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Field Theory of Particles with Arbitrary Spin

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Abstract

It is pointed out that existing field equations for particles of higher spin are unsuitable for the formulation of field theories with interaction. A generalization of the Dirac and Kemmer matrices is discussed in terms of finite-dimensional representations of the de Sitter group. It is shown how to formulate a general field theory in such a way as to exhibit a corresponding dynamical symmetry. The resulting field equation resembles Bhabha's, but is self-consistent in its applications to interacting particles and has a different type of mass spectrum. In the Appendix, it is shown that within any irreducible representation of the Poincaré group there are finite-dimensional representations of the Lorentz group labelled $(s, \pm s)$.

1. Introduction

The theory of particles with spin values greater than 1 has a long and interesting history. At a time when the physical applications had not yet been discovered experimentally, Dirac (1936) was the first to attempt a generalization of his relativistic theory of the electron. Although this seemed adequate for particles without interaction, Fierz and Pauli (1939) found that Dirac's theory led to a contradiction for particles with spin greater than 1 in interaction with an electromagnetic field. They proposed a modified theory which was self-consistent even for interacting particles, insofar as it could be derived by a variational procedure from an action; but it required a set of awkward supplementary conditions, and subsequent experience has revealed problems with quantization and renormalization which are still largely unresolved. Rarita and Schwinger (1941) showed how to reformulate the theory of Fierz and Pauli, for all particles of half-integral spin, in a way which did not require their cumbersome spinor notation.

Lubanski (1942) and Bhabha (1945) attempted a different generalization of Dirac's theory of particles of spin $\frac{1}{2}$, and Kemmer's (1939) theory of particles of spin 0 and 1. This predicted in general a set of particles with different masses and spins, but was physically unsatisfactory since some of these particles proved to have negative probabilities or energies. Although Bhabha was later (1952) able to dispose of this particular difficulty there are others associated with the interactions of the particles; also, it is now clear that the predicted spectrum of masses and spins is different from that appearing in nature (see Trippe *et al.* 1976).

More recent theories have usually been based on the fundamental work of Wigner and Bargmann (Wigner 1939; Bargmann and Wigner 1948) on representations of the Poincaré group (inhomogeneous Lorentz group). The generators $j_{\lambda\mu}, p_\nu$ of the

Poincaré group include the angular momentum $\boldsymbol{j} = (j_{23}, j_{31}, j_{12})$, the energy p^0 and the momentum $\boldsymbol{p} = (p^1, p^2, p^3)$ (in units with Planck's constant $h = 2\pi\hbar = 2\pi$ and the velocity of light $c = 1$; $p^\lambda = g^{\lambda\mu} p_\mu$, where $g^{00} = 1$ and $g^{11} = g^{22} = g^{33} = -1$). They satisfy the commutation relations

$$[j_{\lambda\mu}, j_{\nu\rho}] = \mathrm{i}(g_{\mu\nu} j_{\lambda\rho} - g_{\lambda\nu} j_{\mu\rho} - g_{\mu\rho} j_{\lambda\nu} + g_{\lambda\rho} j_{\mu\nu}), \tag{1a}$$

$$[j_{\lambda\mu}, p_\nu] = \mathrm{i}(g_{\mu\nu} p_\lambda - g_{\lambda\nu} p_\mu), \tag{1b}$$

$$[p_\mu, p_\nu] = 0. \tag{1c}$$

The mass and spin are eigenvalues of operators μ and σ defined by

$$\mu^2 = p^\lambda p_\lambda, \qquad \mu^2 \sigma(\sigma+1) = -w^\lambda w_\lambda, \qquad w_\lambda = \tfrac{1}{2}\varepsilon_{\lambda\mu\nu\rho} j^{\mu\nu} p^\rho. \tag{2}$$

The nice idea that all particles in nature are associated with specific irreducible representations of the Poincaré group was developed in detail by Shirokov (1958), among others. We shall argue here, however, that this particular view of the elementary particles is inadequate, and that the ultimate symmetry group of physics must be one of higher symmetry than the Poincaré group. The latter is not, of course, a new suggestion. From different points of view, Pauli and Solomon (1932), Fronsdahl (1965), Tanikawa (1965), Böhm (1966) and Chakrabarti *et al.* (1968) have all stressed the importance of the de Sitter group SO(4, 1) for the representation of elementary particles. (The present application of the de Sitter group has of course nothing to do with cosmology.) But till now there has been no special relativistic field theory of particles of arbitrary spin incorporating this symmetry.

To appreciate both the merit and the inadequacy of a field theory based on Poincaré invariance, let us consider a quantized field variable $\phi(x, \omega)$ with any number of components, depending on the coordinates x^λ and also a set of parameters $\omega^{\mu\nu}$ specifying the orientation and velocity of the observer. Under changes of the x^λ and $\omega^{\mu\nu}$, $\phi(x, \omega)$ is subject to a unitary transformation:

$$\phi(x+\delta x, \omega+\delta\omega) = U(\delta x, \delta\omega)\, \phi(x, \omega)\, U^*(\delta x, \delta\omega), \tag{3}$$

where, for small δx^λ and $\delta\omega^{\mu\nu}$,

$$U(\delta x, \delta\omega) = 1 + \mathrm{i}\, P_\lambda\, \delta x^\lambda + \tfrac{1}{2}\mathrm{i}\, J_{\mu\nu}\, \delta\omega^{\mu\nu}. \tag{4}$$

Here P_λ and $J_{\mu\nu}$ are universal generators of the Poincaré group, satisfying relations like (1) above. If we define p_λ and $j_{\mu\nu}$ as linear operators on $\phi(x, \omega)$, given by

$$p_\lambda\, \phi(x, \omega) = [\phi(x, \omega), P_\lambda], \qquad j_{\mu\nu}\, \phi(x, \omega) = [\phi(x, \omega), J_{\mu\nu}], \tag{5}$$

we can regard $\phi(x, \omega)$ as carrying a representation of the Poincaré group.

