Aust. J. Phys., 1979, **32**, 137–54

Calculation of $3jm$ Factors and the Matrix Elements of E_7 Group Generators

P. H. Butler, R. W. Haase and B. G. Wybourne

Department of Physics, University of Canterbury, Christchurch 1, New Zealand.

Abstract

The matrix elements of the group generators of E_7 have been calculated in an $E_7 \supset SU_6^{\mathrm{fl}} \times SU_3^{\mathrm{C}} \supset SU_2^{H} \times SU_3^{\mathrm{fl}} \times SU_3^{\mathrm{C}} \supset SU_2^{H} \times SU_2^{I} \times U_1^{Y} \times SU_3^{\mathrm{C}}$ basis for the fundamental and adjoint irreps of E_7. The results were obtained by first calculating the $3jm$ factors for the various group–subgroup combinations. Tables of the relevant $3jm$ factors for $E_7 \supset SU_6 \times SU_3$, $SU_6 \supset SU_2 \times SU_3$ and $SU_3 \supset SU_2 \times U_1$ are given.

1. Introduction

The group–subgroup structure $E_7 \supset SU_6^{\mathrm{fl}} \times SU_3^{\mathrm{C}}$ has been used to develop unified theories of strong, electromagnetic and weak interactions (Gürsey *et al.* 1975; Gürsey and Sikivie 1976; Ramond 1976, 1977; Cung and Kim 1977; Saclioglu 1977; Sikivie and Gürsey 1977; Gell-Mann *et al.* 1978). In these theories the basic fermions (quarks, leptons and their antiparticles) are associated with the 56-dimensional fundamental irreducible representation (irrep) of E_7, and the gauge vector bosons that mediate the interactions are associated with the 133-dimensional adjoint irrep.

There are many possible schemes for breaking the E_7 symmetry down to an appropriate $SU_2^{I} \times U_1^{Y} \times SU_3^{\mathrm{C}}$ subgroup (Ramond 1977; Sikivie and Gürsey 1977). The correct scheme, if indeed there is such a scheme, must be decided by a confrontation with experimental results. In this paper we set ourselves the somewhat modest task of calculating the various $3jm$ factors associated with the group–subgroup structure

$$E_7 \supset SU_6^{\mathrm{fl}} \times SU_3^{\mathrm{C}} \supset SU_2^{H} \times SU_3^{\mathrm{fl}} \times SU_3^{\mathrm{C}} \supset SU_2^{H} \times SU_2^{I} \times U_1^{Y} \times SU_3^{\mathrm{C}}.$$

These $3jm$ factors are then used to calculate the matrix elements of the generators of E_7 in the fermion and boson sectors. These calculations give added insight into two significant problems: (1) the properties of $3jm$ factors and (2) the structure of the fermion and boson mass matrices.

A detailed discussion of the basic properties of the exceptional groups has been given by Wybourne and Bowick (1977) and we refer to that paper for matters of notation. Additional general information has been considered by Butler (1975, 1979) and by Butler and Wybourne (1976*a*, 1976*b*). The calculation of the relevant $6j$ symbols for E_7 has been reported by Butler *et al.* (1978). These $6j$ symbols form the key to obtaining the $3jm$ factors for $E_7 \supset SU_6 \times SU_3$.

2. Irreps of SU_n Groups

The irreps of E_7 and their associated properties have already been given (Wybourne and Bowick 1977; Wybourne 1978; Butler *et al.* 1978) and need not be repeated here. We label the irreps of SU_n by partitions $\{\lambda\}$ of integers into not more than $n-1$ nonzero parts (Wybourne 1970). For the irreps of SU_3 and SU_6 we shall omit the braces and use a dot to separate the irreps for the direct product group $SU_6 \times SU_3$ (e.g. the $\{21\}\times\{32\}$ irrep of $SU_6 \times SU_3$ will be designated as 21.32). In the case of SU_2^I we shall usually label the irrep by $I \equiv \frac{1}{2}\lambda$ while for the product group $SU_2^H \times SU_3^{\mathrm{fl}}$ we shall indicate the SU_2^H irrep as a spectroscopic multiplicity ($H = \lambda + 1$) that appears as a left superscript attached to the SU_3^{fl} irrep (e.g. the 1.21 irrep of $SU_2^H \times SU_2^{\mathrm{fl}}$ will be designated as ${}^{2}21$).

Table 1. Some SU_6 and SU_3 irreps and their associated properties

Irrep λ	Dimension $\|\lambda\|$	Power p_λ	Phase ϕ_λ	$2j_\lambda$ value
		(a) SU_6 irreps		
0	1	0	1	0
1	6	1	-1	1
1^2	15	2	1	0
2	21	2	1	2
21^4	35	2	1	2
1^3	20	3	-1	3
3	56	3	-1	3
21	70	3	-1	3
21^3	84	3	-1	3
31^4	120	3	-1	3
21^2	105	4	1	2
		(b) SU_3 irreps		
0	1	0	1	0
1	3	1	1	2
2	6	2	1	0
21	8	2	1	0
3	10	3	1	2
31	15	3	1	2
4	15	4	1	0

The dimensions $|\lambda|$, power p_λ and $2j$ symbol ϕ_λ associated with each irrep of SU_6 or SU_3 arising in our calculations are given in Tables 1*a* or 1*b* respectively. In the case of contragredient pairs of irreps, we give only one member since the quantities listed are common to both members. All the irreps considered here are simple phase (Butler and King 1974) and may be associated with a j value such that

$$\phi_\lambda = (-1)^{2j_\lambda}, \tag{1}$$

where j_λ is an integer if λ is orthogonal and a half-integer if λ is symplectic. We hasten to add that such a simple phase structure is not always possible (Butler 1975). The j_λ value to be associated with a given irrep λ is found from an analysis of the Kronecker square of λ. The appropriate values of $2j_\lambda$ are included in Table 1. The relevant branching rules for $E_7 \to SU_6 \times SU_3$ are given in Table 2.

