Aust. J. Phys., 1977, **30**, 33–7

An Improved Calculation of the Multiphoton Bremsstrahlung Process in the Presence of a Laser Beam

Rama Acharya and Man Mohan

Department of Physics and Astrophysics, University of Delhi, Delhi 110007, India.

Abstract

It has been noted previously that analyses of free–free transitions in the presence of an ion are appreciably affected by neglecting the Coulomb field of the ion at infinity. We have therefore made here an improved calculation of the cross section for the multiphoton bremsstrahlung process by taking account of the Coulomb field in the initial and final states.

Introduction

In all previous treatments of the problem of bremsstrahlung in a strong laser beam, the wavefunction of the electron at a great distance from the scattering centre has been taken to be a plane wave. However, the Coulomb field due to the ion, in the vicinity of which the free-free transition is considered, is not zero at infinity and its neglect appreciably affects the calculations (Faisal 1973; Geltman and Teague 1974). In an attempt to further explicate the problem, we have modified the analysis here by adopting Coulomb wavefunctions (Akhizer and Berestetskii 1965) for the initial and final states of the electron when scattered in a Coulomb potential, since these forms satisfy the required conditions at infinity, i.e. the wavefunction describing the initial state must be the sum of a plane wave and an outgoing spherical wave at infinity while that for the final state must be the sum of a plane wave and an incoming spherical wave at infinity.

For the processes investigated in nonlinear optics, the intensity of the radiation is less than the critical value so that the state of the medium does not change significantly during the time of interaction with the radiation. However, when the intensity E_0 of the radiation is comparable with the intensity E_a of the interatomic field, that is, $E_0 \sim E_a \gtrsim 5\times10^9\,\mathrm{V\,cm^{-1}}$, the atoms of the medium become rapidly ionized and the substance is converted into plasma. Although processes occur in this region which are analogous to those considered in nonlinear optics, phenomenological methods will not now suffice for their study and a microscopic approach is essential. The problem of scattering of electrons in the presence of laser radiation falls in this region. At still higher intensities, quantum electrodynamical effects may contribute significantly to the process concerned. But with the laser intensities which are presently available these effects need not be taken into account.

Cross-section Calculation for Multiphoton Bremsstrahlung

In the presence of an electromagnetic field of vector potential A, the exact solution of the Schrödinger equation for an electron of momentum $\boldsymbol{p}$ in a background Coulomb potential of the form $V(r) = -Ze^2/r$ is (Faisal 1973)

$$\psi_p = \exp(\tfrac{1}{2}\pi\eta)\,\Gamma(1-\mathrm{i}\eta)\exp(\mathrm{i}\boldsymbol{p}.\boldsymbol{r})\,\mathrm{F}\big(\mathrm{i}\eta, 1, \mathrm{i}(pr - \boldsymbol{p}.\boldsymbol{r})\big)$$

$$\times\exp\left(-(\mathrm{i}/\hbar)\int_0^t \tfrac{1}{2}m_\mathrm{e}^{-1}(\boldsymbol{p}-e\boldsymbol{A})^2\,\mathrm{d}\tau\right),$$

where $\eta = Ze^2/\hbar V$ and the gamma and confluent hypergeometric functions have their standard definitions (Gradshteyn and Ryznik 1965).

If spontaneous emission is considered to be negligible compared with that resulting from induced excitation, the probability of absorption or emission of n quanta is determined by the probability of transition of an electron from a state of momentum $\boldsymbol{p}$ to one of momentum $\boldsymbol{p}'$, with

$$\Delta\varepsilon = (p'^2 - p^2)/2m_\mathrm{e} = \pm n\hbar\omega .$$

The probability amplitude for such a transition is given by

$$C_{pp'}(t) = -\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t'\,\mathrm{d}\boldsymbol{r}\,\psi^*_{p'}\,V(\boldsymbol{r})\,\psi_p$$

