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The Magnetic Field of a Current Loop Encircling an Axisymmetric Iron Core

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Abstract

The effect of an iron transformer core on the field of a current loop is examined for two models of the core: (1) An infinite straight rod of high permeability aligned along the axis of symmetry, for which asymptotic expressions for the effect of the core are obtained and compared with numerical results. (2) A rectangular toroidal iron casing surrounding the loop. The latter model is more realistic because a return path is provided for the flux. For this model, the effect of air gaps is considered, and rapidly convergent series are obtained and numerical results are given. The significance of these results for tokamak equilibrium is indicated.

1. Introduction

An iron core is commonly used to increase the coupling between the primary and secondary (plasma) circuits of toroidal plasma containment devices, such as tokamaks. This gives rise to perturbations in the plasma equilibrium in two ways: (i) due to the static field errors arising from the d.c. bias windings used to premagnetize the core, and (ii) due to the alteration in the Green's function for currents in the vicinity of the plasma arising from the changed boundary conditions. The latter effect may be screened over a short time scale by use of a conducting wall between the primary windings and the core but, in the absence of a copper shell, this time scale may be much less than the plasma containment time (Hugill and Gibson 1974).

It is therefore desirable to be able to estimate, at least qualitatively, the magnitude of the perturbations due to the iron. Mukhovatov and Shafranov (1971) used an infinite rod model, but this is open to the objection that it does not represent, even qualitatively, the effect of a finite transformer core, since the infinite rod forms a simply connected region (excluding a return path at infinity) while a transformer core is topologically toroidal. The importance of topology for this problem has been made clear by Van Bladel (1961) who points out that the frequently used infinite-permeability boundary condition of vanishing tangential $\boldsymbol{B}$ is in general appropriate only for simply connected regions. This is illustrated by comparing Figs 1*a* and 1*b*, where it is seen that the magnetic field lines enter an infinite core at right angles, while they enter obliquely into a core with a return path for the magnetic flux. The infinite core model is not quite as bad as it seems, however, since the boundary condition of vanishing parallel $\boldsymbol{B}$ is recovered when the total current circulating around the core is zero; a condition which is approximately satisfied due to the low resistivity of the plasma.

The simplest model with the correct topology is that of a circular iron cylinder surrounding a straight current, which may be solved by the method of images (Van Bladel 1961; Mukhovatov and Shafranov 1971). The more realistic model of a rectangular iron cylinder has been solved by Kulda (1962, 1964). In the present paper we adopt a model suitable for finite aspect ratio tokamaks with an axisymmetric

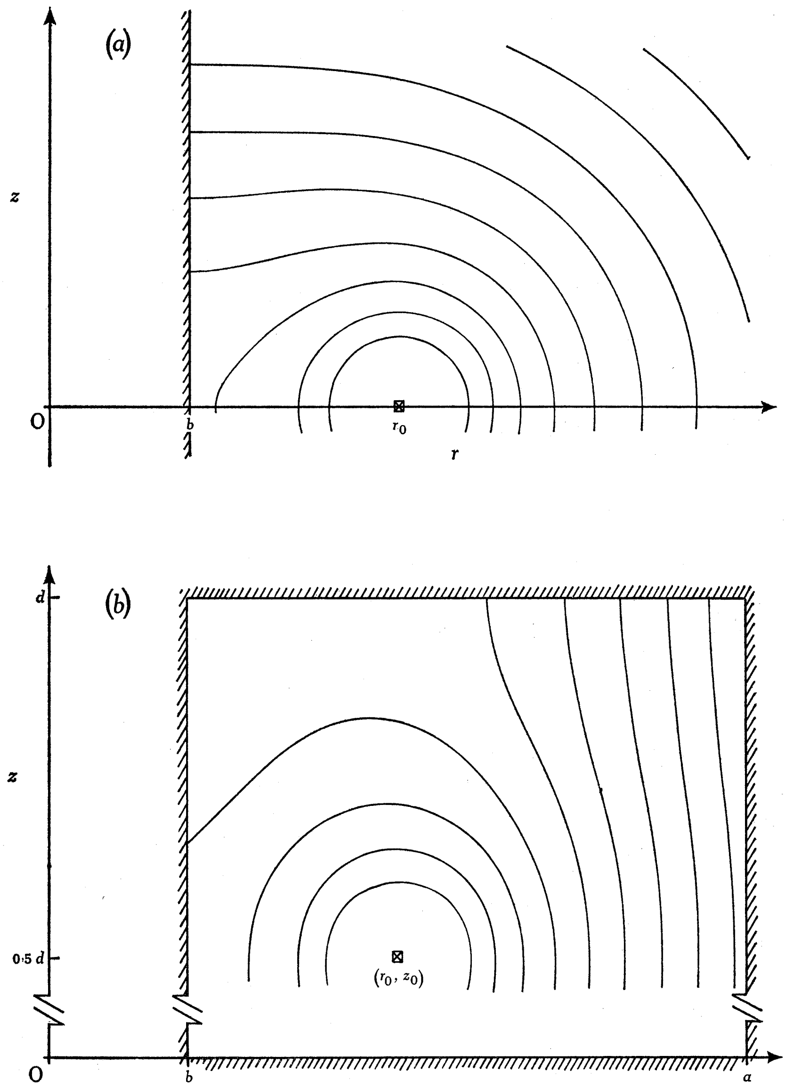


Fig. 1. Section through the axis of symmetry (z axis) of the model transformers (hatched lines), showing the current loop (cross) and the magnetic surfaces (continuous lines):