However, existing field theories cannot be formulated completely in this way. In the theory of particles of spin $\frac{1}{2}$, a fundamental role is played by Dirac's operator $\gamma^\lambda p_\lambda$ and the pseudoscalar γ^5, neither of which is in the enveloping algebra generated by $j_{\lambda\mu}$ and p_ν. It is true that the extension required in this instance is quite simple, and can be achieved by introducing a pair of operators γ_0 and γ_5 which are related to parity. But this in fact implies the extension of a finite dimensional representation of the Lorentz group SO(3, 1) to a corresponding representation of the de Sitter group SO(4, 1). The same thing is found in field theories of particles with higher spin, and there the required extensions are no longer simply related to parity. Thus, the theory of Fierz and Pauli (1939), as formulated by Rarita and Schwinger (1941), for particles of spin $\frac{3}{2}$, requires a 16-component field variable, while that of Bhabha (1945) requires a 20-component field variable; dimensionally and structurally, both correspond to representations of the de Sitter group.

The necessity for extending the Poincaré group can be established as follows. As shown in the Appendix to this paper, it is possible within any irreducible representation of the Poincaré group to resolve $j_{\lambda\mu}$ into two parts,

$$j_{\lambda\mu} = l_{\lambda\mu} + s_{\lambda\mu}, \tag{6}$$

associated with the orbital and the spin angular momentum respectively. However, the spin component $s_{\lambda\mu}$ then belongs to a particular kind of finite-dimensional representation of the Lorentz group, labelled $(s, \pm s)$ in terms of highest weights. All other kinds of finite-dimensional representations are absent, and it is not possible to construct a complete set of states in a unitary representation of the Poincaré group without them. The de Sitter group is the smallest group with irreducible representations containing all required representations of the Lorentz group. On the other hand, it seems reasonable to require of a field theory that it should include irreducible representations of the Poincaré group. From this point of view, the theories of Fierz and Pauli (1939) and Rarita and Schwinger (1941) are defective. It is well known that their theories of particles of spin $\frac{3}{2}$, for instance, are 16-dimensional and contain only finite-dimensional representations of the Lorentz group labelled $(\frac{3}{2}, \pm\frac{1}{2})$ and $(\frac{1}{2}, \pm\frac{1}{2})$, the latter excluded by a supplementary condition; but an irreducible representation of the Poincaré group corresponding to spin $\frac{3}{2}$ contains finite-dimensional representation of the Lorentz group, labelled $(\frac{3}{2}, \frac{3}{2})$ or $(\frac{3}{2}, -\frac{3}{2})$. Some but not all of the representations in the theory of Bhabha (1945) are excluded for a similar reason.

There are besides other reasons for claiming that an elementary particle cannot always be represented with an unextended finite-dimensional irreducible representation of the Poincaré group. A particle in interaction with other particles has no determinate mass, and must therefore be thought of in terms of at least a superposition of such representations. The field theory which we develop below is novel in that it does not associate a definite mass with a particle and has the advantage that it is possible to deal with interactions in a consistent way. This method of formulation could provide another way of looking at the still formidable problems presented by the theory of interacting particles.

2. Finite-dimensional Representations of the de Sitter Group

As Bhabha (1945) has shown, field theories for particles of arbitrary spin can be formulated in terms of a set of linear operators $\alpha_\lambda, \alpha_{\mu\nu}$ satisfying the commutation relations

$$[\alpha_\mu, \alpha_\nu] = \alpha_{\mu\nu}, \qquad [\alpha_\lambda, \alpha_{\mu\nu}] = g_{\lambda\mu}\alpha_\nu - g_{\lambda\nu}\alpha_\mu. \tag{7}$$

In the physical interpretation, the spin angular momentum tensor (as defined, for instance, in the Appendix) is given by

$$s_{\lambda\mu} = \mathrm{i}\,\alpha_{\lambda\mu}\,; \tag{8}$$

it follows from equations (7) that the components satisfy commutation relations similar to those of $j_{\lambda\mu}$ in equations (1) and are therefore generators of the Lorentz group in some representation. If we define s and s', within a suitable extension of the enveloping algebra, by

$$\tfrac{1}{2}\alpha_\mu^\lambda\alpha_\lambda^\mu = s(s+2)+s'^2\,, \qquad \tfrac{1}{8}\varepsilon_{\lambda\mu\nu\rho}\alpha^{\lambda\mu}\alpha^{\nu\rho} = \mathrm{i}\,s'(s+1)\,, \tag{9}$$

we may label irreducible representations in terms of eigenvalues of the invariants s and s' thus: (s, s'). The eigenvalues of s and s' are highest weights in an irreducible representation; in such a representation, s is half a non-negative integer, and $s-|s'|$ is a non-negative integer.

