Aust. J. Phys., 1977, **30**, 109–11

Plane Symmetric Solutions of a Scalar–Tensor Theory of Gravitation

D. R. K. Reddy

Department of Applied Mathematics, Andhra University, Waltair, India.

Abstract

Plane symmetric solutions of a scalar–tensor theory proposed by Dunn have been obtained. These solutions are observed to be similar to the plane symmetric solutions of the field equations corresponding to zero mass scalar fields obtained by Patel. It is found that the empty space–times of general relativity discussed by Taub and by Bera are obtained as special cases.

Recently Dunn (1974) has constructed a scalar–tensor theory of gravitation using a non-Riemannian geometry in which both the metric tensor and the scalar function have an unambiguous geometric interpretation. The scalar is introduced by defining a linear connection with non-vanishing torsion. It is pointed out that the field equations of the theory and the Lagrangian from which they are derived are identical with those given by Dicke (1962) in an alternate formulation of the Brans–Dicke (1961) theory. By using a static spherically symmetric solution to the field equations Dunn (1974) has also found that, with a proper choice of parameter, this theory agrees with experimental results in the three classical tests of redshift, light deflection and perihelion advance. Singh (1975) has investigated cylindrically symmetric solutions of the field equations of this theory for the Einstein–Rosen metric and has given a method by which one can obtain, under certain conditions, solutions of the scalar–tensor theory from known solutions of Einstein's theory of gravitation.

In the present paper we investigate plane symmetric solutions of the field equations of the scalar–tensor theory proposed by Dunn (1974). We find that these solutions are similar to the plane symmetric solutions of zero mass meson fields obtained by Patel (1975) and that, under special conditions, they reduce to the solutions of Taub (1951) and Bera (1969) in Einstein's theory.

Field Equations

In the regions of space–time with zero charge and zero mass densities, the field equations of the scalar–tensor theory of gravitation formulated by Dunn (1974) are

$$R_{ij} - \tfrac{1}{2}g_{ij}R = 6K^2\lambda^{-2}(\lambda_{,i}\lambda_{,j} - \tfrac{1}{2}g_{ij}\lambda_{,s}\lambda^{,s}), \tag{1}$$

$$\partial\{\lambda_{,j}(-g)^{\frac{1}{2}}g^{js}\}/\partial x^s - (\lambda_{,s}\lambda_{,i}/\lambda)g^{si}(-g)^{\frac{1}{2}} = 0, \tag{2}$$

where R_{ij} is the Ricci tensor, R the curvature scalar of the metric g_{ij}, λ a scalar field and K a constant. If either $K = 0$ or $\lambda = \text{const.}$, the connection of the space–time is metric-preserving and torsion-free, i.e. we have a Riemannian geometry.

Static Solution

Let us consider a space–time whose geometry is described by the plane symmetric line element

$$ds^2 = e^{2\alpha}(dt^2 - dx^2) - e^{2\beta}(dy^2 + dz^2), \tag{3}$$

where α and β are functions of x only. The static plane symmetry assumed obviously implies that $\lambda_{,2} = \lambda_{,3} = \lambda_{,4} = 0$, that is, λ is a function of x only. With the metric (3), the field equations (1) and (2) reduce to

$$\beta_1^2 + 2\alpha_1\beta_1 = -3K^2\phi_1^2, \qquad 2\beta_{11} + 3\beta_1^2 - 2\alpha_1\beta_1 = 3K^2\phi_1^2, \tag{4a, b}$$

$$\beta_{11} + \beta_1^2 + \alpha_{11} = 3K^2\phi_1^2, \qquad \phi_{11} + 2\beta_1\phi_1 = 0, \tag{4c, d}$$

where we have put $\lambda = e^{\phi}$, ϕ being a scalar function, and the subscripts 1 denote differentiation with respect to x.