3. Basic Group Structure

The generators of E_7 span the 21^6 irrep of E_7. The various subgroup structures contained in E_7 may be explored by systematically discarding sets of the E_7 generators (cf. Wybourne 1973). Under $E_7 \to SU_6 \times SU_3$ we have (Wybourne and Bowick 1977)

$$21^6 \to 21^4.0 + 1^2.1^2 + 1^4.1 + 0.21\,. \tag{2}$$

The 35 vector bosons are associated with the $21^4.0$ and form the generators of the SU_6 subgroup. The 90 leptoquarks span the $1^2.1^2$ and $1^4.1$ irreps of $SU_6 \times SU_3$ while the 8 gluons span the 0.21 irrep and form the generators of the presumably unbroken colour gauge group SU_3^C.

Table 2. Some $E_7 \to SU_6 \times SU_3$ branching rules

E_7 irrep	Branching to $SU_6 \times SU_3$
(0)	0.0
(1^6)	$1.1 + 1^5.1^2 + 1^3.0$
(21^6)	$21^4.0 + 0.21 + 1^2.1^2 + 1^4.1$
(2^6)	$21^4.21 + 21^4.0 + 21^2.1 + 2^31^2.1^2 + 2^3.0 + 2.2 + 2^5.2^2$ $+ 1^2.1^2 + 1^4.1 + 0.0$
(2^51^2)	$21^4.21 + 21^4.0 + 21^2.1 + 2^31^2.1^2 + 0.21 + 2.1^2 + 2^5.1$ $+ 1^2.2 + 1^4.2^2 + 1^2.1^2 + 1^4.1 + 2^21^2.0 + 0.0$

The SU_6 subgroup may be broken in various ways. Under $SU_6 \to SU_2 \times SU_3$ we have

$$21^4 \to {}^30 + {}^321 + {}^121\,. \tag{3}$$

In this case the three vector bosons associated with ^{30} can be regarded as forming the generators of an SU_2 group and those with ^{121} the generators of the SU_3 group. The SU_3 group may be reduced to $SU_2^I \times U_1^Y$ by noting that under $SU_3 \to SU_2^I \times U_1^Y$

$$21 \to (\tfrac{1}{2}, 1) + (1, 0) + (0, 0) + (\tfrac{1}{2}, -1)\,, \tag{4}$$

where we use (I, Y) to label irreps of $SU_2^I \times U_1^Y$. The three vector bosons transforming as the $(1,0)$ irrep of $SU_2^I \times U_1^Y$ form the generators of SU_2^I while the $(0,0)$ gives the single generator of U_1^Y.

So far we have neglected to give any specific representation of the spin. The n-particle fermion states may be regarded as spanning the antisymmetric $\{1^n\}$ irreps and the n-particle bosons the symmetric $\{n\}$ irreps of $U_{112} \supset SU_2 \times E_7$. Some relevant branching rules are given in Table 3.

We note that the basic fermions span the vector irrep of U_{112}. The objects spanning the $\{1^2\}$ and $\{1^3\}$ irreps of U_{112} can be constructed out of pairs and triplets of the basic fermions. Presumably only objects corresponding to colour singlets will be accessible to observation. This class of objects will include mesons, lepton pairs and massive leptoquark–antileptoquark states in the case of the $\{1^2\}$ irrep and the various baryons and lepton triplets for the $\{1^3\}$ irrep.

Objects spanning the symmetric $\{2\}$ irrep of U_{112} cannot be constructed from the basic fermions and they represent the scalar and vector bosons. These objects can be expected to contribute to the fermion and boson mass matrices.

4. 3*j* Symbols

The 3*j* symbols $\{(\pi)\lambda_1\lambda_2\lambda_3\}_{rr'}$ give the permutational symmetries of the 3*jm* factors (Butler 1975). For simple phase irreps the 3*j* symbol is no more than a phase factor (Butler and King 1974) and we may write (Butler and Wybourne 1976*a*)

$$\{(123)\lambda_1\lambda_2\lambda_3\}_{rr'} = \{(132)\lambda_1\lambda_2\lambda_3\}_{rr'} = \delta_{rr'}, \tag{5}$$

$$\{(12)\lambda_1\lambda_2\lambda_3\}_{rr'} = \{(23)\lambda_1\lambda_2\lambda_3\}_{rr'} = \{(13)\lambda_1\lambda_2\lambda_3\}_{rr'} = \{\lambda_1\lambda_2\lambda_3 r\}\delta_{rr'}. \tag{6}$$

In the cases treated here it was always possible to cast the 3*j* symbols of equation (6) into the form

$$\{\lambda_1\lambda_2\lambda_3 r\} = (-1)^{j\lambda_1+j\lambda_2+j\lambda_3+r}, \tag{7}$$

where r is the product multiplicity index. For the irreps considered here the multiplicity never exceeds 2 and we may restrict r to 0 and 1. The relevant 3*j* symbols may be readily evaluated using Tables 1*a* and 1*b* for SU_6 and SU_3 respectively, and the results given by Butler *et al.* (1978) for E_7.

Table 3. Some $U_{112} \to SU_2 \times E_7$ branching rules

Dimension $\|\lambda\|$	U_{112} irrep λ	Branching to $SU_2 \times E_7$
1	0	${}^{1}0$
112	1	${}^{2}1^6$
6216	1^2	${}^{3}(0+2^51^2)+{}^{1}(21^6+2^6)$
6328	2	${}^{1}(0+2^51^2)+{}^{3}(21^6+2^6)$
227920	1^3	${}^{2}(3^521+32^51+2^7+1^6)+{}^{4}(3^42^3+1^6)$
240464	3	${}^{2}(3^521+32^51+2^7+1^6)+{}^{4}(3^6+32^51+1^6)$
12543	21^{110}	${}^{3}(2^6+21^6+2^51^2+0)+{}^{1}(2^6+21^6+2^51^2)^{A}$

^{A} The generators of U_{112} span this irrep.

5. 6*j* Symbols

The relevant 6*j* symbols for E_7 have been given by Butler *et al.* (1978). In addition, 6*j* symbols for the direct product groups $SU_6 \times SU_3$ and $SU_2 \times SU_3$ were required. These 6*j* symbols are simply products of those of the individual groups. The required 6*j* symbols were calculated in a similar fashion to those for E_7 using the orthogonality, Racah backcoupling and generalized Biedenharn–Elliott relations to construct sets of simultaneous equations which were then systematically solved. Very careful attention was given to the fixing of phases, ensuring that phase choices were made only when a clear freedom to choose them existed.

