\}/h\omega_0 = A_{\omega 0}\,\rho(E_f). \tag{53}$$

If, as an approximation, we write

$$\{\rho(E_n+h\omega_0)-\rho(E_n)\}/h\omega_0 = \mathrm{d}\rho/\mathrm{d}E_n, \tag{54}$$

equation (53) becomes

$$-I(\omega_0)(c\,B_{\omega 0}\,h\omega_0)\,\mathrm{d}\rho/\mathrm{d}E_n = A_{\omega 0}\,\rho.$$

Comparing this with equation (51) we deduce that

$$A_{\omega 0} = B_{\omega 0}\,h\omega_0^3/c\pi^2, \tag{55}$$

which is the usual quantum relation between the coefficients of stimulated and spontaneous emission. This procedure can be reversed. In equation (51) replace $\mathrm{d}\rho/\mathrm{d}W_{\mathrm{p}}$ using the relation (54) and set

$$\rho(E) \propto \exp(-E/KT).$$

Then (51) becomes

$$I(\omega_0) = \frac{h\omega_0^3/\pi^2c^2}{\exp(h\omega_0/KT)-1}, \tag{56}$$

which is Planck's law. This result was also found by Le Roux (1960) who did not, however, draw all the connecting links with the quantum equations.

(c) Quasiclassical Stimulated Emission

Twiss (1958) showed that, for the typical frequencies used in radio astronomy, stimulated emission might be important. His procedure was to take the quantum equations and convert them to a quasiclassical form. Here we recover his results using equations (41) and (42).

If the motion is ergodic we can expect that the time average in equation (41) is the same for all particles with the same constants of the motion. In many cases the only relevant constant will be the energy W_p. Even if the motion is not ergodic, one can expect that the time average will depend only on the constants of the motion. Assuming then that the energy is the only relevant constant of the motion we set

$$\left\langle f_{k\lambda}(t)\int_{t_0}^{t} f_{k\lambda}^*(s)\cos\{kc(t-s)\}\,\mathrm{d}s\right\rangle + \text{c.c.} = F(W_p).$$

The absorption at $\boldsymbol{k}$ is then

$$-W_{k\lambda}\int (\mathrm{d}\rho/\mathrm{d}W_p)\,F(W_p)\,\mathrm{d}V_n. \tag{57}$$

If the volume of phase space between W_p and $W_p+\mathrm{d}W_p$ is $Q(W_p)\,\mathrm{d}W_p$ then the expression (57) becomes

$$-W_{k\lambda}\int (\mathrm{d}\rho/\mathrm{d}W_p)\,F(W_p)\,Q(W_p)\,\mathrm{d}W_p. \tag{58}$$

The number of particles n with energy between W_p and $W_p+\mathrm{d}W_p$ is given by

$$n = \rho\,Q(W_p),$$

so that (58) may be written

$$-W_{k\lambda}\int \frac{\mathrm{d}}{\mathrm{d}W_p}\left(\frac{n}{Q}\right) FQ\,\mathrm{d}W_p.$$

This last expression for the absorption is equivalent to the basic equation of Twiss's (1958) paper. Note that here the quantum statistical weights are replaced, as expected, by Q, the usual classical statistical weighting.

The analysis given here can easily be extended to deal with more general problems involving either other constants of the motion or a frequency-dependent refractive index for the cavity.

7. Relativistic Case

If the particle Hamiltonian (10) is expanded assuming the particle motion is relativistic and the radiation field is a weak perturbation, we find

$$\begin{aligned} H = {} & \sigma U + (m^2c^4 + c^2p_0^2)^{\frac{1}{2}} + \sum_k{}' \sum_\lambda (p_{k\lambda}p_{k\lambda}^* + k^2c^2 q_{k\lambda}q_{k\lambda}^*) \\ & -\sigma c\boldsymbol{p}_0 \cdot \boldsymbol{A}\,(m^2c^4 + c^2p_0^2)^{-\frac{1}{2}}, \end{aligned} \tag{59}$$

