Nuclear Structure of ^{12}C from an α-particle Model

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Abstract

The isoscalar levels with $J^\pi = 0^+$ of the ^{12}C nucleus have been investigated by finite-group theoretical methods using an α-particle model. An estimate has been made of the restoring force parameter in the potential between two α particles from symmetry.

The α-particle model is particularly suitable for description of the low energy levels of nuclei which are composed of a whole number of α particles, e.g. ^{12}C, ^{16}O etc. Iachello and Arima (1975) have described the entire collective spectra of vibrational nuclei in terms of a few interacting elementary excitation modes, while Bergholtz (1975) has used the α-particle model of Block and Brink to study the importance of vibrations and polarizations of clusters in the low lying states of ^{12}C and ^{16}O. In this note, we discuss a (finite) group theoretical approach to the calculation of the vibrational frequencies of transitions between the energy levels of ^{12}C.

Let us consider the ^{12}C nucleus to be composed of three elementary α particles, each of mass m placed at the vertices of an equilateral triangle. We will now investigate the different normal modes of vibration of this triangle. Representing the configuration of the system by a six-dimensional state vector $\boldsymbol{\rho}$, we have the kinetic and potential energies of the system given by

$$T = \tfrac{1}{2}m\sum_i \dot{\rho}_i^2, \qquad V = \tfrac{1}{2}K\sum_{ij} V_{ij}\rho_i\rho_j,$$

where K is the constant restitutive force. The equation of motion is

$$m\ddot{\rho}_i = -\partial V/\partial\rho_i = -K\sum_j V_{ij}\rho_j.$$

For vibration in a normal mode we have

$$\sum_j V_{ij}\rho_j = \varepsilon\rho_i \qquad \text{with} \qquad \varepsilon = m\omega^2/K,$$

ω being the classical angular frequency of vibration. The normal modes are known to be the eigenvectors of the matrix V, the eigenvalues giving the frequencies. For our symmetry group we have the six elements I, R, R^2, P, PR and PR^2, where I is the identity matrix, R rotates the triangle by 120° in a positive sense and P reflects it about a vertical line through the centre. These elements bring about linear trans-

formations of the ρ_i when they act on the triangle. Thus if R operates on the system we have $\boldsymbol{\rho}' = \boldsymbol{D}(R)\boldsymbol{\rho}$, where the matrix $\boldsymbol{D}(R)$ is

$$\boldsymbol{D}(R) = \begin{bmatrix} 0 & 0 & \beta \\ \beta & 0 & 0 \\ 0 & \beta & 0 \end{bmatrix} \quad \text{with} \quad \beta = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}.$$

Similarly

$$\boldsymbol{D}(P) = \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & \gamma \end{bmatrix} \quad \text{with} \quad \gamma = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since the symmetry group of the equilateral triangle can be decomposed into three classes of equivalent elements and the number of representations equals the number of classes, we have

$$\boldsymbol{D} = \boldsymbol{D}^1 \oplus \boldsymbol{D}^2 \oplus 2\boldsymbol{D}^3,$$

the characters of the corresponding classes being given by $\chi = 6, 0$ and 0 respectively. Therefore, in a coordinate system in which $\boldsymbol{V}$ is diagonalized, we get

$$\boldsymbol{V} = \begin{bmatrix} \varepsilon_1 & & & & & \\ & \varepsilon_2 & & & & \\ & & \varepsilon_{31} & & & \\ & & & \varepsilon_{31} & & \\ & & & & \varepsilon_{32} & \\ & & & & & \varepsilon_{32} \end{bmatrix} \begin{matrix} D^1 \\ D^2 \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} D^3 \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} D^3 \end{matrix}.$$

Following standard procedure, we then obtain

$$\mathrm{Tr}\,\boldsymbol{D}(I)\,\boldsymbol{V} = 6, \qquad \mathrm{Tr}\,\boldsymbol{D}(R)\,\boldsymbol{V} = \tfrac{3}{2}, \qquad \mathrm{Tr}\,\boldsymbol{D}(P)\,\boldsymbol{V} = 3.$$

Hence the eigenvalues obey the equations

$$\begin{aligned} \varepsilon_1 + \varepsilon_2 + 2(\varepsilon_{31} + \varepsilon_{32}) &= 6, \\ \varepsilon_1 + \varepsilon_2 - (\varepsilon_{31} + \varepsilon_{32}) &= \tfrac{3}{2}, \\ \varepsilon_1 - \varepsilon_2 &= 3. \end{aligned}$$

Thus $\varepsilon_1 = 3$, $\varepsilon_2 = 0$ and $\varepsilon_{31} + \varepsilon_{32} = \frac{3}{2}$. To calculate ε_{31} and ε_{32} explicitly, we note that there must be three degrees of freedom having a zero eigenvalue, two translational and one rotational. Therefore we have $\varepsilon_{31} = 0$ and $\varepsilon_{32} = \frac{3}{2}$. It follows that the zero eigenvalues ε_2 and ε_{31} correspond to translations and rotations of the system as a whole. The eigenvectors describing the true vibrational modes are orthogonal to these and to each other. This orthogonality dictates that in a vibrational mode the centre of mass is stationary and the angular momentum is zero.

As an alternative geometry to the equilateral triangle, we can assume a reduced symmetry in which there is a linear clustering of the three α particles in ^{12}C. In this case it is easy to calculate that the two nonzero frequencies become equal to $(3K/m)^{\frac{1}{2}}$.

Let us consider now an interpretation of the eigenvalues obtained for the two geometries investigated. The ground state of ^{12}C and the two excited states at 7·65 and 10·3 MeV have zero isospins and angular momenta, while the level at 17·77 MeV has $J^{\pi} = 0^{+}$ but an isospin of 1, and it is of interest to see whether the three lowest isoscalar 0^{+} levels can be reproduced in our models. For the triangular symmetry, the nonzero eigenvalue $\varepsilon_1 = 3$ may be made to correspond to the frequency of vibrational transition or the relative spacing between the ground state and the excited state at 10·3 MeV, that is, $\hbar(3K/m)^{\frac{1}{2}} = 10{\cdot}3$ MeV or $K \approx 0{\cdot}02$ fm^{-3}. The remaining frequency $(3K/2m)^{\frac{1}{2}}$ predicts a 0^{+} state at $10{\cdot}3/\sqrt{2} = 7{\cdot}28$ MeV, which is close to the experimental level at 7·65 MeV. Since K is the restitutive force constant in the harmonic oscillator potential between any two α particles each of mass m, it seems reasonable to assume the same value for K for both the triangular and linear configurations. Therefore the two degenerate frequencies $(3K/m)^{\frac{1}{2}}$ in the linear geometry may be associated with a vibrational transition or an energy gap between the ground state and the level at 10·3 MeV. That is, although we have two very different geometries that give the correct energy for the 10·3 MeV state, the linear model fails to reproduce the 0^{+} state at 7·28 MeV. On the other hand, the equilateral triangle model seems to reproduce fairly well the low lying 0^{+} isoscalar levels in ^{12}C.

In summary, using an α-particle model of the ^{12}C nucleus we have derived the relative values of the nonzero frequencies of vibrational transitions through pure group theoretical considerations of the geometrical symmetry. With a frequency scaling factor of $(K/m)^{\frac{1}{2}}$, determined by the dynamics of the system, we have been able to predict satisfactorily the restitutive force constant in a harmonic oscillator potential between two α particles.

References

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