In some instances nonlinear equations were obtained and it was necessary to find the roots of a quadratic equation. In these cases it was sometimes possible to use the duality between SU_n and S_n to relate the required 6*j* symbol to an SU_2 6*j* symbol. This allowed the correct root to be obtained with the phase being determined by the solution of the quadratic equation.

The calculation of the 6*j* symbols was greatly facilitated by a computer program that constructs all the required equations. In the process of carrying out the calculations reported here several hundred 6*j* symbols for SU_6 and SU_3 were evaluated.

6. 2jm Factors

The 2jm factor is defined as (Butler and Wybourne 1976a)

$$(\lambda)_{a\sigma,a'\sigma'} = |\lambda|^{\frac{1}{2}}|\sigma|^{-\frac{1}{2}}\langle 0|\lambda a\sigma;\lambda^* a'\sigma'\rangle, \tag{8}$$

giving the coupling of a representation and its complex conjugate to the identity irrep 0. Here we use λ to denote the irreps of a group G, and σ the irreps of the subgroup H, with a being a branching multiplicity index. It follows from equation (8) that

$$(\lambda)_{a\sigma,a'\sigma'} = (\lambda)_{a\sigma,a'\sigma^*}\delta_{\sigma'\sigma^*} \tag{9}$$

and we have the symmetry

$$(\lambda)_{a\sigma,a'\sigma^*} = \phi_\lambda\phi_\sigma(\lambda^*)_{a'\sigma^*,a\sigma}. \tag{10}$$

We find $\phi_\lambda\phi_\sigma = +1$ for $E_7 \supset SU_6\times SU_3$ and $SU_6 \supset SU_2\times SU_3$ whereas for $SU_3 \supset SU_2^I \times U_1^Y$ we find $\phi_\lambda\phi_\sigma = (-1)^{2I}$. For the first two cases we can choose

$$(\lambda)_{a\sigma,a'\sigma^*} = \delta_{aa'} \tag{11}$$

with

$$(\lambda)_{a\sigma,a'\sigma^*} = (\lambda^*)_{a'\sigma^*,a\sigma}, \tag{12}$$

remembering also that for E_7 we have $\lambda \equiv \lambda^*$. In the case of $SU_3 \supset SU_2^I\times U_1^Y$ we have

$$(\lambda)_{a\sigma,a'\sigma^*} = (-1)^{2I}(\lambda^*)_{a'\sigma^*,a\sigma}. \tag{13}$$

For I integral there is no difference from the previous cases. If I is a half-integer we can still maintain equation (11) provided we sequence the $a\sigma$ in a definite order.

7. 3jm Factors

A typical 3jm factor may be written symbolically as

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\sigma_1 & a_2\sigma_2 & a_3\sigma_3 \end{pmatrix}^r_s,$$

where r and s are product multiplicity indices for G and H respectively. The 3j symbols give the permutational symmetry relations for the 3jm factors:

$$\begin{pmatrix} \lambda_a & \lambda_b & \lambda_c \\ a_a\sigma_a & a_b\sigma_b & a_c\sigma_c \end{pmatrix}^{r'}_{s'} = \sum_{rs}\{(\pi)\lambda_1\lambda_2\lambda_3\}_{r'r}\{(\pi^{-1})\sigma_1\sigma_2\sigma_3\}_{s's}\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\sigma_1 & a_2\sigma_2 & a_3\sigma_3 \end{pmatrix}^r_s. \tag{14}$$

In all cases considered here an odd permutation results in at most a change of sign. In Table 4 below we use a right superscript plus or minus sign to indicate whether or not a given 3jm factor changes sign under an odd permutation.

Under complex conjugation we have

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\sigma_1 & a_2\sigma_2 & a_3\sigma_3 \end{pmatrix}^{r*}_s = \sum_{a_1'a_2'a_3'}(\lambda_1)_{a_1\sigma_1a_1'\sigma_1^*}(\lambda_2)_{a_2\sigma_2a_2'\sigma_2^*}(\lambda_3)_{a_3\sigma_3a_3'\sigma_3^*} \times\begin{pmatrix} \lambda_1^* & \lambda_2^* & \lambda_3^* \\ a_1'\sigma_1^* & a_2'\sigma_2^* & a_3'\sigma_3^* \end{pmatrix}^r_s. \tag{15}$$

Comparison of equations (14) and (15) often indicates that a given $3jm$ factor is necessarily imaginary. For example, in the case of $E_7 \supset SU_6 \times SU_3$ we find from equation (14) that

$$\begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1.1 & 1^5.1^2 & 0.21 \end{pmatrix} = -\begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1^5.1^2 & 1.1 & 0.21 \end{pmatrix},$$

whereas (15) gives

$$\begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1.1 & 1^5.1^2 & 0.21 \end{pmatrix}^* = \begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1^5.1^2 & 1.1 & 0.21 \end{pmatrix} = -\begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1.1 & 1^5.1^2 & 0.21 \end{pmatrix},$$

leading to the conclusion that this $3jm$ factor is imaginary.

The $3jm$ factors satisfy the orthogonality relations

$$\sum_{\lambda_3 a_3} \frac{|\lambda_3|}{|\sigma_3|} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\sigma_1 & a_2\sigma_2 & a_3\sigma_3 \end{pmatrix}_s^{r*} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1'\sigma_1' & a_2'\sigma_2' & a_3\sigma_3 \end{pmatrix}_{s'}^{r} = \delta_{a_1a_1'}\delta_{a_2a_2'}\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}\delta_{ss'} \quad (16a)$$

and

$$\sum_{a_1\sigma_1a_2\sigma_2 s} \frac{|\lambda_3|}{|\sigma_3|} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\sigma_1 & a_2\sigma_2 & a_3\sigma_3 \end{pmatrix}_s^{r*} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3' \\ a_1\sigma_1 & a_2\sigma_2 & a_3'\sigma_3 \end{pmatrix}_{s}^{r'} = \delta_{a_3a_3'}\delta_{\lambda_3\lambda_3'}\delta_{rr'}. \quad (16b)$$

The orthogonality conditions give equations that will often yield the magnitudes of $3jm$ factors and some phase information but by themselves cannot lead to a complete evaluation of the $3jm$ factors.