The general vacuum solution, for $K \neq 0$, of the field equations (4) is given by

$$\alpha = (c_3/c_1)\ln(c_1 x + c_2) + c_4, \qquad 2\beta = \ln(c_1 x + c_2), \tag{5a, b}$$

with

$$\lambda = \lambda_0(c_1 x + c_2)^{c_5/c_1}, \tag{5c}$$

where λ_0 and the c_i's are constants of integration and c_1, c_3 and c_5 are related by

$$c_1^2 + 4c_1 c_3 = -12K^2c_5^2, \qquad c_1 \neq 0. \tag{6}$$

If $c_5 = 0$ then the scalar field λ is equal to the constant λ_0 and the field equations (4) reduce to Einstein's field equations, giving $c_1 + 4c_3 = 0$. In this case the functions α and β are given by

$$e^{\alpha} = (c_1 x + c_2)^{-\frac{1}{4}}, \qquad e^{2\beta} = c_1 x + c_2, \tag{7a, b}$$

with

$$\lambda = \lambda_0 = \text{const.} \tag{7c}$$

The metric (3), with α and β given by equations (7), describes the empty space–time discussed by Taub (1951). Thus equations (3) and (5) along with (6) constitute a static plane symmetric solution of the scalar–tensor theory proposed by Dunn (1974). This solution is formally similar to the static plane symmetric solution of the field equations for zero mass meson fields obtained by Patel (1975).

Nonstatic Solution

Consider a Riemannian space–time described by the line element

$$ds^2 = e^{2h}(dt^2 - dr^2 - r^2\, d\Phi^2 - S^2\, dz^2), \tag{8}$$

where r, Φ, z are the usual cylindrical polar coordinates and h and S are functions of time t alone. It is well known that this line element is plane symmetric. Taking λ as a function of t only, again putting $\lambda = e^{\phi}$ and using the metric (8), we find that the field equations (1) and (2) in this case are given by

$$\ddot{S}+M = 3K^2\dot{\phi}^2, \qquad 2(\ddot{h}-\dot{h}^2)-M = 3K^2\dot{\phi}^2, \tag{9a, b}$$

$$2\dot{S}\dot{h}/S - M = 3K^2\dot{\phi}^2, \qquad \ddot{\phi}+\dot{\phi}(\dot{S}/S+2\dot{h}) = 0, \tag{9c, d}$$

where

$$M = 2\ddot{h}+\dot{h}^2+2\dot{S}\dot{h}/S$$

and an overhead dot denotes differentiation with respect to t.

The solution of the differential equations (9), for $K \neq 0$, can be expressed as

$$S = A\exp\{(c_1/c_2)h\}, \qquad \exp\{(2+c_1/c_2)h\} = (2+c_1/c_2)(c_2 A^{-1}t+c_3), \tag{10a, b}$$

with

$$\lambda = \lambda_0 \exp\{(c_3/c_2)h\}, \tag{10c}$$

where A, λ_0 and the c_i's are integration constants and

$$3c_2^2+2c_1c_2 = -3K^2c_3^2, \qquad c_3, A \neq 0. \tag{11}$$

If $c_3 = 0$ then λ is the constant λ_0. In this particular case we have $3c_2+2c_1 = 0$ and consequently

$$S = A\exp(-3h/2), \qquad \exp(\tfrac{1}{2}h) = \tfrac{1}{2}(c_2 A^{-1}t+c_3), \qquad \lambda = \lambda_0. \tag{12}$$

The metric (8), with S and h given by equations (12), describes the nonstatic empty space–time discussed by Bera (1969). Thus equations (8) and (10) along with (11) constitute a nonstatic plane symmetric solution of the field equations of the scalar–tensor theory proposed by Dunn (1974). This solution is also formally similar to the nonstatic plane symmetric solutions of the field equations for zero rest-mass scalar fields obtained by Patel (1975).

References

Bera, K. (1969). *J. Phys.* A **2**, 138.
Brans, C., and Dicke, R. H. (1961). *Phys. Rev.* **124**, 925.
Dicke, R. H. (1962). *Phys. Rev.* **125**, 2163.
Dunn, K. A. (1974). *J. Math. Phys.* **15**, 2229.
Patel, L. K. (1975). *Tensor* **29**, 237.
Singh, T. (1975). *J. Math. Phys.* **16**, 2517.
Taub, A. H. (1951). *Ann. Math.* **53**, 472.

Manuscript received 13 July 1976