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The Axially Symmetric Stationary Vacuum Field Equations in Einstein's Theory of General Relativity

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Abstract

All formulations of Einstein's vacuum field equations for which exact solutions have been found in the axially symmetric stationary case are compared, and the inherent restrictions of each are displayed. A measure of their usefulness as theoretical tools is gained by the ease with which they admit Kerr's (1963) solution (it being the simplest asymptotically flat metric of this kind). The solutions found using each formulation are listed and, where possible, suitably classified.

1. Introduction

Exact solutions of Einstein's field equations in general relativity are as yet very rare, and not all of those that have been found are applicable to real physical situations. Before a solution can be found certain simplifying assumptions have to be made, or the problem would remain intractable. The assumptions under consideration in the present paper are those of axial symmetry and stationary character. Even within this seemingly narrow domain, there exists a quite arbitrary choice of form for the metric functions which are to be found on solution of the field equations. It is this freedom of choice that leads to so many different forms of the field equations appearing in the literature. The present paper sets out to demonstrate the unity within this choice: that despite the apparently completely-general freedom, the various forms of the field equations all possess an essentially simple solution (although this may only be obtained in some cases by complicated coordinate transformations).

In Section 2 the general form of the stationary axially symmetric metric is derived, and in Section 3 the various formulations of the field equations are discussed at length. A convenient yardstick is applied to each to assess its usefulness for generating further solutions. This yardstick is the ease with which the formulation admits Kerr's (1963) solution, so chosen because it is the simplest known asymptotically flat metric that can represent a finite bounded source or a black hole. A brief mention is made of the solutions that have arisen from each formulation and, if the formulation provides a suitable classifying feature, a classification of the vacuum solutions is given.

2. General Form of Axially Symmetric Stationary Line Element

Derivation

In the general line element

$$ds^2 = g_{ab}\,dx^a\,dx^b, \tag{1}$$

the metric tensor g_{ab} has 10 independent components. Imposition of Einstein's field equations

$$R_{ab} = 0 \tag{2}$$

produces an under-determined problem. The further assumption of some form of symmetry renders the problem solvable (e.g. static spherical symmetry leaves two components of g_{ab} to be found). The assumption of stationary axially symmetric fields provides the basis for the present paper, and is shown below to leave five components undetermined. This assumption necessitates the metric being invariant under the following transformations (where x^0 is the timelike coordinate and x^3 is the coordinate of axial symmetry)

$$\text{stationary nature:} \qquad x^0 \to x^0 + c_1, \tag{3a}$$

$$\text{axial symmetry:} \qquad x^3 \to x^3 + c_2, \tag{3b}$$

$$\text{simultaneous reflection:} \qquad x^0 \to -x^0 \quad \text{and} \quad x^3 \to -x^3, \tag{3c}$$

where the last two reflections are performed simultaneously.

The transformations (3a) and (3b) are equivalent to the existence of a pair of commuting Killing vectors, which together with Killing's equations lead to the relations

$$g_{ab,0} = 0 \quad \text{and} \quad g_{ab,3} = 0, \tag{4a, b}$$

showing that the g_{ab} are functions of x^1 and x^2 only. The joint transformation (3c) limits the possible components of g_{ab} to the coefficients of $(\mathrm{d}x^0)^2$, $(\mathrm{d}x^3)^2$, $(\mathrm{d}x^0\,\mathrm{d}x^3)$, $(\mathrm{d}x^1)^2$, $(\mathrm{d}x^2)^2$ and $(\mathrm{d}x^1\,\mathrm{d}x^2)$ in equation (1). We make the additional assumption that the metric for the 2-space (x^1, x^2) can be diagonalized so that $g_{12} = 0$, and this gives us the specialized line element:

$$\mathrm{d}s^2 = g_{00}(\mathrm{d}x^0)^2 + 2g_{03}\,\mathrm{d}x^0\,\mathrm{d}x^3 + g_{33}(\mathrm{d}x^3)^2 + g_{11}(\mathrm{d}x^1)^2 + g_{22}(\mathrm{d}x^2)^2. \tag{5}$$

Equation (5) represents the most general form of the axially symmetric stationary line element to be considered here.