$$= -\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t'\exp\left(\frac{\mathrm{i}}{\hbar}\frac{(p'^2-p^2)t'}{2m_\mathrm{e}} - \frac{\mathrm{i}}{\hbar}\frac{(\boldsymbol{p}'-\boldsymbol{p})}{m_\mathrm{e}}.\int_0^{t'} \boldsymbol{A}(\tau)\,\mathrm{d}\tau\right)$$

$$\times\int \mathrm{d}\boldsymbol{r}\,V(\boldsymbol{r})\exp\{\tfrac{1}{2}\pi(\eta+\eta')\}\,\Gamma(1-\mathrm{i}\eta)\,\Gamma(1+\mathrm{i}\eta')\exp\{\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}').\boldsymbol{r}\}$$

$$\times\mathrm{F}\big(\mathrm{i}\eta, 1, \mathrm{i}(pr - \boldsymbol{p}.\boldsymbol{r})\big)\,\mathrm{F}\big(\mathrm{i}\eta', 1, \mathrm{i}(p'r + \boldsymbol{p}'.\boldsymbol{r})\big).$$

The quantity under the integral with respect to t' contains a periodic function of time which can be expanded in a Fourier series:

$$C_{pp'}(t) = -\sum_n\left\{\exp\left(\frac{\mathrm{i}(\Delta\varepsilon+n\hbar\omega)t}{\hbar}\right)-1\right\}\frac{1}{\Delta\varepsilon+n\hbar\omega}\mathrm{J}_n\left(\frac{eE(\boldsymbol{p}-\boldsymbol{p}')}{m_\mathrm{e}\hbar\omega^2}\right)\int \mathrm{d}\boldsymbol{r}\;V(\boldsymbol{r})\;\times\mathrm{T}(\boldsymbol{r}),$$

where $\mathrm{T}(\boldsymbol{r})$ denotes $\boldsymbol{r}$-dependent terms and $\Delta\varepsilon$ is as defined above.

In the calculation of the transition probability per unit time, we have

$$\omega_{pp'} = \lim_{t\to\infty}(|C_{pp'}(t)|^2/t),$$

which leads to a double summation. However, as $t \to \infty$ the crossing terms of the sum ($n \neq n'$) make no contribution and the diagonal terms ($n = n'$) lead to the energy δ function

$$\omega_{pp'} = \sum_{n=-\infty}^{\infty}\omega^{(n)}_{pp'}\,\delta(\Delta\varepsilon + n\hbar\omega).$$

The individual terms of the sum are interpreted as the probabilities of emission or absorption ($n \gtrless 0$) of n photons because of the presence of the δ function. The probability of absorption of n photons per unit time is given by

$$\omega_{pp'}^{(n)} = (2\pi/\hbar)\,\delta(\Delta\varepsilon + n\hbar\omega)\,\mathrm{J}_n^2(eE(\boldsymbol{p}-\boldsymbol{p}')/m_e\hbar\omega^2)$$

$$\times \left| \int \mathrm{d}\boldsymbol{r}\; V(\boldsymbol{r}) \exp\{\tfrac{1}{2}\pi\eta(1+\lambda^{-1})\}\,\Gamma(1-\mathrm{i}\eta)\,\Gamma(1+\mathrm{i}\eta'\lambda)\right.$$

$$\left.\times \exp\{\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{r}\}\,\mathrm{F}(\mathrm{i}\eta, 1, \mathrm{i}(pr-\boldsymbol{p}\cdot\boldsymbol{r}))\,\mathrm{F}(\mathrm{i}\eta\lambda^{-1}, 1, \mathrm{i}(p'\lambda r-\boldsymbol{p}'\cdot\boldsymbol{r}))\right|^2,$$

where

$$\lambda = (1 + 2n\hbar\omega/m_e V^2)^{\frac{1}{2}}.$$

The transition probability should be summed over the final states of the electron. The δ function is removed by integrating over

$$p'^2\,\mathrm{d}p'\,\mathrm{d}o' = \tfrac{1}{2}p'\,\mathrm{d}(p'^2)\,\mathrm{d}o'.$$