(a) Infinite core, for the case $b = 0{\cdot}2$ m, $r_0 = 0{\cdot}5$ m, $z_0 = 0$, $I = 1$ A. The increment in Ψ between surfaces is $2{\cdot}97 \times 10^{-8}$ Wb rad^{-1} and the boundary condition is $B_z(b, z) = 0$.

(b) Rectangular toroidal core, for the case $b = 0{\cdot}2$ m, $a = d = 1{\cdot}0$ m, $r_0 = z_0 = 0{\cdot}5$ m and $I = 1$ A. The increment in Ψ between surfaces is $3{\cdot}19 \times 10^{-8}$ Wb rad^{-1}. The boundary conditions are $B_z(b, z) = -B_z(a, z) = \frac{1}{2}\mu_0 I d^{-1}$ and $B_r(r, 0) = B_r(r, d) = 0$.

return path for the flux: a current loop surrounded by a toroidal iron cavity of rectangular cross section, with the same axis of symmetry as that of the current loop. The effect of air gaps is modelled by suitable choice of boundary conditions on the iron. In Section 2 we review the infinite rod model and derive some new asymptotic approximations. In Section 3 we develop the toroidal model, and in Section 4 we briefly consider the effect on tokamak equilibrium and stability.

2. Infinite Rod

We represent the magnetic field $\boldsymbol{B}$ of a current loop by use of a stream function $\Psi(r,z)$, such that

$$\boldsymbol{B} = \nabla\phi \times \nabla\Psi, \tag{1a}$$

where r, ϕ and z are cylindrical polar coordinates. In terms of the vector potential $\boldsymbol{A}$ we have $\Psi = -rA_\phi$. The r and z components of $\boldsymbol{B}$ are given by

$$B_r = r^{-1}\,\partial\Psi/\partial z \qquad \text{and} \qquad B_z = -r^{-1}\,\partial\Psi/\partial r. \tag{1b}$$

The changes in the fields when an infinite rod of radius b and relative permeability $\mu_r = 1/\varepsilon$ is inserted on the symmetry axis of a circular current loop carrying a current I are given by (Smythe 1950)

$$\psi = \int_0^\infty \mathrm{d}k\,\Delta\Psi_k\{rr_0\,K_1(kr)\,K_1(kr_0)\cos(kZ) - a^2\,K_1^2(ka)\}, \tag{2a}$$

$$b_r = -r_0\int_0^\infty k\,\mathrm{d}k\,\Delta\Psi_k\,K_1(kr)\,K_1(kr_0)\sin(kZ), \tag{2b}$$

$$b_z = r_0\int_0^\infty k\,\mathrm{d}k\,\Delta\Psi_k\,K_0(kr)\,K_1(kr_0)\cos(kZ), \tag{2c}$$

where

$$Z \equiv z - z_0, \qquad \psi \equiv \Psi - \Psi^{\mathrm{v}}, \qquad b_z \equiv B_z - B_z^{\mathrm{v}}, \qquad b_r \equiv B_r - B_r^{\mathrm{v}},$$

and

$$\Delta\Psi_k = -\frac{\mu_0 I(1-\varepsilon)}{\pi}\,\frac{I_0(kb)\,I_1(kb)}{K_0(kb)\,I_1(kb) + \varepsilon\,K_1(kb)\,I_0(kb)}. \tag{3}$$

The vacuum fields Ψ^{v}, B_r^{v}, and B_z^{v} may be represented by similar integrals, for example,

$$\Psi^{\mathrm{v}} = -\frac{\mu_0 Irr_0}{\pi}\int_0^\infty \mathrm{d}k\,\{K_1(kr_0)\,I_1(kr)\,\vartheta(r_0-r) + K_1(kr)\,I_1(kr_0)\,\vartheta(r-r_0)\cos(kZ)\}, \tag{4}$$

where $\vartheta(x)$ is the unit step function $(x+|x|)/2x$. Because the integrals for the vacuum fields diverge as $r \to r_0$, it is much more convenient for numerical purposes to use the alternative forms (Smythe)