The α_λ and $\alpha_{\mu\nu}$ together can be regarded as generators of the de Sitter group. If we write

$$\alpha_{4\lambda} = \alpha_\lambda\,, \qquad g_{44} = -1\,, \tag{10}$$

and agree that $l, m, n, r, \ldots$ shall take the five values $0, 1, 2, 3, 4$, it follows that

$$[\alpha_{lm}, \alpha_{nr}] = g_{mn}\alpha_{lr} - g_{ln}\alpha_{mr} - g_{mr}\alpha_{ln} + g_{lr}\alpha_{mn}\,, \tag{11}$$

which are the commutation relations of SO(4, 1). If s_1 and s_1' are defined by

$$\tfrac{1}{2}\alpha^{lm}\alpha_{ml} = s_1(s_1+3)+s_1'(s_1'+1)\,, \tag{12a}$$

$$\varepsilon^l\varepsilon_l = (s_1+1)(s_1+2)s_1'(s_1'+1)\,, \tag{12b}$$

$$\varepsilon_l = \tfrac{1}{8}\varepsilon_{lmnrs}\alpha^{mn}\alpha^{rs}\,, \tag{12c}$$

then s_1 and s_1' have eigenvalues which can be used to label irreducible representations thus: (s_1, s_1'). As they are again highest weights in a finite-dimensional representation, s_1 and s_1' are half non-negative integers and $s_1 - s_1'$ is a non-negative integer. Within such an irreducible representation, there are all different representations of the Lorentz group labelled (s, s'), where $s_1 \geqslant s \geqslant s_1'$ and $s_1' \geqslant |s'|$. It follows that, if we wish an irreducible representation of the de Sitter group to contain a representation of the Lorentz group labelled $(s, \pm s)$, we must choose $s_1 = s_1' = s$, *and this we shall do.* We thus exclude, for reasons already stated in the Introduction, the theories of Fierz and Pauli (1939) and Rarita and Schwinger (1941) for particles of higher spin. However, we do include Dirac's (1936) theory of particles of spin $\frac{1}{2}$, where the Dirac matrices are given by $\gamma_\lambda = 2\alpha_\lambda$, and Kemmer's (1939) theory of particles of spin 1, where the Kemmer matrices are the same as our α_λ.

An irreducible representation of the de Sitter group labelled (s, s) has dimension $\frac{1}{6}(2s+1)(2s+2)(2s+3)$. It is known from Bhabha's (1945) investigations (see also Harish-Chandra 1947) that the spin within one of the included irreducible representations of the Lorentz group, labelled (s, s'), takes every value σ such that $s-\sigma$ and $\sigma-|s'|$ are non-negative integers. The component of the representation of the de Sitter group corresponding to the spin σ is therefore $(2\sigma+1)^2$-dimensional. Since we intend to make use of the component with $\sigma = s$ only, there would appear to be a considerable redundancy in the number of field components, at least for large s. This redundancy is, however, hard to avoid in field theories of particles with interaction. To cite a familiar example, the electromagnetic field components in the 10-dimensional Kemmer (1939) representation are the field tensor $F_{\lambda\mu}$ ($= \boldsymbol{E}, \boldsymbol{B}$) and the four-vector potential A_ν, where the latter includes one unwanted component of spin 0. This latter component is sometimes eliminated by means of a supplementary condition (the Lorentz condition), but the difficulties in formulating quantum electrodynamics without at least considering all four components of A_ν are well known.

We wish now to examine the structure of the $(2s+1)^2$-dimensional representation, corresponding to spin s, within an irreducible representation of the de Sitter group labelled (s,s). For this purpose we shall need to make use of the identity

$$(\alpha^\lambda_\mu \alpha^\mu_\nu - 2\alpha^\lambda_\nu)\alpha^\nu = (s'^2 - 1)\alpha^\lambda . \tag{13}$$

This can be proved by the method used by Bracken and Green (1971) to establish characteristic identities satisfied by matrices of generators like α^λ_μ. Normally a four-vector operator like α^λ can be resolved into four components changing each of the labels (s,s') of irreducible representations of the Lorentz group by $+1$ or -1. However, because the label s cannot be changed within a representation of the de Sitter group labelled (s,s), only two of the usual four components, namely $\alpha^\lambda_\mu \alpha^\mu - (1 \pm s')\alpha^\lambda$, are different from zero. The reduced identity (13) must therefore hold.

Now there are three scalar operators, $\alpha.p = \alpha^\lambda p_\lambda$, $\alpha.w = \alpha^\lambda w_\lambda$ and s', which can be constructed with the help of the α^λ and which commute with all generators of the Poincaré group. They satisfy the commutation relations

$$[\alpha.p, s'] = \alpha.w/(s+1) , \tag{14a}$$

$$[\alpha.w, \alpha.p] = -s'(s+1)\mu^2 , \tag{14b}$$

$$[s', \alpha.w] = -\alpha.p(s+1) , \tag{14c}$$

the first two (14a, b) on account of the definitions only, and (14c) because of the identity (13). From these relations it follows that s', $\alpha.p/\mu$ and $-i\alpha.w/\mu(s+1)$ are generators of a representation of SO(3), and in view of the known eigenvalues of s', it must be a $(2s+1)$-dimensional representation. The operators w^λ also suffice to determine a $(2s+1)$-dimensional representation of SO(3); thus, we have identified the structure of the $(2s+1)^2$-dimensional component of the finite-dimensional representation of the de Sitter group corresponding to spin s as that of SO(3) × SO(3). In Dirac's theory, $s' = \frac{1}{2}\gamma_5$, $\alpha.p = \frac{1}{2}\gamma^\lambda p_\lambda$ and $\alpha.w = \frac{1}{2}(s+1)\gamma^\lambda p_\lambda \gamma_5$ all play an important role, and it is to be expected that they should play a corresponding role in theories of particles with higher spin.