Two further equations that relate the $3jm$ factors to the $6j$ symbols of the group G and its subgroup H play a crucial role in the calculation of $3jm$ factors. Firstly (Butler and Wybourne 1976*a*)

$$\sum_{r_4} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ a_1\sigma_1 & a_2\sigma_2 & a_3\sigma_3 \end{pmatrix}_{s_4}^{r_4} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4} = \sum (\mu_1)_{b_1\rho_1b_1'\rho_1^*}(\mu_2)_{b_2\rho_2b_2'\rho_2^*}(\mu_3)_{b_3\rho_3b_3'\rho_3^*}$$
$$\times \begin{pmatrix} \lambda_1 & \mu_2^* & \mu_3 \\ a_1\sigma_1 & b_2'\rho_2^* & b_3\rho_3 \end{pmatrix}_{s_1}^{r_1} \begin{pmatrix} \mu_1 & \lambda_2 & \mu_3^* \\ b_1\rho_1 & a_2\sigma_2 & b_3'\rho_3^* \end{pmatrix}_{s_2}^{r_2} \begin{pmatrix} \mu_1^* & \mu_2 & \lambda_3 \\ b_1'\rho_1^* & b_2\rho_2 & a_3\sigma_3 \end{pmatrix}_{s_3}^{r_3} \begin{Bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \rho_1 & \rho_2 & \rho_3 \end{Bmatrix}_{s_1s_2s_3s_4}, \quad (17)$$

where the right-hand summation is over all $b_i b_i' \rho_i s_i$ $(i = 1, 2, 3)$. It is convenient to rearrange the above equation to obtain the second equation

$$\sum_{\lambda_3 r_3 r_4} \frac{|\lambda_3|}{|\rho_3|} \begin{pmatrix} \mu_1^* & \mu_2 & \lambda_3 \\ a_1'\sigma_1^* & a_2\sigma_2 & b_3\rho_3 \end{pmatrix}_{s_3}^{r_3*} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1\rho_1 & b_2\rho_2 & b_3\rho_3 \end{pmatrix}_{s_4}^{r_4} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1r_2r_3r_4}$$
$$= \sum_{\sigma_3 s_1 s_2} \phi_{\mu_1}\phi_{\sigma_1}(\mu_1^*)_{a_1'\sigma_1^*a_1\sigma_1}(\mu_2)_{a_2\sigma_2a_2'\sigma_2^*}(\mu_3)_{a_3\sigma_3a_3'\sigma_3^*}$$
$$\times \begin{pmatrix} \lambda_1 & \mu_2^* & \mu_3 \\ b_1\rho_1 & a_2'\sigma_2^* & a_3\sigma_3 \end{pmatrix}_{s_1}^{r_1} \begin{pmatrix} \mu_1 & \lambda_2 & \mu_3^* \\ a_1\sigma_1 & b_2\rho_2 & a_3'\sigma_3^* \end{pmatrix}_{s_2}^{r_2} \begin{Bmatrix} \rho_1 & \rho_2 & \rho_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{Bmatrix}_{s_1s_2s_3s_4}. \quad (18)$$

The two equations (17) and (18) contain all the information that can be extracted from the orthogonality relations together with additional phase and magnitude information.

The trivial $3jm$ factors that involve the identity irrep 0 in the group G follow immediately from equation (8) to give

$$\begin{pmatrix} \lambda & \lambda^* & 0 \\ a\sigma & a'\sigma^* & 0 \end{pmatrix} = +|\sigma|^{\frac{1}{2}}|\lambda|^{-\frac{1}{2}}\delta_{aa'}. \tag{19}$$

To proceed to the practical calculation of $3jm$ factors we first calculate the required $6j$ symbols of the group G and its subgroup H. In the case of G only primitive $6j$ symbols are required while for H a somewhat larger set is needed. The trivial $3jm$ factors are then found via equation (19).

The next stage is to systematically calculate the primitive $3jm$ factors (Butler and Wybourne 1976*a*). Here it is essential to give careful attention to the choice of phases associated with each primitive $3jm$ factor. Once these phases are chosen, those of the non-primitive $3jm$ factors are implied via equation (17). Some of the phases of the primitive $3jm$ factors may be freely chosen while others may not. These latter must be determined by use of equations (16)–(18). It is important to note that the orthogonality relations alone do not give sufficent phase information. In fixing the phases of the primitive $3jm$ factors we note that there is a free choice for each new ket vector.

The calculation of the primitive $3jm$ factors proceeds by first determining the permutational symmetry of each of the $3jm$ factors followed by use of equation (15) to determine which $3jm$ factors are necessarily imaginary. The orthogonality relations are used to establish a set of simultaneous equations in the $3jm$ factors. In many cases these equations together with the phase freedom allow us to fix a number of $3jm$ factors, but not all of them. Additional equations are required and these are found by a judicious use of equations (17) or (18). For example, the magnitude of the $E_7 \supset SU_6 \times SU_3$ $3jm$ factor

$$\begin{pmatrix} 1 & 1 & 21^6 \\ 1.1 & 1.1 & 1^4.1 \end{pmatrix}^+$$

was determined by first evaluating the $3jm$ factors

$$\begin{pmatrix} 1^6 & 1^6 & \lambda \\ 1.1 & 1^5.1^2 & 0.21 \end{pmatrix}$$

and then choosing in equation (18) $\mu_1 = \mu_2 = \lambda_1 = \lambda_2 \equiv 1^6$, $\mu_3 \equiv 21^6$ and $\sigma_1 = \sigma_2^* = \rho_1^* = \rho_2 \equiv 1.1$. This choice then made $\rho_3 \equiv 0.21$ and $\sigma_3 \equiv 1^4.1$, leading to

$$\left|\begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1.1 & 1.1 & 1^4.1 \end{pmatrix}^+\right|^2 = \tfrac{15}{133}.$$

Since the $|21^6, 1^4.1\rangle$ ket arose here for the first time, we used our phase freedom to fix the $3jm$ factor as real and positive. This then allowed the magnitudes of all the remaining primitive $3jm$ factors to be immediately determined. Where a phase freedom existed it was chosen. The remaining phases were found by again making a

