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Energy Principle for Compressible Dissipative Magnetoplasma

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Abstract

Qualitative evaluation of the influence of dissipation on plasma stability follows from a generalized energy principle.

Ideal hydromagnetic energy principles have been exploited to define stability criteria in complicated geometry (e.g. toroidal) since the classical publication by Bernstein *et al.* (1958), in well-established analytical tradition for ideal conservative dynamical systems. However, it is possible to include dissipation (Furth *et al.* 1963; Barston 1970; Tasso 1977), and in this note the formalism for a compressible magneto-plasma with viscosity and resistivity is outlined.

The plasma has volume V bounded by surface S, and perturbation of any magneto-hydrostatic equilibrium is presumed to be governed by the linearized system

$$\partial\rho_1/\partial t + \nabla . (\rho_0 \boldsymbol{v}_1) = 0, \tag{1a}$$

$$\rho_0 \partial\boldsymbol{v}_1/\partial t + \nabla p_1 + \nabla . \mathbf{t} = \mu_0^{-1}\{(\nabla\times\boldsymbol{B}_1)\times\boldsymbol{B}_0 + (\nabla\times\boldsymbol{B}_0)\times\boldsymbol{B}_1\} + \rho_1\boldsymbol{g}, \tag{1b}$$

$$\partial\boldsymbol{B}_1/\partial t = \nabla\times(\boldsymbol{v}_1\times\boldsymbol{B}_0) - \nabla\times\{\eta_0(\nabla\times\boldsymbol{B}_1) + \eta_1(\nabla\times\boldsymbol{B}_0)\}, \tag{1c}$$

$$\partial p_1/\partial t + \boldsymbol{v}_1 . \nabla p_0 + \gamma p_0 \nabla . \boldsymbol{v}_1 = 0, \tag{1d}$$

$$\partial\eta_1/\partial t + \boldsymbol{v}_1 . \nabla\eta_0 = 0, \tag{1e}$$

where $\mathbf{t}$ is the nonhydrostatic stress tensor (Hosking and Marinoff 1973) and zero and unit subscripts denote equilibrium and perturbation quantities respectively. Also here, η is the resistivity, which is assumed to be convective, μ_0 is the permeability, $\boldsymbol{g}$ denotes gravity, and the other notation is as used by Bernstein *et al.* (1958).

Let us introduce Lagrangian displacement and magnetic vectors defined by

$$\partial\boldsymbol{\xi}/\partial t = \boldsymbol{v}_1(\boldsymbol{r}_0, t), \qquad \partial\boldsymbol{R}/\partial t = \boldsymbol{B}_1(\boldsymbol{r}_0, t),$$

and observe that no distinction between Lagrangian and Eulerian variables need be made for small displacements. Integration of equations (1a), (1d) and (1e) with respect to time, followed by elimination of ρ_1, p_1 and η_1 from equation (1b) and the integral of (1c), eventually yields the reduced system

$$P\ddot{\underline{\xi}} + K\dot{\underline{\xi}} + H\underline{\xi} = 0, \tag{2}$$

where the six-vector $\underline{\xi}$ now introduced is given by

$$\underline{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi \\ \boldsymbol{R} \end{bmatrix},$$

and the coefficient matrices are

$$P = \begin{bmatrix} \rho_0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad K = \begin{bmatrix} L_0 & 0 \\ 0 & L_2 \end{bmatrix}, \qquad H = \begin{bmatrix} L_1 L_3 + L_4 & -L_1 L_2 \\ -L_2 L_3 & L_2^2 \end{bmatrix},$$

with the implicit linear operators

$$\begin{aligned}
L_0 \dot{\xi} &\equiv \nabla \cdot \mathbf{t}(\dot{\xi}), \\
L_1 \dot{\boldsymbol{R}} &\equiv \mu_0^{-1}\{\boldsymbol{B}_0 \times (\nabla \times \dot{\boldsymbol{R}}) + \dot{\boldsymbol{R}} \times (\nabla \times \boldsymbol{B}_0)\}, \\
L_2 \boldsymbol{R} &\equiv \nabla \times (\eta_0 \nabla \times \boldsymbol{R}), \\
L_3 \xi &\equiv \nabla \times \{(\xi \times \boldsymbol{B}_0) + (\nabla \times \boldsymbol{B}_0)\xi \cdot \nabla \eta_0\}, \\
L_4 \xi &\equiv \boldsymbol{g} \nabla \cdot (\rho_0 \xi) - \nabla(\gamma p_0 \nabla \cdot \xi + \xi \cdot \nabla p_0).
\end{aligned}$$