3. Field Theory for Arbitrary Spin

Because of the fundamental role of finite-dimensional representations of the de Sitter group in field theories of particles with spin, it is natural to seek corresponding dynamical representations. In Section 4 we shall define a set of generators j_{lm}, including $j_{4\lambda} = -j_{\lambda 4}$, extending the dynamical generators $j_{\lambda\mu}$ of the Lorentz group. But we shall first formulate the field theory in a general way so as to provide suitable vector spaces to carry representations of both the α_{lm} and the j_{lm}.

The traditional formulation of relativistic field theories (Pauli 1941) is in terms of a field variable ϕ and its derivatives $\phi_{,\lambda}$ with respect to the coordinates x^λ. The field variable may have any number of components, which without loss of generality may be assumed to be hermitian (real, in the unquantized theory). Thus a non-hermitian field $\phi_a + \mathrm{i}\phi_b$ is replaced by its two hermitian components (ϕ_a, ϕ_b). However, we shall then need to introduce linear operators C and i, such that

$$C(\phi_a, \phi_b) = (\phi_a, -\phi_b), \qquad \mathrm{i}(\phi_a, \phi_b) = (-\phi_b, \phi_a).$$

As Bhabha (1945) has shown, there is always an element η of the spin algebra such that $\eta^2 = 1$ and $\eta\alpha_\lambda = \alpha^\lambda\eta$. We can choose a matrix representation of the α_λ in which the transpose of α^λ is $-\alpha_\lambda$, and so η will be antisymmetric for half-odd-integral spin and symmetric for integral spin. Then we may define a conjugate $\bar{\phi}$ of ϕ in the usual way by $\bar{\phi} = (C\phi)\eta$. Lagrangian field theory does not display any obvious symmetry of the de Sitter group when formulated solely in terms of ϕ and $\phi_{,\lambda}$, and we therefore introduce a set of conjugate variables ψ^l, defined by

$$\psi^\lambda = -\mathrm{i}\,\partial L/\partial\phi_{,\lambda}, \qquad \psi^4 = \bar{\phi}, \tag{15}$$

where L is the Lagrangian density. If we write

$$R = \mathrm{i}\psi^\lambda\phi_{,\lambda} - L, \tag{16}$$

we have

$$\mathrm{d}R = \mathrm{i}(\mathrm{d}\psi^\lambda\phi_{,\lambda} + \mathrm{d}\psi^4\phi_{,4}), \tag{17}$$

where, for the sake of uniformity, we have written

$$\phi_{,4} = \mathrm{i}\,\partial L/\partial\bar{\phi}. \tag{18}$$

This purely notational device does *not*, of course, carry the implication that $\phi_{,4}$ is a derivative with respect to a new 'coordinate' x^4; if there were such a coordinate in the present formalism, it would be given the value zero. The identity (17) shows, however, that R can always be regarded as a function of ψ_l, instead of the Lagrangian variables ϕ and $\phi_{,\lambda}$, and that the $\phi_{,l}$ are then given by

$$\mathrm{i}\phi_{,l} = \partial R/\partial\psi^l. \tag{19}$$

The field equations in this notation take the form

$$\psi^l{}_{,l} = 0. \tag{20}$$

As an example of the application of this formalism, suppose that the Lagrangian density is

$$L = \mathrm{i}\bar{\phi}\alpha^\lambda\phi_{,\lambda} - \bar{\phi}(\kappa + e\alpha^\lambda A_\lambda)\phi, \tag{21}$$

as in Bhabha's (1945) generalization of Dirac's equation. Then, in terms of the conjugate variables,

$$L = \mathrm{i}\psi^l\phi_{,l} - (\psi^\lambda - \psi^4\alpha^\lambda)\chi_\lambda, \tag{22}$$

with

$$\phi_{,4} = -(\kappa + e\alpha^{\lambda} A_{\lambda})\phi ,$$

where the χ_{λ} are Lagrangian parameters, required because of the identical relations

$$\psi^{\lambda} = \psi^{4} \alpha^{\lambda} \tag{23}$$

between the conjugate variables.

It is evident that any field theory can be shaped in this way, in terms of a set of five conjugate variables ψ_l and a set of five derived variables $\phi_{,l}$. This formalism is indeed similar to the Hamiltonian formalism of ordinary mechanics. It will be assumed here that each of the components of ψ_l carries an irreducible finite-dimensional representation of the de Sitter group of the type discussed in Section 2. In advance of a complete definition of the generators j_{lm}, we may define a set of linear operators η_{lm} (written here, for convenience, *after* the row vector $\psi_l = (\psi_0, \psi_1, \psi_2, \psi_3, \psi_4)$), by means of

$$\psi_l \eta_{mn} = \psi_l \alpha_{mn} - g_{lm} \psi_n + g_{ln} \psi_m . \tag{24}$$

It is easy to verify that they satisfy the commutation relations, analogous to (11), of SO(4, 1).