General Formulation

The line element (5) can be rewritten in the form

$$\begin{aligned}\mathrm{d}s^2 = {} & A(x^1, x^2)(\mathrm{d}x^0)^2 + B(x^1, x^2)(\mathrm{d}x^1)^2 + C(x^1, x^2)(\mathrm{d}x^2)^2 + D(x^1, x^2)(\mathrm{d}x^3)^2 \\ & + 2F(x^1, x^2)\,\mathrm{d}x^0\,\mathrm{d}x^3,\end{aligned} \tag{6}$$

with x^0 the timelike coordinate and x^3 the coordinate of axial symmetry. An explicit representation of the metric tensor associated with the line element (6) and expressions for the components of the Ricci tensor are set out in Appendix 1. As will be appreciated, these expressions do not readily lend themselves to direct solution.

As the line element (6) has the same form as (5) it will contain no hidden symmetries or constraints beyond those already made explicit, and so may be used for verification of the Kerr metric (though not for finding it initially). The generality of the form (6) virtually precludes the possibility of using it for derivation of new exact solutions, although this approach was used by Bach (1922) to give an approximate solution to the rotating body problem, which concurred with an expansion of the Kerr solution to first order. Also, because of its generality, the form (6) provides no feature that could be used as a tool for classifying the exact solutions concisely. We now turn to the specific forms of (6) that have been used to yield exact solutions.

3. Comparison Between Existing Formulations

(*a*) *Weyl–Lewis–Papapetrou form*

Line Element

Lewis (1932), by extending the form of the static line element used by Weyl (1917), proposed the following form for the stationary axially symmetric line element

$$ds^2 = f\,dx^0 - \{\exp\mu\,(dx^1)^2 + \exp\nu\,(dx^2)^2 + l(dx^3)^2\} - 2m\,dx^0\,dx^3\,. \tag{7}$$

After deriving expressions for the Ricci tensor he made two assumptions:

$$\nu = \mu \qquad \text{and} \qquad fl + m^2 = r^2\,, \tag{8a, b}$$

where $x^1 = r$ and $x^2 = z$. These give the so-called 'canonical coordinates' of the Weyl–Lewis–Papapetrou (or WLP) formulation, as used by Papapetrou (1953) and later in a modified form by Levy (1968) and others.

Restrictions of WLP Form

It was not until recently that the implications of Lewis's (1932) assumption (8a) could be fully realized. Chandrasekhar (1978: see subsection (*e*) below) managed to put the equations associated with the general formulation (6) in a form specifically designed to facilitate derivation of the Kerr metric. He showed that, if an event horizon is assumed to exist, it will be the same as that which occurs in Kerr's metric. If a similar analysis is done for Levy's line element (equation (9) below), which is a modified form of (7) also in canonical coordinates, it is seen that the only event horizon that may exist in WLP space is the origin. The implication of the other constraint (8b), which has been preserved more for historic than analytic reasons, is that it imposes a 'cylindrical' symmetry. This makes it extremely difficult to find 'spheroidal' solutions like those of Kerr and of Tomimatsu and Sato (1972, 1973) without resort to complicated coordinate transformations.

Derivation of Kerr Metric from WLP Form

The derivation of the Kerr metric is impossible in canonical coordinates without prior knowledge of the complicated transformations between the r and z of canonical coordinates and the R and θ of Boyer–Lindquist coordinates, in which Kerr's solution is normally written. This is mainly a consequence of the relation (8b). Canonical coordinates, while providing simplified field equations (see Appendix 2 for the field equations in Levy's formulation), do so at the expense of 'spheroidal' solutions such as Kerr's.