$$\Psi^{\mathrm{v}} = -\frac{\mu_0 I}{2\pi}\,\frac{r_0^2+r^2+Z^2}{\{(r_0+r)^2+Z^2\}^{\frac{1}{2}}}\left(K - \frac{\{(r_0+r)^2+Z^2\}E}{r_0^2+r^2+Z^2}\right), \tag{5a}$$

$$B_r^{\mathrm{v}} = \frac{\mu_0 I}{2\pi}\,\frac{Z}{r\{(r_0+r)^2+Z^2\}^{\frac{1}{2}}}\left(-K + \frac{(r_0^2+r^2+Z^2)E}{(r_0-r)^2+Z^2}\right), \tag{5b}$$

$$B_z^{\mathrm{v}} = \frac{\mu_0 I}{2\pi} \frac{1}{\{(r_0+r)^2+Z^2\}^{\frac{1}{2}}} \left(K + \frac{(r_0^2-r^2-Z^2)E}{(r_0-r)^2+Z^2}\right), \tag{5c}$$

where $K(\kappa)$ and $E(\kappa)$ are complete elliptic integrals of the first and second kind, with

$$\kappa^2 = 4rr_0\{(r_0+r)^2+Z^2\}^{-1}. \tag{6}$$

In equation (2a), the function $\psi \equiv \Psi - \Psi^{\mathrm{v}}$ has been defined to vanish at $r = r_0 = a$ and $z = z_0$, where a is arbitrary. The second term of the integrand makes the singularity at $k = 0$ integrable even in the infinite permeability limit, $\varepsilon \to 0$. Fig. 2 shows graphs of $b_z \equiv B_z - B_z^{\mathrm{v}}$ versus r for several values of ε. It is seen that the convergence to the $\varepsilon = 0$ result is quite slow; this being due to the logarithmic behaviour of the denominator in equation (3) near $k = 0$. Nevertheless, $\varepsilon = 0$ is a reasonable approximation for large μ_{r}, and we study this limit to determine the asymptotic behaviour of ψ in the limit $\{(r+r_0)^2+Z^2\}^{\frac{1}{2}} \gg b$.

For $\varepsilon = 0$, equation (2a) becomes

$$\psi = -\frac{\mu_0 I}{\pi} \int_0^\infty \mathrm{d}k\, \{I_0(kb)/K_0(kb)\}\{rr_0\, K_1(kr)\, K_1(kr_0)\cos(kZ) - a^2 K_1^2(ka)\}. \tag{7}$$

Because of the exponential decay of $K_\nu(x)$ for $x \gg 1$ and the oscillatory behaviour of $\cos(kz - kz_0)$, it is clear that the principal contribution to the integral in equation (7) must come from the region

$$kb \approx b\{(r+r_0)^2+Z^2\}^{-\frac{1}{2}} \ll 1. \tag{8}$$

Now, for small kb, we have

$$K_0(kb) = \{\ln(2/kb) - \gamma\}\, I_0(kb) + O(k^2 b^2),$$

γ being Euler's constant. Thus we may make an asymptotic expansion in the parameter

$$\lambda \equiv \ln\!\left(2\alpha\{(r+r_0)^2+Z^2\}^{\frac{1}{2}} b^{-1}\right) \gg 1 \tag{9}$$

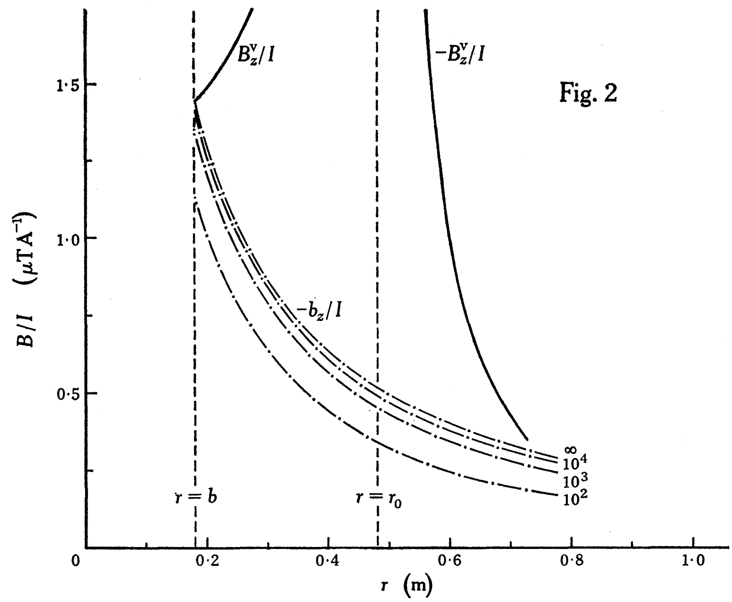
to give

$$\psi \approx -\frac{\mu_0 I}{\pi\lambda} \sum_{n=0} \lambda^{-n} \int_0^\infty \mathrm{d}k\, \{\ln\!\left(\alpha \exp(\gamma)\, k\{(r+r_0)^2+Z^2\}^{\frac{1}{2}}\right)\}^n$$
$$\times \{rr_0\, K_1(kr)\, K_1(kr_0)\cos(kZ) - a^2 K_1^2(ka)\}, \tag{10}$$

where α is an arbitrary constant of order unity.