If the representation of the de Sitter group with generators η_{mn}, as defined by equation (24), is not already irreducible, ψ_l can be resolved into at most three components, each of which carries an irreducible representation. This can be seen by noting that, as a consequence of equation (24),

$$\psi_l \eta_n^m \eta_m^n = \psi_l(\alpha_n^m \alpha_m^n + 8) - 4\psi_m \alpha_l^m , \tag{25}$$

where $\alpha_n^m \alpha_m^n$ has the eigenvalues $4s(s+2)$ in this representation. The only possible eigenvalues of $\eta_n^m \eta_m^n$ are $4s(s+2)$, for a representation labelled (s,s); $4(s+1)(s+2)$, for a representation labelled $(s+1,s)$; and $4s(s+1)$, for a representation labelled $(s,s-1)$. Consequently, the irreducible representations must be of the following types:

(*a*) that labelled (s,s), with an eigenvector ψ_l satisfying

$$\psi_m \alpha_l^m = 2\psi_l ; \tag{26a}$$

(*b*) that labelled $(s+1,s)$, with an eigenvector satisfying

$$\psi_m \alpha_l^m = -s\psi_l ; \tag{26b}$$

(*c*) that labelled $(s,s-1)$, with an eigenvector satisfying

$$\psi_m \alpha_l^m = (s+2)\psi_l . \tag{26c}$$

The matrix α whose elements are α_l^m therefore satisfies the cubic identity

$$(\alpha-2)(\alpha+s)(\alpha-s-2) = 0 , \tag{27}$$

a degenerate form of the characteristic identity for SO(4, 1), which in general is of the fifth degree (Bracken and Green 1971; Green 1971). The simpler identity is satisfied in representations labelled (s,s) because there are no representations labelled

$(s-1, s)$ or $(s, s+1)$. We shall now consider each of the three types of representations in turn.

For type (a), the form of the eigenvector is easily identified, since it can be verified, directly from the definition of ε_l in equation (12c), that

$$\varepsilon_m \alpha_l^m = 2\varepsilon_l. \tag{28}$$

Thus, ψ_l has the form $\psi\varepsilon_l$. Dirac's theory of particles of spin $\frac{1}{2}$ makes use of this representation: for $s = \frac{1}{2}$, $\varepsilon_l = \frac{1}{2}\mathrm{i}(s+1)(\gamma_\lambda\gamma_5, \gamma_5)$, whence $\psi^\lambda = \psi^4\gamma^\lambda$. In the special representations labelled (s, s) adopted here for the α_{lm}, there is a partial generalization for arbitrary spin. It follows from the identity (27) that any antisymmetric tensor which can be constructed from α_{lm} must be proportional to α_{lm} itself, in particular that

$$[\varepsilon_l, \varepsilon_m] = (s+1)^2\alpha_{lm}\,; \tag{29}$$

the factor $(s+1)^2$ results from the fact that $-\mathrm{i}\,\varepsilon_4/(s+1) = s'$, where s' is defined by equations (9) and has the half-integral eigenvalues $-s, -s+1, \ldots, s$. Thus, $\varepsilon_\lambda/(s+1)$ and $\alpha_{\lambda\mu}$ are generators of a de Sitter group similar to that generated by α_λ and $\alpha_{\lambda\mu}$. However, as there is in general no simple linear relation between the components of ψ_l, it does not seem possible to construct a proper generalization of Dirac's theory on this basis.

We shall see in Section 4 that the dynamical representation is of type (b). It will be evident that ψ_l cannot belong to this type of representation, except when $s = 0$, and for this reason we do not consider it further in this context.

In spin representations of type (c), ψ^λ must satisfy an identity

$$\psi^\nu(\alpha_\nu{}^\mu\alpha_\mu{}^\lambda - 2\alpha_\nu{}^\lambda) = \psi^\lambda(s'^2 - 1), \tag{30}$$

with

$$s'^2 = s(s+2) - \alpha^\lambda\alpha_\lambda,$$

like (13). Then it follows from equation (26c) that ψ^λ and ψ^4 are related by

$$\psi^\lambda(\alpha^\mu\alpha_\mu + 1) = \psi^4(\alpha^\mu\alpha_\mu{}^\lambda + s\alpha^\lambda). \tag{31}$$

This representation is not available for spin $\frac{1}{2}$, but is consistent with Kemmer's (1939) theory of particles of spin 1: for $s = 1$, equation (31) reduces to $\psi^\lambda = \psi^4\alpha^\lambda$ when $\psi^\lambda\alpha^\mu\alpha_\mu = 2\psi^\lambda$ and $\psi^4\alpha^\mu\alpha_\mu = 3\psi^4$. However, again there is no simple analogue for particles of higher spin.

It follows that for $s > 1$ a synthesis of representations of types (a) and (c) must be used to secure a generalization of Dirac's theory of spin $\frac{1}{2}$ and Kemmer's theory of spin 1. In such representations, the characteristic identity (27) shows that ψ_l has the form

$$\psi^l = \theta^m\alpha_m{}^l + s\,\theta^l. \tag{32}$$

If we choose $\theta^\lambda = 0$, this yields $\psi^\lambda = \psi^4\alpha^\lambda/s$, consistent with both the theories of Dirac and Kemmer, though not with the Lagrangian density of equation (22) except when $s = 1$. In Section 5 below we shall, however, construct a Lagrangian density compatible with equation (32).