Solutions for WLP Form

The solutions found using the WLP formulation fall into two classes: those of Lewis and those of Papapetrou. The Lewis solutions (as well as those of Gürses and Güven 1975; see Cohen 1976) are taken to be fields exterior to infinite cylinders of matter, while the Papapetrou solutions are found to be massless if the space is assumed to be asymptotically flat and asymptotically flat if the space is assumed to be massless. The solution of Newman *et al.* (1963; the so-called Taub–NUT space) belongs to the Papapetrou class. Thus the derivation of solutions using the WLP canonical coordinates has proved to be difficult (if not impossible) beyond the case of cylindrical symmetry, to which these coordinates are best suited.

Classification of Solutions by WLP Form

The canonical coordinates do not in themselves provide a means for classifying the WLP solutions. However, it is shown in subsection (*b*) that the relations between a and u and their first partial derivatives in the Levy line element (equation (9) below) may be used, with limited success, for this purpose.

(b) Levy's form

Line Element

The modified form of the line element for canonical coordinates proposed by Levy (1968) was

$$\mathrm{d}s^2 = \exp(2u)(\mathrm{d}t + a\,\mathrm{d}\phi)^2 - \exp(2k-2u)(\mathrm{d}r^2 + \mathrm{d}z^2) - r^2\exp(-2u)\,\mathrm{d}\phi^2\,. \qquad (9)$$

This is still of the WLP form (7) and satisfies the relations (8a) and (8b), with $\nu = \mu = 2k-2u$. Only the form of the resulting field equations has been altered; not the functions on which they depend. In the canonical line element (9), the function k is found by quadrature when u and a are known, so that the essential problem is to find u and a. Levy's formulation depends on the two vectors $\mathbf{\Omega}$ and $\mathbf{\Lambda}$, which are defined in a naturally associated 3-space and which depend on u and a through

$$\mathbf{\Omega} = \exp(-G)\nabla a \qquad \text{and} \qquad \mathbf{\Lambda} = \nabla G\,, \qquad (10\text{a, b})$$

where $G = \ln(r-2u)$, while ∇ is defined relative to the flat-space metric

$$\mathrm{d}r^2 + \mathrm{d}z^2 + r^2\,\mathrm{d}\phi^2\,.$$

These give the field equations as

$$\nabla\times\mathbf{\Omega} = \mathbf{\Omega}\times\mathbf{\Lambda} \qquad \text{and} \qquad \nabla.\mathbf{\Omega} = \mathbf{\Omega}.\mathbf{\Lambda}\,, \qquad (11\text{a, b})$$

together with the integrability conditions

$$\nabla\times\mathbf{\Lambda} = \mathbf{0} \qquad \text{and} \qquad \nabla.\mathbf{\Lambda} = \Omega^2\,.$$

The restrictions in the Levy formulation are identical with those already described for the canonical coordinates.

Solutions for Levy's Form

The Kerr metric is shrouded even further in the Levy formulation, and although the field equations may now be written with an apparent simplicity (see Appendix 2) they do not lend themselves to solution easily. The only solutions to be found from this formulation are those due to Marek (1968) who assumed a specific form for $\mathbf{\Omega}$

(see Table 1) and found the corresponding $\mathbf{\Lambda}$. Marek's solutions proved to be unphysical and impossible to express in elementary functions.

Classification of Solutions by Levy's Form

The main use of the Levy formulation has been as a classification scheme but, due to the implicit cylindrical symmetry of the canonical coordinates, it has proved impossible to include the TS and Kerr solutions in this scheme. The classification, based on the $\mathbf{\Omega}, \mathbf{\Lambda}$ relation, is given in Table 1. The $\mathbf{\Omega}, \mathbf{\Lambda}$ relation for the Weyl solution is equivalent to putting $a = 0$ in the line element (9) or in the metric tensor (A8) given in Appendix 2. The $\mathbf{\Omega}, \mathbf{\Lambda}$ relations for the Lewis, Papapetrou and Marek solutions can also be written as relations between the first partial derivatives of u and a. The equivalent forms given in Table 1 show how the field equations have been further simplified for solution. The fact that no simple relation exists between the first partial derivatives of u and a for the Kerr solution compounds the difficulty of including it in the classification scheme. Although no other solution can be put in the appropriate form and the classification remains incomplete, it is useful because of its simplicity.