Table 4. 3*jm* factors for E_7 group generators

3*jm* factor	Value	3*jm* factor	Value	3*jm* factor	Value
(a) $E_7 \supset SU_6 \times SU_3$					
$\begin{pmatrix} 1^6 & 1^6 & 0 \\ 1^3.0 & 1^3.0 & 0.0 \end{pmatrix}^+$	$\sqrt{\frac{5}{14}}$	$\begin{pmatrix} 1^6 & 1^6 & 0 \\ 1.1 & 1^5.1^2 & 0.0 \end{pmatrix}^+$	$\frac{3}{2}\sqrt{\frac{1}{7}}$		
$\begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1^3.0 & 1^3.0 & 21^4.0 \end{pmatrix}^+$	$-\sqrt{\frac{5}{38}}$	$\begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1.1 & 1^5.1^2 & 21^4.0 \end{pmatrix}^+$	$\frac{1}{2}\sqrt{\frac{5}{19}}$	$\begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1.1 & 1^5.1^2 & 0.21 \end{pmatrix}^-$	$2i\sqrt{\frac{1}{133}}$
$\begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1^3.0 & 1^5.1^2 & 1^4.1 \end{pmatrix}^+$	$\sqrt{\frac{15}{133}}$	$\begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1.1 & 1.1 & 1^4.1 \end{pmatrix}^+$	$\sqrt{\frac{15}{133}}$		
$\begin{pmatrix} 1^6 & 1^6 & 2^6 \\ 1^3.0 & 1^3.0 & 0.0 \end{pmatrix}^-$	0	$\begin{pmatrix} 1^6 & 1^6 & 2^6 \\ 1.1 & 1^5.1^2 & 0.0 \end{pmatrix}^-$	$i\sqrt{\frac{1}{2926}}$	$\begin{pmatrix} 1^6 & 1^6 & 2^6 \\ 1.1 & 1^5.1^2 & 21^4.21 \end{pmatrix}^+$	$2\sqrt{\frac{5}{209}}$
$\begin{pmatrix} 1^6 & 1^6 & 2^6 \\ 1^3.0 & 1^3.0 & 21^4.0 \end{pmatrix}^+$	$\sqrt{\frac{5}{418}}$	$\begin{pmatrix} 1^6 & 1^6 & 2^6 \\ 1.1 & 1^5.1^2 & 21^4.0 \end{pmatrix}^+$	$\frac{1}{2}\sqrt{\frac{5}{209}}$	$\begin{pmatrix} 1^6 & 1^6 & 2^6 \\ 1.1 & 1.1 & 2.2 \end{pmatrix}^+$	$3\sqrt{\frac{2}{209}}$
$\begin{pmatrix} 1^6 & 1^6 & 2^6 \\ 1^3.0 & 1^3.0 & 2^3.0 \end{pmatrix}^+$	$5\sqrt{\frac{1}{209}}$	$\begin{pmatrix} 1^6 & 1^6 & 2^6 \\ 1.1 & 1.1 & 1^4.1 \end{pmatrix}^+$	$\sqrt{\frac{30}{1463}}$	$\begin{pmatrix} 1^6 & 1^6 & 2^6 \\ 1^3.0 & 1^5.1^2 & 21^2.1 \end{pmatrix}^+$	$\sqrt{\frac{5}{418}}$
$\begin{pmatrix} 1^6 & 1^6 & 2^6 \\ 1^3.0 & 1^5.1^2 & 1^4.1 \end{pmatrix}^+$	$-\sqrt{\frac{15}{2926}}$				
$\begin{pmatrix} 1^6 & 1^6 & 2^51^2 \\ 1^3.0 & 1^3.0 & 0.0 \end{pmatrix}^+$	$-\frac{1}{3}\sqrt{\frac{1}{266}}$	$\begin{pmatrix} 1^6 & 1^6 & 2^51^2 \\ 1.1 & 1^5.1^2 & 0.0 \end{pmatrix}^+$	$\frac{1}{18}\sqrt{\frac{5}{133}}$	$\begin{pmatrix} 1^6 & 1^6 & 2^51^2 \\ 1.1 & 1^5.1^2 & 21^4.21 \end{pmatrix}^-$	$\frac{2}{9}i\sqrt{\frac{35}{19}}$
$\begin{pmatrix} 1^6 & 1^6 & 2^51^2 \\ 1^3.0 & 1^3.0 & 21^4.0 \end{pmatrix}^-$	0	$\begin{pmatrix} 1^6 & 1^6 & 2^51^2 \\ 1.1 & 1^5.1^2 & 21^4.0 \end{pmatrix}^-$	$\frac{1}{9}i\sqrt{\frac{35}{38}}$	$\begin{pmatrix} 1^6 & 1^6 & 2^51^2 \\ 1.1 & 1^5.1^2 & 0.21 \end{pmatrix}^+$	$\frac{2}{9}\sqrt{\frac{1}{19}}$
$\begin{pmatrix} 1^6 & 1^6 & 2^51^2 \\ 1^3.0 & 1^3.0 & 2^21^2.0 \end{pmatrix}^+$	$\sqrt{\frac{7}{57}}$	$\begin{pmatrix} 1^6 & 1^6 & 2^51^2 \\ 1^3.0 & 1^5.1^2 & 1^4.1 \end{pmatrix}^-$	$\frac{1}{3}\sqrt{\frac{5}{38}}$	$\begin{pmatrix} 1^6 & 1^6 & 2^51^2 \\ 1.1 & 1.1 & 1^4.1 \end{pmatrix}^-$	0
$\begin{pmatrix} 1^6 & 1^6 & 2^51^2 \\ 1^3.0 & 1^5.1^2 & 21^2.1 \end{pmatrix}^+$	$\frac{1}{3}\sqrt{\frac{35}{38}}$	$\begin{pmatrix} 1^6 & 1^6 & 2^51^2 \\ 1.1 & 1.1 & 2^5.1 \end{pmatrix}^+$	$\frac{1}{3}\sqrt{\frac{7}{19}}$	$\begin{pmatrix} 1^6 & 1^6 & 2^51^2 \\ 1.1 & 1.1 & 1^4.2^2 \end{pmatrix}^+$	$\frac{1}{3}\sqrt{\frac{10}{19}}$
$\begin{pmatrix} 21^6 & 21^6 & 21^6 \\ 21^4.0 & 21^4.0 & 21^4.0 \end{pmatrix}^-_1$	0	$\begin{pmatrix} 21^6 & 21^6 & 21^6 \\ 21^4.0 & 21^4.0 & 21^4.0 \end{pmatrix}^+_0$	$\sqrt{\frac{5}{57}}$	$\begin{pmatrix} 21^6 & 21^6 & 21^6 \\ 0.21 & 0.21 & 0.21 \end{pmatrix}^-_0$	0
$\begin{pmatrix} 21^6 & 21^6 & 21^6 \\ 0.21 & 0.21 & 0.21 \end{pmatrix}^+_1$	$2\sqrt{\frac{1}{399}}$	$\begin{pmatrix} 21^6 & 21^6 & 21^6 \\ 1^2.1^2 & 1^2.1^2 & 1^2.1^2 \end{pmatrix}^+$	$-\sqrt{\frac{15}{133}}$	$\begin{pmatrix} 21^6 & 21^6 & 21^6 \\ 21^4.0 & 1^2.1^2 & 1^4.1 \end{pmatrix}^+$	$\sqrt{\frac{5}{57}}$
$\begin{pmatrix} 21^6 & 21^6 & 21^6 \\ 0.21 & 1^2.1^2 & 1^4.1 \end{pmatrix}^-$	$-\sqrt{\frac{10}{399}}$				
$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 0.21 & 0.21 & 0.21 \end{pmatrix}^-_1$	0	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 0.21 & 0.21 & 0.21 \end{pmatrix}^+_0$	$-\frac{2}{9}\sqrt{\frac{1}{19}}$	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 0.21 & 0.21 & 0.0 \end{pmatrix}^+$	$\frac{2}{9}\sqrt{\frac{1}{133}}$
$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 0.21 & 1^2.1^2 & 1^4.1 \end{pmatrix}^+$	$\frac{2}{3}i\sqrt{\frac{1}{57}}$	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 0.21 & 1^2.1^2 & 1^4.2^2 \end{pmatrix}^-$	$\frac{1}{3}i\sqrt{\frac{2}{19}}$	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 0.21 & 21^4.0 & 21^4.21 \end{pmatrix}^+$	$\frac{2}{9}\sqrt{\frac{7}{19}}$
$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 21^4.0 & 21^4.0 & 21^4.0 \end{pmatrix}^+_1$	$\frac{2}{9}\sqrt{\frac{7}{19}}$	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 21^4.0 & 21^4.0 & 21^4.0 \end{pmatrix}^-_0$	0	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 21^4.0 & 21^4.0 & 0.0 \end{pmatrix}^+$	$-\frac{2}{9}\sqrt{\frac{1}{190}}$
$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 21^4.0 & 21^4.0 & 2^21^2.0 \end{pmatrix}^+$	$\sqrt{\frac{14}{285}}$	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 21^4.0 & 1^2.1^2 & 2^5.1 \end{pmatrix}^+$	$-\frac{1}{3}\sqrt{\frac{7}{38}}$	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 21^4.0 & 1^4.1 & 2^31^2.1^2 \end{pmatrix}^+$	$\frac{1}{3}\sqrt{\frac{21}{38}}$
$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 21^4.0 & 1^2.1^2 & 1^4.1 \end{pmatrix}^-$	$\frac{1}{3}\sqrt{\frac{7}{114}}$				