The leading term in the expansion (10) may be evaluated in terms of Legendre functions using the following identity, which is valid for $|\operatorname{Re}\nu| < \frac{3}{2}$,

$$\int_0^\infty \{(ab)^\nu K_\nu(ax)\, K_\nu(bx)\cos(cx) - (a'b')^\nu K_\nu(a'x)\, K_\nu(b'x)\cos(c'x)\}\, \mathrm{d}x$$
$$= \frac{\pi^2}{4\cos(\nu\pi)}\left\{(ab)^{\nu-\frac{1}{2}} P_{\nu-\frac{1}{2}}\!\left(\frac{a^2+b^2+c^2}{2ab}\right) - (a'b')^{\nu-\frac{1}{2}} P_{\nu-\frac{1}{2}}\!\left(\frac{a'^2+b'^2+c'^2}{2a'b'}\right)\right\}. \tag{11}$$



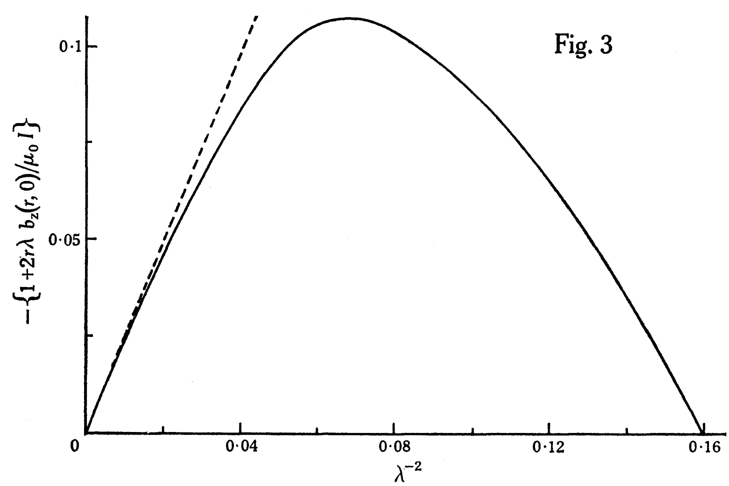


Fig. 2. Distributions of the vertical field $B_z^v(r,0)$ per unit current, in the absence of a core, and the negative of the changes $b_z(r,0)$ in it (dot–dash curves) which arise in the presence of an infinite core of the indicated relative permeabilities $\mu_r = \mu/\mu_0$. The conductor is assumed to be located at $(r_0, z_0) = (0{\cdot}48\ \mathrm{m}, 0)$ and the core radius is $b = 0{\cdot}18$ m.

Fig. 3. Comparison of $-\{1 + 2r\lambda\, b_z(r,0)/\mu_0 I\}$ as a function of λ^{-2} with the asymptotic result as $\lambda \to \infty$, namely, $2{\cdot}46\,\lambda^{-2}$ (dashed line). The curves are for an infinite core of radius b surrounded by a current loop at $(r_0, z_0) = (2{\cdot}5\,b, 0)$, with λ given by $\ln(4r/b)$.

Equation (11) may be obtained from a well-known identity (Gradshteyn and Ryzhik 1965) by analytic continuation in the variable ν. Applying equation (11) to the $n = 0$ term of equation (10) we find (to within a constant in the case of ψ)

$$\psi = \frac{\pi\mu_0 I(rr_0)^{\frac{1}{2}}}{4\lambda} P_{\frac{1}{2}}(\zeta) + O(\lambda^{-2}), \tag{12a}$$

$$b_r = \frac{\pi\mu_0 IZ}{8\lambda r(rr_0)^{\frac{1}{2}}} \frac{\zeta P_{\frac{1}{2}}(\zeta) - P_{-\frac{1}{2}}(\zeta)}{\zeta^2 - 1} + O(\lambda^{-2}), \tag{12b}$$

$$b_z = -\frac{\pi\mu_0 I}{8\lambda r} \frac{r_0^{\frac{1}{2}}}{r} \frac{(\zeta r/r_0 - 1) P_{\frac{1}{2}}(\zeta) - (r/r_0 - \zeta) P_{-\frac{1}{2}}(\zeta)}{\zeta^2 - 1} + O(\lambda^{-2}), \tag{12c}$$

where

$$\zeta = (r^2 + r_0^2 + Z^2)/2rr_0 \geqslant 1 .$$

The behaviour of $\psi(r, z \,|\, r_0, z_0)$ in the neighbourhood of the conductor, that is, for $r \approx r_0$, $z \approx z_0$ and $\zeta \approx 1$, can be obtained from the hypergeometric series for the Legendre functions (Gradshteyn and Ryzhik). The first few terms in the series are

$$P_{\frac{1}{2}}(\zeta) = 1 + \tfrac{3}{8}(\zeta - 1) - \tfrac{15}{128}(\zeta - 1)^2 + \dots . \tag{13}$$

A comparison of equation (12c) with the exact result (2c) is included in Fig. 4 (below), in which the parameter α has been chosen to give a reasonable fit.