where $\boldsymbol{A}$ is given by equation (1). In the unperturbed case

$$(m^2c^4+c^2p_0^2)^{\frac{1}{2}} = mc^2(1-v^2/c^2)^{-\frac{1}{2}},$$

since

$$p_0 = mv(1-v^2/c^2)^{-\frac{1}{2}}.$$

The interaction term therefore becomes

$$-(\sigma/c)\,\boldsymbol{v}.\boldsymbol{A},$$

which is identical with the interaction term in the non-relativistic calculation. The expression (20) for the emission is unaltered if we simply replace $\boldsymbol{p}_0$ by $m\boldsymbol{v}$. The absorption is also unaltered, for the absorption rate is still $\sigma\boldsymbol{E}.\delta\boldsymbol{v}$ and the interaction term causing the perturbation in v has the form $-\sigma\,\boldsymbol{v}.\boldsymbol{A}/c$. In short equations (23) and (37) remain true with $f_{k\lambda}$ being given by

$$f_{k\lambda} = (\sigma/c)\boldsymbol{v}.\boldsymbol{u}_{k\lambda}.$$

The Rayleigh–Jeans law and the Planck law (granted the ansatz 54) therefore remain true in the relativistic case.

References

Bethe, H. (1964). 'Intermediate Quantum Mechanics' (Benjamin: New York).
Born, M. (1969). 'Atomic Physics', 8th edn (Blackie & Son: Glasgow).
Dirac, P. A. M. (1958). 'The Principles of Quantum Mechanics', 4th edn (Oxford Univ. Press).
Einstein, A. (1917). *Phys. Z.* **18**, 121.
Fermi, E. (1932). *Rev. Mod. Phys.* **4**, 87.
Feynman, R. P. (1964). 'Lectures on Physics', Vol. 2 (Addison–Wesley: Reading, Mass.).
Jackson, J. D. (1962). 'Classical Electrodynamics' (Wiley: New York).
Kilmister, C. N., and Reeves, J. E. (1966). 'Rational Mechanics' (Longmans: London).
Le Roux, E. (1960). *Ann. Astrophys.* **23**, 1010.
Lorentz, H. A. (1915). 'Theory of Electrons', reprinted, 2nd edn 1952 (Dover: New York).
McLaren, S. (1911). *Philos. Mag.* **21**, 15.
McLaren, S. (1912). *Philos. Mag.* **23**, 513.
Monaghan, J. J. (1968). *J. Phys.* A **1**, 112.
Planck, M. (1912). 'The Theory of Heat Radiation', reprinted, 2nd edn 1959 (Dover: New York).
Twiss, R. Q. (1958). *Aust. J. Phys.* **11**, 564.

Appendix 1

We require the average of $\sum_\lambda (\boldsymbol{a}.\boldsymbol{\varepsilon}_{k\lambda})(\boldsymbol{b}.\boldsymbol{\varepsilon}_{k\lambda})$. We use polar coordinates with $\boldsymbol{k}$ as the polar axis and recall that $\boldsymbol{\varepsilon}_{k1}$ and $\boldsymbol{\varepsilon}_{k2}$ are perpendicular to $\boldsymbol{k}$ and to each other. The various angles are:

	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{\varepsilon}_{k1}$	$\boldsymbol{\varepsilon}_{k2}$
Polar angle	θ_a	θ_b	$\frac{1}{2}\pi$	$\frac{1}{2}\pi$
Azimuthal angle	ϕ_a	ϕ_b	ϕ	$\phi+\frac{1}{2}\pi$

Therefore

$$\frac{1}{2\pi}\int_0^{2\pi}\sum_{\lambda=1,2}(\boldsymbol{\varepsilon}_{k\lambda}\cdot\boldsymbol{a})(\boldsymbol{\varepsilon}_{k\lambda}\cdot\boldsymbol{b})\,\mathrm{d}\phi = \frac{ab}{2\pi}\int_0^{2\pi}\{\sin\theta_a\cos(\phi-\phi_a)\}\{\sin\theta_b\cos(\phi-\phi_b)\}\,\mathrm{d}\phi$$

$$+\frac{ab}{2\pi}\int_0^{2\pi}\{\sin\theta_a\sin(\phi-\phi_a)\}\{\sin\theta_b\sin(\phi-\phi_b)\}\,\mathrm{d}\phi$$

$$= ab\{\sin\theta_a\sin\theta_b\cos(\phi_a-\phi_b)\}$$

$$= (\boldsymbol{a}\times\hat{\boldsymbol{k}})\cdot(\boldsymbol{b}\times\hat{\boldsymbol{k}}).$$