4. Dynamical Generators of the de Sitter Group

As we have already noted, the idea that the de Sitter group, rather than the Poincaré group, should be regarded as the fundamental group of physics, is an old one. It originated in cosmology, but the fundamental length there, the radius of the universe, is too big to be of immediate use in particle physics. However, several authors including Chakrabarti *et al.* (1968) have noticed that if

$$g_\lambda = \tfrac{1}{2}\{j_{\lambda\nu}, p^\nu\}/\mu, \tag{33}$$

where μ is defined in equations (2), then g_λ and $j_{\lambda\mu}$ satisfy the same commutation relations as $\mathrm{i}\alpha_\lambda$ and $\mathrm{i}\alpha_{\lambda\mu}$, shown in equations (7). The operator g_λ/μ may be interpreted as the position vector of a particle without spin in the barycentric frame, though its components commute with one another only in a non-relativistic approximation. But there are difficulties in extending the Lorentz group with generators $j_{\lambda\mu}$ to a de Sitter group with generators g_λ and $j_{\lambda\mu}$. The most serious of these is that, if $g_\lambda^{(1)}$ and $g_\lambda^{(2)}$ are generators for two different particles, the generators for the composite system cannot be assumed to be $g_\lambda = g_\lambda^{(1)} + g_\lambda^{(2)}$, even if the particles are not in interaction.* We therefore propose what appears to be a more satisfactory solution to the problem.

In a field theory, the generators of the Lorentz group can be expressed in terms of the coordinates x_λ thus:

$$j_{\lambda\nu} = x_\lambda p_\nu - x_\nu p_\lambda + \mathrm{i}\alpha_{\lambda\nu}, \tag{34}$$

where $p_\lambda = \mathrm{i}\,\partial/\partial x^\lambda$. In a similar way, we shall write

$$j_{4\nu} = -x_\nu p_4 + \mathrm{i}\alpha_\nu, \tag{35}$$

where

$$p_4 = \mu = (p^\lambda p_\lambda)^{\frac{1}{2}} \tag{36}$$

is the mass. Since the mass is not an invariant of the de Sitter group, it cannot be replaced by a numerical eigenvalue in general. In specific circumstances, it is possible to interpret p_4 as a rational square root of $p^\lambda p_\lambda$ within the present extension of the Poincaré algebra. Thus, if ϕ is an eigenvector of $\alpha . p$ corresponding to the 'eigenvalue' $s\mu$, we can write

$$j_{4\nu}\phi = (x_\nu \alpha . p/s + \mathrm{i}\alpha_\nu)\phi\,; \tag{37}$$

but this can obviously be accepted as a definition of $j_{4\nu}$ only within a restricted domain. In practice, the definition (37) will in fact be used in connection with irreducible representations of a particular type. Within all such representations, the commutation relations

$$[j_{lm}, j_{nr}] = \mathrm{i}(g_{mn} j_{lr} - g_{ln} j_{mr} - g_{mr} j_{ln} + g_{lr} j_{mn}), \tag{38a}$$

$$[j_{lm}, p_n] = \mathrm{i}(g_{mn} p_l - g_{ln} p_m), \tag{38b}$$

are satisfied for all five values 0, 1, 2, 3, 4 of the subscripts.

* The author is much indebted to Dr A. J. Bracken for discussions and correspondence on this point.

To determine the irreducible representations of the j_{lm} thus defined, we calculate the invariant

$$\tfrac{1}{2}j^{lm}j_{lm} = (-\mathrm{i}\,x.p)(-\mathrm{i}\,x.p+3)+2s(s+2)+2\mathrm{i}\,x_\lambda\alpha^\lambda_n p^n, \tag{39}$$

where $x.p = x^\lambda p_\lambda$. In an irreducible representation, this must reduce to a quadratic function of two labelling invariants of the type shown by equations (12). Therefore, if ϕ belongs to such a representation, we must have

$$\alpha^\lambda_n p^n\phi = s_p p^\lambda\phi, \tag{40}$$

where s_p is some invariant. If we multiply this equation by p_λ and divide by p^4, we have also

$$\alpha^4_n p^n\phi = s_p p^4\phi, \tag{41}$$

showing that $p^n\phi$ is an eigenvector of the matrix of generators α^l_n, and s_p the corresponding eigenvalue. From the characteristic identity (27), we see that the only possible values of s_p are 2, $-s$ and $(s+2)$. The value $s_p = 2$ can be excluded, because the components of p^n commute with one another, and the value $s_p = s+2$ is excluded by equation (41), because $\alpha.p$ cannot have the 'eigenvalue' $(s+2)\mu$. Hence

$$\alpha^l_n p^n\phi = -sp^l\phi, \tag{42}$$

if ϕ belongs to this type of irreducible representation of the dynamical de Sitter group.