In order to gain an idea of the convergence of the λ^{-n} expansion we examine the case of large r or Z, so that r_0 is of the same order as b, that is, $r_0/r = \mathrm{O}(\exp(-\lambda))$. Thus, to all orders in λ^{-1}, the explicit r dependence of ψ is that of the lowest order term in a multipole expansion. By examining the behaviour of equation (12a) as $r \to \infty$, we see that the lowest order behaviour of ψ is as $(r^2 + Z^2)^{\frac{1}{2}}$. A general procedure for constructing the terms of the series in inverse powers of λ which multiplies the factor $(r^2 + Z^2)^{\frac{1}{2}}$ is given in the Appendix, and the expansion (dashed curve) is compared with numerical results in Fig. 3. The expansions are summarized here:

$$\Psi \approx \tfrac{1}{2}\mu_0 I (r^2 + Z^2)^{\frac{1}{2}} \left(\frac{1}{\lambda} + \frac{1 + \sin^2(\frac{1}{2}\theta) \ln(\sin^2(\frac{1}{2}\theta)) + \cos^2(\frac{1}{2}\theta) \ln(\cos^2(\frac{1}{2}\theta))}{\lambda^2} + \dots \right), \tag{14a}$$

$$B_r \approx \tfrac{1}{2}\mu_0 I (r^2 + Z^2)^{-\frac{1}{2}} \left(\frac{\cot(\theta)}{\lambda} - \frac{\sin^2(\frac{1}{2}\theta) \ln(\sin^2(\frac{1}{2}\theta)) - \cos^2(\frac{1}{2}\theta) \ln(\cos^2(\frac{1}{2}\theta))}{\lambda^2} + \dots \right), \tag{14b}$$

$$B_z \approx -\tfrac{1}{2}\mu_0 I (r^2 + Z^2)^{-\frac{1}{2}} \left(\frac{1}{\lambda} + \frac{\ln(\frac{1}{2}\sin(\theta))}{\lambda^2} + \dots \right), \tag{14c}$$

where $\theta = \arctan(r/Z)$. We have taken $\alpha = 1$, so that $\lambda = \ln\{2b^{-1}(r^2 + Z^2)^{\frac{1}{2}}\}$, since r_0 is exponentially small. Note that the vacuum fields are also exponentially small in this limit, so that $\psi \approx \Psi$, $b_r \approx B_r$, and $b_z \approx B_z$ to all orders in λ^{-1}.

3. Rectangular Toroid

We assume the transformer to be symmetric about the z axis, and to have the cross-sectional dimensions indicated in Fig. 1*b*. Outside the iron, the stream function obeys the equation

$$\Delta^* \Psi = r\mu_0\,\sigma(z)\,\delta(r-r_0), \tag{15}$$

where $\sigma(z)$ is a sheet current, which in our case is

$$\sigma(z) = I\,\delta(z-z_0), \tag{16}$$

and Δ^* is the Stokes operator

$$\Delta^* \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{17}$$

In principle we should solve Maxwell's equations for $\boldsymbol{H}$ within the iron, in order to find the tangential component of $\boldsymbol{B}$ at the boundary with the iron (Van Bladel 1961). However, the latter is a function that is independent of r_0 and z_0, and we shall simply prescribe it to be a convenient function, subject only to the constraint provided by Ampere's law.

For mathematical convenience we take

$$B_r(0,r) = B_r(d,r) = 0. \tag{18}$$

Physically this means that we assume the reluctance of the vertical sections to be much greater than that of the horizontal sections, an assumption that is not unreasonable if there are air gaps in the vertical sections. We specify the tangential fields on the vertical sections in terms of their Fourier components

$$B_z(b,z) = \sum_k B_k^b \cos(kz) \qquad \text{and} \qquad B_z(a,z) = -\sum_k B_k^a \cos(kz), \tag{19}$$

where $k = n\pi/d$ and $n = 0, 1, 2, \ldots$. From Ampere's law we have

$$B_0^a(I) + B_0^b(I) = \mu_0 I/d. \tag{20}$$

For $k > 0$, $B_k^a(I)$ and $B_k^b(I)$ are proportional to I, but otherwise they are arbitrary.