Appendix 2

From equations (3a) and (14) of Sections 2 and 4 we have

$$\boldsymbol{E}(\boldsymbol{x},t) = -c^{-1}(4\pi c^2/v)^{\frac{1}{2}}\sum_{k\lambda}\left(\boldsymbol{\varepsilon}_{k\lambda}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\int_{-\infty}^{t} f_{k\lambda}^{*}\cos\{kc(\xi-t)\}\,\mathrm{d}\xi\right.$$

$$\left.+\boldsymbol{\varepsilon}_{k\lambda}\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\int_{-\infty}^{t} f_{k\lambda}\cos\{kc(\xi-t)\}\,\mathrm{d}\xi\right), \qquad \text{(A1)}$$

with $\boldsymbol{x}$ being the position vector of the point where $\boldsymbol{E}$ is measured at time t. Taking $\boldsymbol{r}$ as the position vector of the particle at time ξ, we can also write $f_{k\lambda}$ in the form

$$f_{k\lambda}(\xi) = \sigma(4\pi/V)^{\frac{1}{2}}\,\boldsymbol{v}\cdot\boldsymbol{\varepsilon}_{k\lambda}\exp\{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}(\xi)\}. \qquad \text{(A2)}$$

Substituting equation (A2) into (A1) and setting $\boldsymbol{R} = \boldsymbol{x}-\boldsymbol{r}$, we find

$$\boldsymbol{E} = -(8\pi\sigma/V)\sum_{k\lambda}\boldsymbol{\varepsilon}_{k\lambda}\int_{-\infty}^{t}\boldsymbol{v}(\xi)\cdot\boldsymbol{\varepsilon}_{k\lambda}\cos(\boldsymbol{k}\cdot\boldsymbol{R})\cos\{kc(\xi-t)\}\,\mathrm{d}\xi.$$

It is convenient to examine $\boldsymbol{E}\cdot\boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is any unit vector. Forming $\boldsymbol{E}\cdot\boldsymbol{\varepsilon}$ and averaging over polarizations as in Appendix 1 we obtain

$$\boldsymbol{E}\cdot\boldsymbol{\varepsilon} = -(8\pi\sigma/V)\sum_{k}(\boldsymbol{\varepsilon}\times\hat{\boldsymbol{k}})\cdot\int_{-\infty}^{t}(\boldsymbol{v}\times\hat{\boldsymbol{k}})\cos(\boldsymbol{k}\cdot\boldsymbol{R})\cos\{kc(\xi-t)\}\,\mathrm{d}\xi. \qquad \text{(A3)}$$

The summation over $\boldsymbol{k}$ can be transformed into an integral using the expression (19) for the number of modes. Accordingly, for large V,

$$\boldsymbol{E}\cdot\boldsymbol{\varepsilon} = -\frac{\sigma}{\pi^2}\int_{-\infty}^{t}\iint k^2\,\mathrm{d}k\,\mathrm{d}\Omega\,(\boldsymbol{\varepsilon}\times\hat{\boldsymbol{k}})\cdot(\boldsymbol{v}\times\hat{\boldsymbol{k}})\cos(\boldsymbol{k}\cdot\boldsymbol{R})\cos\{kc(\xi-t)\}\,\mathrm{d}\xi, \qquad \text{(A4)}$$

or

$$\boldsymbol{E}\cdot\boldsymbol{\varepsilon} = -\frac{\sigma}{\pi^2}\int_{-\infty}^{t}\iint k^2\,\mathrm{d}k\,\mathrm{d}\Omega\left(\boldsymbol{\varepsilon}\cdot\boldsymbol{v}-\frac{(\boldsymbol{\varepsilon}\cdot\boldsymbol{k})(\boldsymbol{v}\cdot\boldsymbol{k})}{k^2}\right)\cos(\boldsymbol{k}\cdot\boldsymbol{R})\cos\{kc(\xi-t)\}\,\mathrm{d}\xi. \qquad \text{(A5)}$$