Table 1. Classification scheme based on Levy's formulation

Solution	$\mathbf{\Omega}, \mathbf{\Lambda}$ relation
Weyl (1917)	$\mathbf{\Omega} = \mathbf{0}$
Lewis (1932)	$\mathbf{\Omega} \times \mathbf{\Lambda} = \mathbf{0}$
Papapetrou (1953)	$\mathbf{\Omega} . \mathbf{\Lambda} = \Omega_r / r$
Marek (1968)	$\mathbf{\Omega} = f(r)\hat{\mathbf{r}} + g(r)\hat{\mathbf{z}}$

(c) *Matzner–Misner and Matzner–Nutku forms*

Line Elements

Matzner and Nutku (1971) put forward a formulation that is closely related to the canonical coordinates of WLP. They applied the transformation

$$u_1 = \rho^{-1}\exp(2u), \qquad v_1 = \rho\exp(-2u) - a^2\rho^{-1}\exp(2u), \qquad w_1 = -a\rho^{-1}\exp(2u) \tag{12a, b, c}$$

to Levy's line element (9), making the identification $\rho = r$, to obtain the following line element

$$\mathrm{d}s^2 = \rho(u\,\mathrm{d}t^2 - 2w\,\mathrm{d}t\,\mathrm{d}\phi - u\,\mathrm{d}\phi^2) - (\rho u)^{-1}\exp(-2\gamma)(\mathrm{d}\rho^2 + \mathrm{d}z^2). \tag{13}$$

The field equations corresponding to this line element are readily shown to be

$$\nabla^2 u_1 = \lambda u_1, \qquad \nabla^2 v_1 = \lambda v_1, \qquad \nabla^2 w_1 = \lambda w_1, \tag{14a, b, c}$$

together with

$$w_1^2 + u_1 v_1 = 1, \tag{14d}$$

where ∇^2 is a flat-space operator and λ is a Lagrange multiplier. In the earlier formulation of Matzner and Misner (1967), the following transformation was used instead of (12)

$$u_1 = \cos\alpha\cosh\beta + \sinh\beta, \qquad v_1 = \cos\alpha\cosh\beta - \sinh\beta, \qquad w_1 = \sin\alpha\cosh\beta. \tag{15a, b, c}$$

These relations automatically satisfy (14d) and produce field equations of the form

$$\nabla . \{\cosh^2(\beta)\nabla\alpha\} = 0, \qquad \nabla^2\beta + \tfrac{1}{2}\sinh(2\beta)(\nabla\alpha)^2 = 0, \tag{16a, b}$$

k being determined by quadrature in both cases. The Matzner–Misner and Matzner–Nutku formulations both suffer from the same drawbacks as the WLP form.

Solutions for the Matzner–Misner and Matzner–Nutku Forms

As is to be expected, the Kerr solution does not lend itself to precise derivation within the Matzner–Misner and Matzner–Nutku formulations. This may be seen from the form of the expression for λ given by Matzner and Nutku (1971):

$$\lambda = \frac{1}{\rho^2} - \frac{4m}{R^3}\left(1 - \frac{3a^2\cos^2\theta}{R^2}\right)\left(1 + \frac{a^2\cos^2\theta}{R^2}\right)^{-2}\left(1 - \frac{2m}{R} + \frac{m^2}{R^2}\sin^2\theta + \frac{a^2}{R^2}\cos^2\theta\right), \tag{17}$$

where

$$\rho^2 = (R^2 + a^2 - 2mR)\sin\theta, \qquad z = (R-m)\cos\theta.$$

It is highly unlikely that Kerr's solution would have been derived using these formulations. No other solutions have been found for these forms of the field equations. There does remain the possiblility that λ might be used as a classifying feature, although such a scheme would doubtless confront the same difficulties as Levy's.