Table 4 (*Continued*)

3*jm* factor	Value	3*jm* factor	Value	3*jm* factor	Value
$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 1^2.1^2 & 1^4.1 & 0.0 \end{pmatrix}^{+}$	$\frac{1}{9}\sqrt{\frac{1}{1330}}$	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 1^2.1^2 & 1^4.1 & 0.21 \end{pmatrix}^{+}$	$\frac{1}{9}\sqrt{\frac{2}{19}}$	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 1^2.1^2 & 1^4.1 & 2^21^2.0 \end{pmatrix}^{+}$	$\sqrt{\frac{7}{190}}$
$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 1^2.1^2 & 1^4.1 & 21^4.0 \end{pmatrix}^{-}$	$-\frac{1}{9}i\sqrt{\frac{7}{38}}$	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 1^2.1^2 & 1^4.1 & 21^4.21 \end{pmatrix}^{-}$	$\frac{4}{9}i\sqrt{\frac{7}{19}}$	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 1^2.1^2 & 1^2.1^2 & 1^2.1^2 \end{pmatrix}^{-}$	0
$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 1^2.1^2 & 1^2.1^2 & 1^2.2 \end{pmatrix}^{+}$	$\sqrt{\frac{2}{57}}$	$\begin{pmatrix} 21^6 & 21^6 & 2^51^2 \\ 1^2.1^2 & 1^2.1^2 & 2^31^2.1^2 \end{pmatrix}^{+}$	$-\frac{1}{3}\sqrt{\frac{14}{19}}$		
		(*b*) $SU_6 \supset SU_2 \times SU_3$			
$\begin{pmatrix} 0 & 0 & 0 \\ {}^{1}0 & {}^{1}0 & {}^{1}0 \end{pmatrix}^{+}$	1	$\begin{pmatrix} 1 & 1^5 & 0 \\ {}^{2}1 & {}^{2}1^2 & {}^{1}0 \end{pmatrix}^{+}$	1	$\begin{pmatrix} 1^2 & 1^4 & 0 \\ {}^{3}1^2 & {}^{3}1 & {}^{1}0 \end{pmatrix}^{+}$	$\sqrt{\frac{3}{5}}$
$\begin{pmatrix} 1 & 1^5 & 21^4 \\ {}^{2}1 & {}^{2}1^2 & {}^{3}0 \end{pmatrix}^{+}$	$\sqrt{\frac{3}{35}}$	$\begin{pmatrix} 1 & 1^5 & 21^4 \\ {}^{2}1 & {}^{2}1^2 & {}^{3}21 \end{pmatrix}^{+}$	$2\sqrt{\frac{6}{35}}$	$\begin{pmatrix} 1 & 1^5 & 21^4 \\ {}^{2}1 & {}^{2}1^2 & {}^{1}21 \end{pmatrix}^{-}$	$2i\sqrt{\frac{2}{35}}$
$\begin{pmatrix} 1 & 1 & 1^4 \\ {}^{2}1 & {}^{2}1 & {}^{3}1 \end{pmatrix}^{+}$	$\sqrt{\frac{3}{5}}$	$\begin{pmatrix} 1 & 1 & 1^4 \\ {}^{2}1 & {}^{2}1 & {}^{1}2^2 \end{pmatrix}^{+}$	$\sqrt{\frac{2}{5}}$		
$\begin{pmatrix} 1 & 1^2 & 1^3 \\ {}^{2}1 & {}^{3}1^2 & {}^{4}0 \end{pmatrix}^{-}$	$\sqrt{\frac{1}{5}}$	$\begin{pmatrix} 1 & 1^2 & 1^3 \\ {}^{2}1 & {}^{3}1^2 & {}^{2}21 \end{pmatrix}^{+}$	$\sqrt{\frac{2}{5}}$	$\begin{pmatrix} 1 & 1^2 & 1^3 \\ {}^{2}1 & {}^{1}2 & {}^{2}21 \end{pmatrix}^{+}$	$\sqrt{\frac{2}{5}}$
$\begin{pmatrix} 1^3 & 1^3 & 21^4 \\ {}^{4}0 & {}^{4}0 & {}^{3}0 \end{pmatrix}^{+}$	$\sqrt{\frac{1}{21}}$	$\begin{pmatrix} 1^3 & 1^3 & 21^4 \\ {}^{4}0 & {}^{2}21 & {}^{3}21 \end{pmatrix}^{-}$	$-4\sqrt{\frac{1}{105}}$	$\begin{pmatrix} 1^3 & 1^3 & 0 \\ {}^{2}21 & {}^{2}21 & {}^{1}0 \end{pmatrix}$	$2\sqrt{\frac{1}{5}}$
$\begin{pmatrix} 1^3 & 1^3 & 21^4 \\ {}^{2}21 & {}^{2}21 & {}^{3}0 \end{pmatrix}^{+}$	$-2\sqrt{\frac{1}{105}}$	$\begin{pmatrix} 1^3 & 1^3 & 21^4 \\ {}^{2}21 & {}^{2}21 & {}^{3}21 \end{pmatrix}^{+}_{0}$	$4\sqrt{\frac{1}{42}}$	$\begin{pmatrix} 1^3 & 1^3 & 21^4 \\ {}^{2}21 & {}^{2}21 & {}^{3}21 \end{pmatrix}^{-}_{1}$	0
$\begin{pmatrix} 1^3 & 1^3 & 21^4 \\ {}^{2}21 & {}^{2}21 & {}^{1}21 \end{pmatrix}^{+}_{1}$	$2\sqrt{\frac{2}{35}}$	$\begin{pmatrix} 1^3 & 1^3 & 21^4 \\ {}^{2}21 & {}^{2}21 & {}^{1}21 \end{pmatrix}^{-}_{0}$	0	$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}0 & {}^{3}0 & {}^{3}0 \end{pmatrix}^{0+}$	$-\sqrt{\frac{1}{105}}$