For $l = 4$, equation (42) yields

$$\alpha.p\,\phi = s\mu\phi, \tag{43}$$

a generalization of Dirac's equation for spin $\tfrac{1}{2}$. We can also show that equation (43) implies (42), as follows. Let us resolve ϕ into eigenvectors $\phi(m)$ of $\alpha.p$, such that

$$\alpha.p\,\phi(m) = sm\,\phi(m), \tag{44}$$

where m is now a number. Then, if

$$\alpha_{\pm\lambda} = \alpha_\lambda - \alpha.p\,p_\lambda/m^2 \pm \alpha_{\lambda\nu}p^\nu/m, \tag{45}$$

it is readily verified that

$$\alpha.p(\alpha_{-\lambda}\phi) = (s+1)m(\alpha_{-\lambda}\phi). \tag{46}$$

But, as we have already seen, the eigenvalues of $\alpha.p$ are bounded by $\pm sm$, so $\alpha_{-\lambda}\phi$ must vanish. It follows from this that equation (42) must be satisfied by any component $\phi(m)$ of ϕ, and hence by ϕ itself, provided that ϕ satisfies equation (43). It also follows that any solution of (43) corresponds to the maximum eigenvalue s of the spin σ. For

$$\begin{aligned}\alpha_{+\lambda}\alpha_{-\lambda}\phi(m) &= [\tfrac{1}{2}\alpha^l_n\alpha^n_l - \sigma(\sigma+1) - s(s+3)]\phi(m)\\ &= [s(s+1)-\sigma(\sigma+1)]\phi(m),\end{aligned} \tag{47}$$

and, as we have already seen, the left-hand side of this equation must vanish.

The above argument shows that a generalized form of Dirac's equation for spin s results from the requirement that the field variable ϕ should carry an irreducible representation of the de Sitter group with dynamical generators given by equations (34) and (35). The resulting equation (43) is similar to that of Lubanski (1942) and Bhabha (1945), but is restricted to a particular kind of finite-dimensional representation of the α^λ, and instead of predicting a discrete spectrum of masses, replaces the mass m with a mass-operator μ with an arbitrary spectrum. The equation is therefore potentially applicable to particles in interaction as well as free particles. We note that the definition (37) of $j_{4\nu}$ is admissible, provided that ϕ is a solution of equation (43).

From equations (39) and (40) it follows that

$$\tfrac{1}{2}j^{lm}j_{lm} = \mathrm{i}a(\mathrm{i}a+3)+s(s+1)\,, \qquad a = -(x\,.p+\mathrm{i}s)\,, \tag{48}$$

so the dynamical representation is labelled $(\mathrm{i}a, s)$, where a can be interpreted as an action associated with the particle, and s of course as the spin. From equation (42) it also follows that

$$j_n^l p^n \phi = a p^l \phi\,. \tag{49}$$

This result exhibits the fundamental relationship between the Poincaré and de Sitter groups, and the role of the energy–momentum vector as an eigenvector of the matrix of generators of the de Sitter group. This way of defining the generators of a non-semi-simple Lie group within the enveloping algebra of a semi-simple Lie algebra can be generalized. The author (Green 1976) has shown recently, *inter alia*, that as a consequence of the characteristic identity satisfied by the generators of SO(n), and corresponding pseudo-orthogonal groups, one can always define within the enveloping algebra an $(n-1)$-dimensional vector (in this instance, p_λ) whose components commute with one another. From this point of view, there is no reason to think of the Poincaré group as more fundamental than the associated de Sitter group.

5. Self-consistent Field Theory

We consider next how the field equation (43) may be incorporated in a field theory of the general type formulated in Section 3. It is natural to require that

$$p_l \phi = \mathrm{i}\phi_{,l} \tag{50}$$

for $l = 4$ as well as the other four values of the subscript, with $\phi_{,4}$ defined as in equation (18). According to the relation (43), the field equation then reduces to

$$\alpha^\lambda \phi_{,\lambda} = -s\phi_{,4}\,. \tag{51}$$

This is similar to the equation resulting from the Lagrangian density (22), apart from the factor s on the right side. However, it should be noticed that there are in general difficulties in formulating a self-consistent theory of interacting particles of higher spin, of the same type as found by Fierz and Pauli (1939) in connection with Dirac's theory. In terms of the preceding discussion, the problem is that the five-vector $\phi_{,l}$ derived from the Lagrangian density might not belong to the same representation of the de Sitter group as $p_l \phi$; then a contradiction could be deduced from equation (50). This difficulty can be met by adopting the field equation (51), in conjunction with the other components of (42), which can be regarded as supplementary conditions.

We therefore adopt the Lagrangian density

$$L = \mathrm{i}\psi^l \phi_{,l} - \psi^l \beta_l^{\,n} \chi_n , \tag{52}$$

with

$$\beta_l^{\,n} = [\alpha_l^{\,m} \alpha_m^{\,n} - (s+4)\alpha_l^{\,n} + 2(s+2)\delta_l^{\,n}]/2(s+1)(s+2),$$

where the χ_n are Lagrangian parameters which, as usual, turn out to have a physical interpretation. Variation with respect to the conjugate variables ψ_l yields

$$\mathrm{i}\phi_{,l} = \beta_l^{\,n} \chi_n , \tag{53}$$

but, because of the characteristic identity (27), it follows that not only the field equation (51), but all five components of

$$\alpha_l^{\,n} \phi_{,n} = -s\phi_{,l} \tag{54}$$

are satisfied. Also, equation (53) reduces to an identity, provided

$$\chi_n = \mathrm{i}\phi_{,n} ; \tag{55}$$

this solution is unique in the representation of type (*b*) selected by the idempotent matrix $\beta_l^{\,n}$.