(*d*) *Ernst's form*

Line Element and Field Equations

Ernst (1968) completely reformulated the problem by using a complex potential method. His form for the line element,

$$ds^2 = f^{-1}\{\exp(2\gamma)(dz^2 + d\rho^2) + \rho^2 d\phi^2\} - f(dt - w\,d\phi)^2, \tag{18}$$

is related to the canonical coordinates of WLP by the transformation

$$\rho = r, \qquad f = -\exp(2u), \qquad w = -a, \qquad \gamma = k. \tag{19}$$

He introduced the complex 'Ernst' potential ε, which is related to the field variables f and w by the simple expression

$$\varepsilon = f + i\phi, \tag{20}$$

where ϕ is related to w by

$$f^{-2}\nabla\phi = -\rho^{-1}\hat{\boldsymbol{n}} \times \nabla w, \tag{21}$$

with $\hat{\boldsymbol{n}}$ the unit vector in the ϕ direction. The transformation (20) leads to a field equation of the form

$$(\mathrm{Re}\,\varepsilon)\nabla^2\varepsilon = \nabla\varepsilon . \nabla\varepsilon, \tag{22}$$

which is a single equation in a complex unknown instead of a pair of equations in two real unknowns. Ernst noted a useful transformation of this equation, obtained by the following change of variable

$$\xi = (1+\varepsilon)/(1-\varepsilon), \tag{23}$$

which gives the field equation as

$$(\xi\xi^* - 1)\nabla^2\xi = 2\xi^*\nabla\xi . \nabla\xi . \tag{24}$$

Symmetries of Ernst's Form

Ernst's form (22) for the field equation is very concise, although it still suffers from the effects of having the coefficients of dr^2 and dz^2 equal, which makes some transformations easy to perform at the expense of making others extremely difficult. Equation (24) has a spheroidal character imparted by the specific form of the transformation used to obtain it and thus, while canonical coordinates are useful for cylindrical solutions, Ernst's formulation is more suited to spheroidal situations.

Kerr Metric in Ernst's Form

In Ernst's formulation the Kerr metric is represented by the following solution of equation (24) for ξ

$$\xi = px - iqy, \tag{25}$$

where $p^2 + q^2 = 1$, and x and y are prolate spheroidal coordinates related to cylindrical polar coordinates by the transformations:

$$\rho^2 = (x^2 - 1)(1 - y^2), \qquad z = xy. \tag{26a, b}$$

It is also useful to note that in Ernst's formulation the Schwarzschild solution has the form

$$\xi = x, \tag{27}$$

showing further that this form allows spheroidal solutions to be written simply.

Solutions for Ernst's Form

Ernst's formulation has been by far the most fruitful in producing new solutions in recent years. The first was found by Tomimatsu and Sato (1972) and was of the form

$$\xi = \frac{p^2x^4 + q^2y^4 - 2ipqxy(x^2 - y^2) - 1}{2px(x^2 - 1) - 2iqy(1 - y^2)}.$$

This was followed (Tomimatsu and Sato 1973) by a series of such solutions, all of which were generalized and extended by Cosgrove (1977). Another set of solutions found by this method, due to Ernst (1977), takes the form

$$\varepsilon = \frac{r^k}{p - iq}\frac{N_k(\cos\theta)}{D_k(\cos\theta)},$$

where $D_k = N_{k-1}^*$ and N_k are certain polynomials, $p^2 + q^2 = 1$, and r, θ and z are spheroidal polar coordinates.