$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}0 & {}^{3}0 & {}^{3}0 \end{pmatrix}^{1-}$	0	$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}21 & {}^{3}21 & {}^{3}0 \end{pmatrix}^{0+}$	$-4\sqrt{\frac{1}{210}}$	$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}21 & {}^{3}21 & {}^{3}0 \end{pmatrix}^{1-}$	0
$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}21 & {}^{3}21 & {}^{3}21 \end{pmatrix}^{1-}_{0}$	0	$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}21 & {}^{3}21 & {}^{3}21 \end{pmatrix}^{1+}_{1}$	$-3\sqrt{\frac{3}{70}}$	$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}21 & {}^{3}21 & {}^{3}21 \end{pmatrix}^{0+}_{0}$	$-2\sqrt{\frac{1}{21}}$
$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}21 & {}^{3}21 & {}^{3}21 \end{pmatrix}^{0-}_{1}$	0	$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}21 & {}^{3}21 & {}^{1}21 \end{pmatrix}^{0-}_{0}$	0	$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}21 & {}^{3}21 & {}^{1}21 \end{pmatrix}^{0+}_{1}$	$\sqrt{\frac{6}{35}}$
$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}21 & {}^{3}21 & {}^{1}21 \end{pmatrix}^{1+}_{0}$	$-\frac{1}{2}\sqrt{\frac{3}{7}}$	$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}21 & {}^{3}21 & {}^{1}21 \end{pmatrix}^{1-}_{1}$	0		
$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}21 & {}^{1}21 & {}^{3}0 \end{pmatrix}^{0-}$	0	$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{3}21 & {}^{1}21 & {}^{3}0 \end{pmatrix}^{1+}$	$-\sqrt{\frac{3}{70}}$		
$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{1}21 & {}^{1}21 & {}^{1}21 \end{pmatrix}^{0-}_{0}$	0	$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{1}21 & {}^{1}21 & {}^{1}21 \end{pmatrix}^{0+}_{1}$	$\sqrt{\frac{2}{35}}$		
$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{1}21 & {}^{1}21 & {}^{1}21 \end{pmatrix}^{1+}_{0}$	$-\frac{1}{2}\sqrt{\frac{1}{7}}$	$\begin{pmatrix} 21^4 & 21^4 & 21^4 \\ {}^{1}21 & {}^{1}21 & {}^{1}21 \end{pmatrix}^{1-}_{1}$	0		
$\begin{pmatrix} 1^2 & 1^4 & 21^4 \\ {}^{3}1^2 & {}^{3}1 & {}^{3}0 \end{pmatrix}^{+}$	$\sqrt{\frac{3}{35}}$	$\begin{pmatrix} 1^2 & 1^4 & 21^4 \\ {}^{3}1^2 & {}^{3}1 & {}^{3}21 \end{pmatrix}^{+}$	$-\sqrt{\frac{6}{35}}$	$\begin{pmatrix} 1^2 & 1^4 & 21^4 \\ {}^{3}1^2 & {}^{3}1 & {}^{1}21 \end{pmatrix}^{-}$	$-i\sqrt{\frac{3}{35}}$
$\begin{pmatrix} 1^2 & 1^4 & 21^4 \\ {}^{1}2 & {}^{3}1 & {}^{3}21 \end{pmatrix}^{+}$	$\sqrt{\frac{9}{35}}$	$\begin{pmatrix} 1^2 & 1^4 & 21^4 \\ {}^{1}2 & {}^{1}2^2 & {}^{1}21 \end{pmatrix}^{-}$	$i\sqrt{\frac{1}{7}}$	$\begin{pmatrix} 1^2 & 1^4 & 21^4 \\ {}^{3}1^2 & {}^{1}2^2 & {}^{3}21 \end{pmatrix}^{+}$	$\sqrt{\frac{9}{35}}$
$\begin{pmatrix} 1^2 & 1^2 & 1^2 \\ {}^{3}1^2 & {}^{3}1^2 & {}^{3}1^2 \end{pmatrix}^{+}$	$-\sqrt{\frac{1}{5}}$	$\begin{pmatrix} 1^2 & 1^2 & 1^2 \\ {}^{1}2 & {}^{1}2 & {}^{1}2 \end{pmatrix}^{+}$	$-\sqrt{\frac{1}{5}}$	$\begin{pmatrix} 1^2 & 1^2 & 1^2 \\ {}^{3}1^2 & {}^{3}1^2 & {}^{1}2 \end{pmatrix}^{+}$	$\sqrt{\frac{1}{5}}$