Carrying out the integration and replacing the scattering probability by the scattering cross section, we then obtain

$$\sigma_{pp'}^{(n)} = \int \mathrm{d}o\, \frac{m_e^2 Z^2 e^4}{\hbar^4}\,\mathrm{J}_n^2(\gamma(\hat{\boldsymbol{n}}-\lambda\hat{\boldsymbol{n}}')\cdot\boldsymbol{\varepsilon})\,\frac{4\pi\eta^2}{\lambda\{\exp(2\pi\eta)-1\}\{1-\exp(-2\pi\eta\lambda^{-1})\}}$$

$$\times \left| \int \frac{\mathrm{d}\boldsymbol{r}}{r} \exp\{\mathrm{i}(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{r}\}\,\mathrm{F}(\mathrm{i}\eta, 1, \mathrm{i}(pr-\boldsymbol{p}\cdot\boldsymbol{r}))\,\mathrm{F}(\mathrm{i}\eta\lambda^{-1}, 1, \mathrm{i}(p'\lambda r-\boldsymbol{p}'\cdot\boldsymbol{r}))\right|^2.$$

In the argument of the Bessel function here, $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$ are unit vectors in the directions of $\boldsymbol{p}$ and $\boldsymbol{p}'$, $\boldsymbol{\varepsilon}$ is the unit vector in the direction of polarization of the electromagnetic field and $\gamma = eVE/\hbar\omega^2$.

For absorption, the quantity required is the absorption coefficient $\sigma_a^{(n)}$, which is the number of photons absorbed per unit volume, $n\rho V\sigma_{pp'}^{(n)}$, where ρ is the electron number density and the factor n accounts for the fact that n photons are absorbed in each electron collision (Pert 1972). The r integral in the expression for $\sigma_{pp'}^{(n)}$ is evaluated by putting the confluent hypergeometric functions in contour integral form (Nordsieck 1954; Landau and Lifshitz 1958; Gradshteyn and Ryznik 1965). We finally obtain

$$\sigma_a^{(n)} = \frac{8\pi^2\rho Z^2 e^8 \omega n}{\hbar E^2 c m_e^2 V^5} \int \frac{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}{\{\exp(2\pi\eta)-1\}\{1-\exp(-2\pi\eta\lambda^{-1})\}}$$

$$\times \frac{\mathrm{J}_n^2(\gamma\{(1-\lambda\cos\theta)\cos\theta_0 - \lambda\sin\theta_0\cos\phi\})}{\lambda^2\{\cos\theta-(1+\lambda^2)/2\lambda\}^2}$$

$$\times \left|\mathrm{F}(\mathrm{i}\eta\lambda^{-1}, \mathrm{i}\eta, 1, -2\lambda(1-\cos\theta)/(1-\lambda)^2)\right|^2. \tag{1}$$

Approximations

For small fields, that is, $\gamma = eVE/\hbar\omega^2 \ll 1$, only the first terms in the expansion of the Bessel function in equation (1) are required (Gradshteyn and Ryznik 1965, equation 8.440), and we get

$$\sigma_{\mathrm{a}}^{(n)} = \frac{8\pi^2\rho Z^4 e^8 \omega n}{\hbar E^2 c m_{\mathrm{e}}^2 V^5} \frac{1}{\{\exp(2\pi\eta)-1\}\{1-\exp(-2\pi\eta\lambda^{-1})\}}$$

$$\times \int \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, (\tfrac{1}{2}\gamma)^{2n} \frac{\{(1-\lambda\cos\theta)\cos\theta_0 - \lambda\sin\theta_0\sin\theta\cos\phi\}^{2n}}{\lambda^2\{\cos\theta - (1+\lambda^2)/2\lambda\}^2}$$

$$\times |\mathrm{F}(\mathrm{i}\eta\lambda^{-1}, \mathrm{i}\eta, 1, 2\lambda(\cos\theta-1)/(1-\lambda)^2)|^2. \tag{2}$$