Classification of Solutions by Ernst's Form

Because Ernst's formulation is expressed in terms of a *single* complex potential, a useful classification scheme is possible. For each ξ or ε there corresponds a solution of the field equations, and this provides a very concise form of tabulating known solutions of the vacuum field equations for the stationary axially symmetric case. The classification given in Table 2, based on the form of the potential function, provides a fuller list of solutions than that given in Table 1, based on the $\boldsymbol{\Omega}, \boldsymbol{\Lambda}$ relation of Levy's form. This is because the basic field equations of Ernst's form have been transformed to take account of sphericity in the solutions; the feature which proved impossible for the earlier scheme to accommodate.

Table 2. Classification scheme based on Ernst's formulation

Solution	Potential
Weyl (1917)	$\xi = -\tanh u$, where u satisfies $\nabla^2 u = 0$
Lewis (1932)	Example: $$\xi = \frac{1+\alpha_1^2 r^{1-k} - \beta_1^2 r^{1+k} + 2ik\alpha_1\beta_1 z}{1-\alpha_1^2 r^{1-k} + \beta_1^2 r^{1+k} - 2ik\alpha_1\beta_1 z},$$ where α_1, β_1 and k are real constants. This class includes the solutions of Gürses and Güven (1975; see Cohen 1976). Also see Bergamini *et al.* (1976)
Papapetrou (1953)	$\xi = \exp(\mathrm{i}\alpha_1)\coth\psi$ and $\exp(\mathrm{i}\alpha_1)\tanh\psi$, where α_1 and α_2 are constants and ψ is an arbitrary harmonic function
Kerr (1963)	$\xi = px - iqy$
Marek (1968)	These solutions are not expressible as elementary functions, as the metric functions depend on Painlevé transcendents
Tomimatsu and Sato (1972, 1973)	Example: $$\xi = \frac{p^2x^4 - q^2y^4 - 2ipqxy(x^2-y^2) - 1}{2px(x^2-1) - 2iqy(1-y^2)}$$ (Taken from Tomimatsu and Sato 1972)
Ernst (1977)	$\varepsilon = r^k Y_k(\cos\theta)$, with r, θ spherical polar coordinates and $$Y_k(y) = \frac{1}{p-\mathrm{i}q}\frac{N_k(y)}{D_k(y)} \quad \text{for} \quad k \geqslant 1,$$ $$D_k(y) = N_k^*(y).$$ The N_k are complex polynomials in y, with expressions for $k = 0, 1, ..., 5$ given by Ernst (1977)

(*e*) *Chandrasekhar's form*

Line Element and Field Equations

In a recent paper, Chandrasekhar (1978) returned to the more general approach adopted by Bach (1922). Choosing the line element to be

$$\mathrm{d}s^2 = -\exp(2\nu)(\mathrm{d}t)^2 + \exp(2\psi)(\mathrm{d}\phi - \omega\,\mathrm{d}t)^2 + \exp(2\mu_2)(\mathrm{d}x^2)^2 + \exp(2\mu_3)(\mathrm{d}x^3)^2, \quad (28)$$

he then transformed this into

$$\mathrm{d}s^2 = (\Delta\,\delta)^{\frac{1}{2}}\{-\chi(\mathrm{d}t)^2+\chi^{-1}(\mathrm{d}\phi-\omega\,\mathrm{d}t)^2\}+\Delta^{-\frac{1}{2}}\exp(\mu_2+\mu_3)\{(\mathrm{d}r)^2+\Delta(\mathrm{d}\theta)^2\}, \quad (29)$$

giving the field equations as

$$\tfrac{1}{2}(X+Y)\{(\Delta\,X_{,r})_{,r}+(\delta\,X_{,\mu})_{,\mu}\} = \Delta(X_{,r})^2+\delta(X_{,\mu})^2, \quad (30a)$$

$$\tfrac{1}{2}(X+Y)\{(\Delta\,Y_{,r})_{,r}+(\delta\,Y_{,\mu})_{,\mu}\} = \Delta(Y_{,r})^2+\delta(Y_{,\mu})^2. \quad (30b)$$

Definitions of the variables introduced in equations (29) and (30) are

$$X = \chi+\omega, \qquad Y = \chi-\omega, \qquad \mu = \cos\theta, \qquad \Delta^{\frac{1}{2}} = \exp(\mu_3-\mu_2), \qquad \delta = 1-\mu^2,$$

where μ_2 and μ_3 are determined by quadrature. This approach by Chandrasekhar has the advantage that it does not initially assume the cylindrical symmetry of canonical coordinates.