For an arbitrary sheet current $\sigma(z)$ we can expand Ψ in a series of the form

$$\Psi = \tfrac{1}{2}(r^2 - r_0^2)\{B_0^a\,\vartheta(r-r_0) - B_0^b\,\vartheta(r_0-r)\} + \sum_{k>0} \Phi_k(r)\cos(kz), \tag{21}$$

where $\vartheta(x)$ is the unit step function, and $\Phi_k(r)$ is of the form

$$\Phi_k(r) = \{\alpha_k + [\alpha_k]\,\vartheta(r-r_0)\}\,r\,I_1(kr) + \{\beta_k + [\beta_k]\,\vartheta(r-r_0)\}\,r\,K_1(kr). \tag{22}$$

The jumps $[\alpha_k]$ and $[\beta_k]$ are determined from the requirements that Ψ be continuous at $r = r_0$, and that the jump in the derivative be given by

$$[\partial\Psi/\partial r]_{r=r_0} = \mu_0\, r_0\, \sigma(z). \tag{23}$$

Using the Wronskian relation for modified Bessel functions, and performing a Fourier analysis on equation (16), we find

$$[\alpha_k] = 2\mu_0 I r_0 d^{-1} K_1(kr_0)\cos(kz_0) \qquad \text{and} \qquad [\beta_k] = -2\mu_0 I r_0 d^{-1} I_1(kr_0)\cos(kz_0). \tag{24}$$

The coefficients α_k and β_k are now determined from equations (1b) and (22) as

$$\begin{aligned}\alpha_k = {} & 2\mu_0 Ir_0 d^{-1}D_k^{-1}\{K_1(kr_0)I_0(ka)+I_1(kr_0)K_0(ka)\}K_0(kb)\cos(kz_0) \\ & -k^{-1}D_k^{-1}\{B_k^a K_0(kb)+B_k^b K_0(ka)\},\end{aligned} \tag{25a}$$

$$\begin{aligned}\beta_k = {} & 2\mu_0 Ir_0 d^{-1}D_k^{-1}\{K_1(kr_0)I_0(ka)+I_1(kr_0)K_0(ka)\}I_0(kb)\cos(kz_0) \\ & -k^{-1}D_k^{-1}\{B_k^a I_0(kb)+B_k^b I_0(ka)\},\end{aligned} \tag{25b}$$

where

$$D_k \equiv I_0(kb)K_0(ka)-I_0(ka)K_0(kb). \tag{26}$$

Although equations (21), (22), (24) and (25) represent the solution of the problem, they are unsuitable for numerical computation owing to the poor convergence of the series in the neighbourhood of r_0, z_0. This may be rectified by subtraction of the stream function for a current loop in the gap between two infinite iron slabs

$$\begin{aligned}\Psi^\infty = {} & -\mu_0 I(2d)^{-1}\vartheta(r_0-r)(r^2-r_0^2) \\ & -2\mu_0 Irr_0 d^{-1}\sum_{k>0} I_1(kr_<)K_1(kr_>)\cos(kz_0)\cos(kz),\end{aligned} \tag{27}$$

where $r_>$ ($r_<$) is the larger (smaller) member of the pair (r, r_0). The total field is then reconstructed by adding on the alternative form for Ψ^∞ given by the method of images,

$$\Psi^\infty = \sum_{n=-\infty}^{\infty} \Psi^{\mathrm{v}}(r,z\,|\,r_0,z_0+2nd) + \sum_{n=-\infty}^{\infty} \Psi^{\mathrm{v}}(r,z\,|\,r_0,-z_0+2nd), \tag{28}$$

where Ψ^{v} is the vacuum field given by equation (5a).

The expression for Ψ may be further decomposed into contributions $\Psi^a+\Psi^b$ from the $k>0$ components of the boundary fields, plus a residual term ψ for which the series is rapidly convergent

$$\Psi = \Psi^\infty+\Psi^a+\Psi^b+\psi. \tag{29}$$

In this expression, we have

$$\Psi^a = -r\sum_{k>0}\frac{K_0(kb)I_1(kr)+I_0(kb)K_1(kr)}{kD_k}B_k^a\cos(kz), \tag{30a}$$

$$\Psi^b = -r\sum_{k>0}\frac{K_0(ka)I_1(kr)+I_0(ka)K_1(kr)}{kD_k}B_k^b\cos(kz), \tag{30b}$$

$$\psi = \tfrac{1}{2}B_0^a(r^2-r_0^2)+r\sum_{k>0}\{\Psi_k^K K_1(kr)+\Psi_k^I I_1(kr)\}\cos(kz_0)\cos(kz), \tag{31}$$

where

$$\Psi_k^K \equiv (2\mu_0 Ir_0/dD_k)\{I_0(ka)K_1(kr_0)+K_0(ka)I_1(kr_0)\}I_0(kb), \tag{32a}$$

$$\Psi_k^I \equiv (2\mu_0 Ir_0/dD_k)\{I_0(kb)K_1(kr_0)+K_0(kb)I_1(kr_0)\}K_0(ka). \tag{32b}$$

It is interesting to observe that, setting $z_0 = \frac{1}{2}d$ and $z = z_0+Z$, and allowing a and d to approach infinity, we may show that ψ approaches the infinite rod result of equation (7). Thus the topological difference is not as serious as at first it appeared.