Comparison of equation (2) with the expression obtained by Bunkin and Federov (1966) shows that the constant terms are identical when the exponentials $\exp(2\pi\eta)$ are approximated as $1+2\pi Ze^2/\hbar V$. This condition is valid when $\eta = Ze^2/\hbar V \ll 1$, which is the Born approximation. The only term that is different is the hypergeometric function which is θ-dependent. However, when the quantity $\xi = \hbar\omega/mV^2$ is large, that is, $\lambda \approx (2n\xi)^{\frac{1}{2}}$, this θ-dependency disappears and the results are identical with the expressions of Bunkin and Federov multiplied by a factor $|\mathrm{F}(\mathrm{i}\eta\lambda^{-1}, \mathrm{i}\eta, 1)|^2$. Thus, for $\gamma \ll 1$ and $\xi \gg 1$, we have

$$\sigma_{\mathrm{a}}^{(n)} = \frac{8\pi^2\rho Z^4 e^8 \omega n}{\hbar E^2 c m_{\mathrm{e}}^2 V^5} (\tfrac{1}{2}\gamma)^{2n} |\mathrm{F}(\mathrm{i}\eta\lambda^{-1}, \mathrm{i}\eta, 1)|^2 \int \mathrm{d}\theta \sin\theta \times \mathrm{T}(\theta),$$

where the $\mathrm{T}(\theta)$ are θ-dependent terms. We finally obtain

$$\sigma_{\mathrm{a}}^{(n)} = \frac{8\pi^2\rho Z^4 e^8 \omega n}{\hbar E^2 c m_{\mathrm{e}}^2 V^5} (\tfrac{1}{2}\gamma)^{2n} (2n\xi)^{2n-4} |\mathrm{F}(\mathrm{i}\eta\lambda^{-1}, \mathrm{i}\eta, 1)|^2$$

$$\times \frac{4\pi}{(2n+1)\{\exp(2\pi\eta)-1\}\{1-\exp(-2\pi\eta\lambda^{-1})\}}$$

$$= \frac{2\pi^3 Z^4 e^8 n}{(2n+1)(2n)^4 \hbar^5 \omega^3 E^2 c} \frac{(eEn/m_{\mathrm{e}}V\omega)^{2n}}{\{\exp(2\pi\eta)-1\}\{1-\exp(-2\pi\eta\lambda^{-1})\}}. \tag{3}$$

For large fields and slow electrons, that is, $\gamma \gg 1$ and $\xi \gg 1$, the asymptotic value of the Bessel function is used in equation (1), with the result

$$\sigma_{\mathrm{a}}^{(n)} = \frac{8\pi^2\rho Z^4 e^8 \omega n}{E^2 c \hbar m_{\mathrm{e}}^2 V^5} \frac{\mathrm{F}(\mathrm{i}\eta/2n\xi, \mathrm{i}\eta, 1)}{\{\exp(2\pi\eta)-1\}\{1-\exp(-2\pi\eta/2n\xi)\}} \frac{2\ln 2}{\gamma(2n\xi)^2}.$$

Thus it is seen that the results of the present calculation differ from that obtained using perturbation theory which predicts $\sigma \sim I^n$, where I is the laser intensity and n the number of quanta absorbed or emitted.

Acknowledgment

We are grateful to Professor S. N. Biswas for encouragement and stimulating discussions during the course of this work.

References

Akhizer, A. I., and Berestetskii, V. B. (1965). 'Quantum Electrodynamics', Sections 29.1, 29.8 (Interscience: New York).
Bunkin, F. V., and Federov, M. V. (1966). *Sov. Phys. JETP* **22**, 844–7.
Faisal, F. H. M. (1973). *J. Phys.* B **6**, L313.
Geltman, S., and Teague, M. R. (1974). *J. Phys.* B **7**, L22–7.
Gradshteyn, I. S., and Ryznik, I. M. (1965). 'Table of Integrals, Series, and Products', Equations 7.622, 8.440 (Academic: New York).
Landau, L. D., and Lifshitz, E. M. (1958). 'Quantum Mechanics' (Pergamon: London–Paris).
Nordsieck, A. (1954). *Phys. Rev.* **93**, 785.
Pert, G. J. (1972). *J. Phys.* A **5**, 1221–32.

Manuscript received 22 December 1975