Solutions for Chandrasekhar's Form

Chandrasekhar's (1978) paper intended to display a derivation of Kerr's solution, and it achieved this better than any other of the formulations considered here, although the value of hindsight cannot be overlooked in assessing his achievement. Chandrasekhar's work has also provided new solutions generated from old ones, but these have not yet been looked into in great detail. The equations (30a) and (30b) are closely related to (22) and so any classification scheme using X and Y would be no more concise or useful than that given in Table 2.

4. Conclusions

This paper has provided a catalogue of the simplifications of the general vacuum field equations for stationary axially symmetric fields that have been used by various authors in the past. Most of the formulations have inherent difficulties or are based on assumptions that severely restrict their usefulness. It is probable that much research has been hampered by the use of canonical coordinates, and indeed the fact that Kerr's solution was not found until 1963 (and then by different means) would appear to bear this out. The more useful of the formulations are those of Ernst and Chandrasekhar. That of Ernst, while based on canonical coordinates, has the inherent difficulties transformed away, while that of Chandrasekhar starts without the inhibiting assumption of WLP. These two formulations should provide suitable mechanisms for further investigations of the exterior field of a finite bounded source.

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Appendix 1. Components of Ricci Tensor for General Metric

The general metric tensor in the axially symmetric stationary case, corresponding to the line element (6), has the form

$$g_{ab} = \begin{bmatrix} A & 0 & 0 & F \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ F & 0 & 0 & D \end{bmatrix}. \tag{A1}$$

Expressions for the components of the associated Ricci tensor are:

$$\begin{aligned} 4R_{00} &= \frac{2A_{11}}{B} + \frac{2A_{22}}{C} + \frac{-DA_1 + AA_1D_1 + 2FA_1F_1 - 2AF_1^2}{\Delta B} \\ &\quad + \frac{-DA_2^2 + AA_2D_2 + 2FA_2F_2 - 2AF_2^2}{\Delta C} \\ &\quad + \frac{-B^2A_2C_2 + BC(A_1C_1 + A_2B_2) - C^2A_1B_1}{B^2C^2}, \end{aligned} \tag{A2}$$

$$\begin{aligned} 4R_{11} &= \frac{2B_{22}}{C} + \frac{2C_{11}}{C} + \frac{2DA_{11} - 4FF_{11} + 2AD_{11}}{\Delta} - \frac{B_1C_1 + B_2^2}{BC} - \frac{B_2C_2 + C_1^2}{C^2} \\ &\quad + \frac{-AD_1C_1 + 2FB_1F_1 - DA_1B}{\Delta B} + \frac{AD_2B_2 - 2FB_2F_2 + DA_2B_2}{\Delta C} \\ &\quad + \frac{-A^2D_1^2 - 2ADF_1^2 - D^2A_1^2 + 4AFD_1F_1 + 4DFA_1F_1 - 2F^2(A_1D_1 + F_1^2)}{\Delta^2}, \end{aligned} \tag{A3}$$

$$\begin{aligned} 4R_{22} &= \frac{2C_{11}}{B} + \frac{2B_{11}}{B} + \frac{2DA_{11} - 4FF_{11} + 2AD_{11}}{\Delta} - \frac{B_2C_2 + C_1^2}{BC} - \frac{B_1C_1 + B_2^2}{B^2} \\ &\quad + \frac{-AD_2B_2 + 2FC_2F_1 - DA_2C}{\Delta C} + \frac{AD_1C_1 - 2FC_1F_1 + DA_1C_1}{\Delta B} \\ &\quad + \frac{-A^2D_2^2 - 2ADF_2^2 - D^2A_2^2 + 4AFD_2F_2 + 4DFA_2F_2 - 2F^2(A_2D_2 + F_2^2)}{\Delta^2}, \end{aligned} \tag{A4}$$