The conjugate variables ψ^l satisfy the equation

$$\psi^l \beta_l^{\,n} = 0, \tag{56}$$

obtained by variation of the Lagrangian with respect to χ_n. It follows, again by virtue of the characteristic identity, that ψ^l must have the proposed form (32). With the help of the field equation (20), we have

$$\psi^4{}_{,\lambda} \alpha^\lambda = -s\psi^4{}_{,4} , \tag{57}$$

which is the conjugate of equation (51).

The above equations are all obviously self-consistent and in agreement with the requirements of the two previous sections. The precise form of $\phi_{,4}$ is determined by the usual requirements of gauge invariance; thus, with electromagnetic interactions only,

$$\mathrm{i}\phi_{,4} = e\alpha^\lambda A_\lambda \phi/s + m\phi , \tag{58}$$

where m is the 'bare' mass; but the principle of gauge invariance has been extended to all other types of interactions (see Taylor 1976). There is also no difficulty in the quantization of the field theory by the usual methods. For particles satisfying Bose statistics the equal-time commutation relations are

$$[\phi(x), C\phi(x')] = \varepsilon\,\delta(\boldsymbol{x}-\boldsymbol{x}') , \tag{59}$$

where $C\phi(x') = \psi^4(x')\eta$ and ε is the idempotent of the representation of type (*b*) to which $\phi(x)$ belongs. For particles satisfying Fermi statistics, the commutator is replaced by the anticommutator.

To summarize, we have developed a field theory of particles of arbitrary spin which can be regarded as a generalization of the theories of Dirac (1936) for spin $\frac{1}{2}$

and Kemmer (1939) for spin 1. It differs from the theories of Fierz and Pauli (1939) and Rarita and Schwinger (1941) in that it allows irreducible representations of the Poincaré group. It differs from the theory of Bhabha (1945) in substituting a possibly continuous mass spectrum for the discrete spectrum of that theory, and permits the introduction of gauge-invariant interactions in a self-consistent way.

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Appendix

Here we wish to show that there are two representations of the spin angular momentum $s_{\lambda\mu}$, and two corresponding representations for the orbital angular momentum $l_{\lambda\mu}$, within an irreducible representation of the Poincaré group. The spin representations are labelled $(s, \pm s)$ and are therefore mirror images. The problem of defining $s_{\lambda\mu}$ and $l_{\lambda\mu}$ in terms of $j_{\lambda\mu}$ and p_ν was discussed recently by Lorentz and Rondu (1974), but by a method depending on the introduction of an arbitrary time-like four-vector, which does not, of course, lead to a unique result.

As elsewhere in the text of this paper, we assume that μ, as defined in equations (2) of Section 1, is not singular—photons and neutrinos would require a separate discussion. The spin vector is then defined by $s_\lambda = w_\lambda/\mu$; as a nontrivial consequence of the commutation relations (1) it satisfies

$$[j_{\lambda\mu}, s_\nu] = \mathrm{i}(g_{\mu\nu} s_\lambda - g_{\lambda\nu} s_\mu), \tag{A1}$$

like any other four-vector. We introduce the pseudoscalar γ, having the value $+1$ or -1, and verify that if

$$\mathrm{i} s_{\lambda\nu} = [s_\lambda, s_\nu] + \gamma(s_\lambda p_\nu - s_\nu p_\lambda)/\mu \tag{A2}$$

then $s_{\lambda\mu}$ satisfies commutation relations like $j_{\lambda\mu}$ in equations (1), as required of the spin angular momentum tensor. It obviously commutes with p_ν. As an appropriate generalization of equation (33), let us write

$$g'_\lambda = \tfrac{1}{2}\{j_{\lambda\nu}, p^\nu\}/\mu + \mathrm{i}\,\gamma s_\lambda . \tag{A3}$$

Then g'_λ commutes with $s_{\mu\nu}$, and yields

$$[g'_\lambda, g'_\mu] = \mathrm{i}\, l_{\lambda\mu}, \qquad l_{\lambda\mu} = (g'_\lambda p_\nu - g'_\nu p_\lambda)/\mu . \tag{A4}$$

It is a straightforward matter to check that $l_{\lambda\mu}$, defined in this way, also satisfies commutation rules like $j_{\lambda\mu}$ in equations (1), and

$$\varepsilon_{\lambda\mu\nu\rho}\, l^{\mu\nu} p^\rho = 0 , \tag{A5}$$

as required of the orbital angular momentum. It will be noticed that, although we have not found it necessary to introduce coordinates x^λ to define $l_{\lambda\mu}$, all the above relations are verified if

$$l_{\lambda\mu} = x_\lambda p_\mu - x_\mu p_\lambda , \qquad g'_\lambda = x_\lambda \mu - (x^\nu p_\nu + 5\mathrm{i}/2) p_\lambda \mu^{-1} . \tag{A6}$$

Since, with the above definitions,

$$\tfrac{1}{2}\varepsilon_{\lambda\mu\nu\rho}\, s^{\nu\rho} = \mathrm{i}\,\gamma s_{\lambda\mu} , \tag{A7}$$

it is evident from equations (8) and (9) that, for $\gamma = -1$ and $+1$, $s_{\lambda\mu}$ belongs to representations labelled (s, s) and $(s, -s)$ respectively. We have thus shown that any irreducible representation of the Poincaré group contains finite-dimensional representations of the Lorentz group labelled $(s, \pm s)$ within the enveloping algebra.

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