Equation (2) is a generalized form of equation for a dissipative system, with the viscosity and resistivity represented by the coefficient matrix K, and the resistivity also rendering the otherwise ideal hydromagnetic coefficient matrix H nondiagonal. The perturbation forcing function could also be included on the right-hand side of equation (2) (cf. Barston 1970), but it plays no part in the subsequent analysis. If resistivity is omitted we may return to the familiar formalism (Bernstein *et al.* 1958), perhaps modified to allow for viscosity.

We define an inner product over the solution space by

$$(\underline{\xi}, \underline{\eta}) = \int_V \xi_1^* \cdot \eta_1 \, d\tau + \int_V \xi_2^* \cdot \eta_2 \, d\tau,$$

where the asterisk denotes a complex conjugate and the integration is over the plasma volume V, with the vector elements satisfying appropriate conditions on the boundary S. We may suppose that S is a perfectly conducting rigid surface, for which ξ_1 $(= \xi)$ vanishes everywhere, and ξ_2 $(= \boldsymbol{R})$ vanishes on all parts of S at infinity while $\xi_2 \cdot \boldsymbol{n}$ vanishes on the remainder of S ($\boldsymbol{n}$ denoting a unit normal on S).

Provided the matrix operator H is self-adjoint, the inner product of $\dot{\underline{\xi}}$ with equation (2) gives

$$\tfrac{1}{2}\{(\dot{\underline{\xi}}, P\dot{\underline{\xi}}) + (\underline{\xi}, H\underline{\xi})\}^{\cdot} = -(\dot{\underline{\xi}}, K\dot{\underline{\xi}}), \tag{3}$$

so that if the right-hand side is not positive the condition

$$(\underline{\xi}, H\underline{\xi}) \geqslant 0 \tag{4}$$

is necessary and sufficient for stability (Tasso 1977). For a magnetoplasma, parallel viscosity is normally dominant (Hosking and Marinoff 1973) so that

$$
\begin{aligned}
(\underset{-}{\xi}, K\underset{-}{\xi}) &= \int_V \xi_1^* \cdot L_0 \xi_1 \, \mathrm{d}\tau + \int_V \xi_2^* \cdot L_2 \xi_2 \, \mathrm{d}\tau \\
&= \int_V \mu_{\|} |(\boldsymbol{bb} - \tfrac{1}{3}\mathbf{I}) : \nabla\xi|^2 \, \mathrm{d}\tau + \int_V \eta_0 |\nabla \times \boldsymbol{R}|^2 \, \mathrm{d}\tau \\
&\geqslant 0,
\end{aligned}
$$

where $\boldsymbol{b} = \boldsymbol{B}_0/|\boldsymbol{B}_0|$, $\mathbf{I}$ is the unit dyadic and $\mu_{\|}$ is the parallel viscous coefficient. Indeed, magnetoviscosity is generally stabilizing if H is self-adjoint, since it can be shown that the matrix K is positive definite for the closed form of the nonhydrostatic stress derived by Liley (1972; see also Hosking and Marinoff 1973). With respect to the stability criterion (4), one may observe that resistive tearing and magnetic interchange instabilities are driven by terms in the linear operator L_1, rippling by a term in L_3 and gravitational interchange by a term in L_4, while plasma compressibility is apparently also generally stabilizing.

In the derivation of equation (3) and the criterion (4) it was essential that H be self-adjoint but no reference to normal modes was made, and it is possible to obtain estimates of growth rates by variational methods. On the other hand, for detailed knowledge of *any* perturbation spectrum it is common to invoke normal mode analysis from the outset.

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