The two models are compared in Fig. 4. Another interesting fact may be derived from equations (24) to (29). If we define a flux Ψ' with boundary terms subtracted,

$$\Psi' = \Psi - \Psi^a - \Psi^b - \tfrac{1}{4}(r^2 - r_0^2)(B_0^a - B_0^b),$$

then we may prove that Ψ' is symmetric under interchange of r, r_0 and/or z, z_0. This is essentially the usual symmetry of mutual inductances.

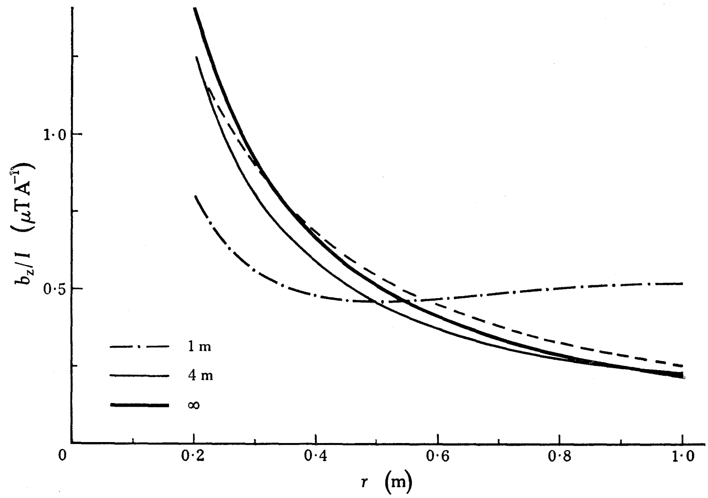


Fig. 4. Distributions of the negative of the changes b_z, per unit current, in the vertical field of a current loop at $(r_0, z_0) = (0\cdot5\ \mathrm{m}, \tfrac{1}{2}d)$ which arise in the presence of a rectangular core of inner radius $b = 0\cdot2$ m and outer dimensions $a = d$ given in the legend to the curves. The dashed curve represents the asymptotic approximation (equation 12c), with $\alpha = 0\cdot6$, while the thick solid curve is identical to the infinite core result (equation 2c).

We also note that the effect of air gaps in the vertical sections can be modelled by taking delta function tangential fields. For example, an air gap at $r = b$, $z = z_g$, with a magnetic potential drop of g relative to that in the whole $r = b$ section, would give

$$B_z(b, z) = B_0^b\{1 - g + gd\,\delta(z - z_g)\},$$

and hence

$$B_k^b = 2B_0^b\, g\cos(kz_g)\,. \tag{33}$$

However, an equivalent (and numerically better) method is to model the air gap by a current loop at $r = b$ and $z = z_g$ carrying a current $-dgB_0^b/\mu_0$. This equivalence can be demonstrated either by application of Ampere's law around an infinitesimal semicircle centred on b, z_g, or by use of equations (21), (30) and (33), and the Wronskian.

4. Plasma Equilibrium

As mentioned in the Introduction, we may distinguish two classes of problems arising from the use of an iron core. The first, the effect of bias windings situated on the core, may be minimized in the early stage of the discharge by use of the equivalence (mentioned at the end of the previous section) of a negative current loop to an air gap. If most of the reluctance resides in the air gaps, the initial stray field may be cancelled out by situating the bias windings near the air gaps.

The effect of the core on the stability of a plasma column is discussed elsewhere (Dewar 1976). For the present we simply note that the existence of a system of external conductors that will give a stable equilibrium depends on the subtracted self field of the plasma column

$$\boldsymbol{b}^{\mathrm{s}}(r,z) \equiv \boldsymbol{b}(r,z\,|\,r,z)\,, \tag{34}$$

where the vacuum contribution has been subtracted, as this depends on the detailed structure of the plasma (Mukhovatov and Shafranov 1971). The existence of a stable system depends on the quantity

$$\sigma \equiv \left(\frac{\partial b_z^{\mathrm{s}}}{\partial r} - \frac{\partial b_r^{\mathrm{s}}}{\partial z}\right) I^{-1}\,. \tag{35}$$

If σ exceeds a critical amount σ_{c} (where $\sigma_{\mathrm{c}} > 0$), depending on the vacuum self field, then no system of external conductors with fixed currents will stabilize the system simultaneously against both vertical and horizontal displacements. In the case of the infinite rod, σ is given by

$$\sigma = \frac{\mu_0}{\pi}\int_0^\infty \{I_0(kb)/K_0(kb)\}\{K_0^2(kr) + K_1^2(kr)\}k^2\,\mathrm{d}k \tag{36}$$

and is thus seen to be positive definite. Thus the iron core tends to be destabilizing. The same conclusion is borne out in the case of the rectangular core.