$$4R_{33} = \frac{2D_{11}}{B} + \frac{2D_{22}}{C} + \frac{-AD_1^2 + DD_1 A_1 + 2FD_1 F_1 - 2DF_1^2}{\Delta B}$$

$$+ \frac{-AD_2^2 + DD_2 A_2 + 2FD_2 F_2 - 2DF_2^2}{\Delta C}$$

$$+ \frac{-B^2 D_2 C_2 + BC(D_1 C_1 + D_2 B_2) - C^2 D_1 B_1}{B^2 C^2}, \tag{A5}$$

$$4R_{12} = \frac{2DA_{12} + 2AD_{12} - 4FF_{12}}{\Delta} + \frac{-AD_1 B_1 - DA_1 B_2 + 2FF_1 B_2}{\Delta B}$$

$$+ \frac{-AD_2 C_1 - DA_2 C_1 + 2FF_2 C_1}{\Delta C}$$

$$+ \frac{1}{\Delta^2}\Big(-A^2 D_1 D_2 - D^2 A_1 A_2 - 2ADF_1 F_2 + 2AF(D_2 F_1 + D_1 F_2)$$

$$+ 2DF(A_1 F_2 + A_2 F_1) - F^2(A_1 D_2 + A_2 D_1 + 2F_1 F_2)\Big)$$

$$= \frac{2\Delta_{12}}{\Delta} - \frac{\Delta_2 \Delta_1}{\Delta^2} - \frac{B_2 \Delta_1}{B\Delta} - \frac{C_1 \Delta_2}{C\Delta} - \frac{A_2 D_1 + A_1 D_2 - 2F_1 F_2}{\Delta}, \tag{A6}$$

$$4R_{03} = \frac{2F_{11}}{B} + \frac{2F_{22}}{C} + \frac{-AD_1 F_1 + 2FA_1 D_1 - DA_1 F_1}{\Delta B}$$

$$+ \frac{-AD_1 F_1 + 2FA_2 D_2 - DA_2 F_2}{\Delta C}$$

$$+ \frac{-B^2 C_2 F_2 + BC(C_1 F_1 + B_2 F_2) - C^2 B_1 F_1}{B^2 C^2}, \tag{A7}$$

where the suffixes 1 and 2 denote differentiation with respect to x^1 and x^2 respectively, while $\Delta = AD - F^2$.

Appendix 2. Field Equations in Canonical Coordinates: Levy's Form

In Levy's formulation the metric tensor has the form

$$g_{ab} = \begin{bmatrix} \exp(2u) & 0 & 0 & a\exp(2u) \\ 0 & -\exp(2k-2u) & 0 & 0 \\ 0 & 0 & -\exp(2k-2u) & 0 \\ a\exp(2u) & 0 & 0 & a^2\exp(2u) - r^2\exp(-2u) \end{bmatrix}, \tag{A8}$$

and the field equations are:

$$R_{00} = \nabla^2 u + \{\exp(4u)/2r^2\}\{(a_{,r})^2 + (a_{,z})^2\} = 0, \tag{A9}$$

$$R_{03} - aR_{00} = \nabla^2 a - 2a_{,r}/r + 4a_{,r}u_{,r} + 4a_{,z}u_{,z} = 0, \tag{A10}$$

$$R_{22}-R_{11} = k_{,r}/2r + 2(u_{,z})^2 - 2(u_{,r})^2 + \{\exp(4u)/2r\}\{(a_{,r})^2 - (a_{,z})^2\} = 0\,, \tag{A11}$$

$$R_{12} = -k_{,z}/r + 2u_{,r}\,u_{,z} - a_{,r}\,a_{,z}\exp(4u)/2r^2 = 0\,, \tag{A12}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

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