On recalling that the integration over Ω includes only a hemisphere, we find

$$\iint k^2\,\mathrm{d}k\,\mathrm{d}\Omega\,\boldsymbol{\varepsilon}.\boldsymbol{v}\cos(\boldsymbol{k}.\boldsymbol{R})\cos\{kc(\xi-t)\} = (\boldsymbol{\varepsilon}.\boldsymbol{v})\pi^2R^{-1}\,\delta'\big(R+c(\xi-t)\big), \quad \text{(A6)}$$

where the prime denotes a derivative with respect to the function's argument and a delta function has been omitted from the right-hand side of (A6) since $\xi < t$. Also

$$\iint \mathrm{d}k\,\mathrm{d}\Omega\,(\boldsymbol{\varepsilon}.\boldsymbol{k})(\boldsymbol{v}.\boldsymbol{k})\cos(\boldsymbol{k}.\boldsymbol{R})\cos\{kc(\xi-t)\}$$

$$= -\varepsilon_i v_j\frac{\partial^2}{\partial x_i\,\partial x_j}\bigg(\iint\cos(\boldsymbol{k}.\boldsymbol{R})\cos\{kc(\xi-t)\}\,\mathrm{d}k\,\mathrm{d}\Omega\bigg). \quad \text{(A7)}$$

If $\boldsymbol{n} = \boldsymbol{R}/|\boldsymbol{R}|$ we can write the last expression as

$$= -(\boldsymbol{n}.\boldsymbol{\varepsilon})(\boldsymbol{v}.\boldsymbol{n})\frac{\partial^2}{\partial R^2}\bigg(\iint\cos(\boldsymbol{k}.\boldsymbol{R})\cos\{kc(\xi-t)\}\,\mathrm{d}k\,\mathrm{d}\Omega\bigg)$$

$$= -(\boldsymbol{n}.\boldsymbol{\varepsilon})(\boldsymbol{v}.\boldsymbol{n})\frac{\partial^2}{\partial R^2}\bigg(\frac{2\pi}{R}\int_0^\infty\frac{\sin(kR)}{k}\cos\{kc(\xi-t)\}\,\mathrm{d}k\bigg). \quad \text{(A8)}$$

Since we are only interested in the radiation field it is legitimate to assume $Rk \gg 1$. Therefore when evaluating $\partial^2/\partial R^2$ we only retain terms which vary as R^{-1}. We find that the result (A8) is approximately

$$= (\boldsymbol{n}.\boldsymbol{\varepsilon})(\boldsymbol{n}.\boldsymbol{v})\pi^2R^{-1}\,\delta'\big(R+c(\xi-t)\big). \quad \text{(A9)}$$

Substituting the expressions (A6) and (A9) into (A5) shows that

$$\boldsymbol{E}.\boldsymbol{\varepsilon} = -\sigma\int_{-\infty}^{t} R^{-1}\boldsymbol{\varepsilon}.\big(\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{v})\big)\,\delta'\big(R+c(\xi-t)\big)\,\mathrm{d}\xi$$

which, according to the rules for integrating delta functions, becomes

$$\boldsymbol{E}.\boldsymbol{\varepsilon} = \frac{\sigma}{c^2}\bigg[\frac{\mathrm{d}}{\mathrm{d}\xi}\bigg(\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{v})}{kR}\bigg)\bigg].\boldsymbol{\varepsilon}, \quad \text{(A10)}$$

where the quantities in the square brackets are to be evaluated at the retarded time $\xi = t-R/c$, and

$$k = 1+c^{-1}\,\mathrm{d}R/\mathrm{d}\xi.$$

The result (A10) is equivalent to the Heaviside–Feynman form of the radiation field discussed by Feynman (1964) and derived by Monaghan (1968).

Manuscript received 29 May 1975