5. Conclusions

We have examined here the highly idealized infinite rod model, and simple approximate formulae suitable for order of magnitude estimates have been presented. We have also examined a more accurate model (especially in axisymmetric configurations), suitable for numerical computation in design studies. The simplest model of plasma equilibrium suggests that use of an iron core need not affect the equilibrium strongly, provided the external conductors are appropriately positioned around the plasma, but may have an appreciable adverse effect on its stability.

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Appendix

We may obtain the $O(r\lambda^{-n})$ correction terms for the large r limit of the equations (12) by requiring that Ψ must satisfy the equation for magnetic fields in a vacuum, namely,

$$\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2} = 0. \tag{A1}$$

We find that equation (A1) can be satisfied asymptotically in λ by a series of the form

$$\Psi \approx (r^2+Z^2)^{\frac{1}{2}} \sum_{n=1}^{\infty} \lambda^{-n} f_n(v), \tag{A2}$$

where

$$v = \cos\theta = Z/(r^2+Z^2)^{\frac{1}{2}} \qquad \text{and} \qquad \lambda = \ln\big(2\alpha(r^2+Z^2)^{\frac{1}{2}}/b\big),$$

while $f_n(v)$ obeys the recurrence relation

$$(1-v^2)f_n'' - (n-1)f_{n-1} + (n-1)(n-2)f_{n-2} = 0. \tag{A3}$$

By symmetry, $f_n(v)$ is an even function of v. The general solution of equation (A3) that is even in v is

$$f_n = (n-1)! \sum_{k=1}^{n} C_k S_{n-k}(v), \tag{A4}$$

where the C_k are arbitrary constants to be determined from the boundary condition $B_z = 0$ at $r = b$. The $S_n(v)$ are given for all n by

$$S_n(v) = \vartheta(n+0) \sum_{\text{odd } m} \left(\frac{1}{(m+1)^{n+1}} + \frac{(-1)^n}{m^{n+1}}\right)(1-v^2)P_m'(v), \tag{A5}$$

where the $P_m(v)$ are Legendre polynomials. The first two $S_n(v)$ are, explicitly,

$$S_0(v) = 1 \qquad \text{and} \qquad S_1(v) = \tfrac{1}{2}\{(1+v)\ln(1+v) + (1-v)\ln(1-v) - 2\ln 2\}. \tag{A6}$$

To apply the boundary condition we need the behaviour of $S_n(v_0)$ and $S_n'(v_0)$, where $v_0 = \cos\theta_0$, θ_0 being $\arctan\big(b(r^2+Z^2)^{-\frac{1}{2}}\big) = b(r^2+Z^2)^{-\frac{1}{2}}$ to all orders in λ^{-1}. In terms of θ_0 we have

$$\lambda = \ln(2\alpha/\theta_0).$$

The asymptotic behaviour on the boundary is, to all orders in λ^{-1},

$$S_0(v_0) = 1, \qquad S_n(v_0) \simeq 0 \quad \text{for} \quad n \geqslant 1, \qquad S_0'(v_0) = 0,$$

$$S_1'(v_0) \simeq \lambda - \ln\alpha, \qquad S_2'(v_0) \simeq -\pi^2/12 - S_1'(v_0),$$

$$S_n'(v_0) \simeq \big(-[(-1)^n + \{1-(-1)^n\}2^{1-n}]\,\zeta(n-1) + [(-1)^n - \{1+(-1)^n\}2^{-n}]\,\zeta(n)\big) \qquad \text{for} \qquad n \geqslant 3.$$

Applying the boundary condition at each order in λ^{-1} we find

$$C_2 = (1+\ln\alpha)C_1,$$

$$C_3 = \tfrac{1}{2}\{(1+\ln\alpha)(2+\ln\alpha) - \ln\alpha + \tfrac{1}{12}\pi^{\frac{1}{2}}\}C_1, \qquad \text{etc.}$$

From Ampere's law we find

$$C_1 = \tfrac{1}{2}\mu_0 I.$$

From the second of the equations (1b) together with (A2) and (A4) we find the series for B_z, given $Z = 0$, to be

$$B_z = -\frac{\mu_0 I}{2(r^2+Z^2)^{\frac{1}{2}}}\left(\frac{1}{\lambda}+\frac{\ln\alpha-\ln 2}{\lambda^2}+\frac{(\ln\alpha)^2-2\ln(2)\ln(\alpha)+2\cdot 94}{\lambda^3}+O(\lambda^{-4})\right).$$

We note that the $O(\lambda^{-2})$ term can be made zero, and the $O(\lambda^{-3})$ term minimized, by taking $\alpha = 2$. It is apparent from Fig. 4 that the leading term gives an approximation accurate to about $\pm 10\%$ for $r \gtrsim r_0 = 2\cdot 5b$. The $O(\lambda^{-2})$ term does not, however, become a good estimate for the error until $r \gtrsim 6r_0$.

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