Table 4 (*Continued*)

$3jm$ factor	Value	$3jm$ factor	Value	$3jm$ factor	Value
		(*c*) $SU_3 \supset SU_2^I \times U_1^Y$			
$\begin{pmatrix} 0 & 0 & 0 \\ 0.0 & 0.0 & 0.0 \end{pmatrix}^{+}$	$\frac{1}{2}$				
$\begin{pmatrix} 1 & 1^2 & 0 \\ \frac{1}{2}.\frac{1}{3} & \frac{1}{2}.-\frac{1}{3} & 0.0 \end{pmatrix}^{-}$	$\sqrt{\frac{2}{3}}$	$\begin{pmatrix} 1 & 1^2 & 0 \\ 0.-\frac{2}{3} & 0.\frac{2}{3} & 0.0 \end{pmatrix}^{+}$	$\sqrt{\frac{1}{3}}$		
$\begin{pmatrix} 1 & 1^2 & 21 \\ \frac{1}{2}.\frac{1}{3} & \frac{1}{2}.-\frac{1}{3} & 0.0 \end{pmatrix}^{-}$	$\frac{1}{2}\sqrt{\frac{1}{6}}$	$\begin{pmatrix} 1 & 1^2 & 21 \\ \frac{1}{2}.\frac{1}{3} & \frac{1}{2}.-\frac{1}{3} & 1.0 \end{pmatrix}^{+}$	$\frac{1}{4}\mathrm{i}\sqrt{6}$	$\begin{pmatrix} 1 & 1^2 & 21 \\ 0.-\frac{2}{3} & 0.\frac{2}{3} & 0.0 \end{pmatrix}^{+}$	$-\frac{1}{2}\sqrt{\frac{1}{3}}$
$\begin{pmatrix} 1 & 1^2 & 21 \\ \frac{1}{2}.\frac{1}{3} & 0.\frac{2}{3} & \frac{1}{2}.-1 \end{pmatrix}^{-}$	$\frac{1}{2}$	$\begin{pmatrix} 1 & 1^2 & 21 \\ 0.-\frac{2}{3} & \frac{1}{2}.-\frac{1}{3} & \frac{1}{2}.1 \end{pmatrix}^{-}$	$\frac{1}{2}$		
$\begin{pmatrix} 1 & 1 & 2^2 \\ \frac{1}{2}.\frac{1}{3} & \frac{1}{2}.\frac{1}{3} & 1.-\frac{2}{3} \end{pmatrix}^{+}$	$\sqrt{\frac{1}{2}}$	$\begin{pmatrix} 1 & 1 & 2^2 \\ \frac{1}{2}.\frac{1}{3} & 0.-\frac{2}{3} & \frac{1}{2}.\frac{1}{3} \end{pmatrix}^{-}$	$\sqrt{\frac{1}{3}}$	$\begin{pmatrix} 1 & 1 & 2 \\ 0.-\frac{2}{3} & 0.-\frac{2}{3} & 0.\frac{4}{3} \end{pmatrix}^{+}$	$\sqrt{\frac{1}{6}}$
$\begin{pmatrix} 21 & 21 & 0 \\ \frac{1}{2}.1 & \frac{1}{2}.-1 & 0.0 \end{pmatrix}^{-}$	$\frac{1}{2}$	$\begin{pmatrix} 21 & 21 & 0 \\ 1.0 & 1.0 & 0.0 \end{pmatrix}^{+}$	$\frac{1}{4}\sqrt{6}$	$\begin{pmatrix} 21 & 21 & 0 \\ 0.0 & 0.0 & 0.0 \end{pmatrix}^{+}$	$\frac{1}{2}\sqrt{\frac{1}{2}}$
$\begin{pmatrix} 21 & 21 & 21 \\ 0.0 & 0.0 & 0.0 \end{pmatrix}^{1-}$	0	$\begin{pmatrix} 21 & 21 & 21 \\ 0.0 & 0.0 & 0.0 \end{pmatrix}^{0+}$	$-\frac{1}{2}\sqrt{\frac{1}{10}}$		
$\begin{pmatrix} 21 & 21 & 21 \\ 1.0 & 1.0 & 0.0 \end{pmatrix}^{1-}$	0	$\begin{pmatrix} 21 & 21 & 21 \\ 1.0 & 1.0 & 0.0 \end{pmatrix}^{0+}$	$\frac{1}{2}\sqrt{\frac{3}{10}}$		
$\begin{pmatrix} 21 & 21 & 21 \\ 1.0 & 1.0 & 1.0 \end{pmatrix}^{1+}$	$\frac{1}{2}$	$\begin{pmatrix} 21 & 21 & 21 \\ 1.0 & 1.0 & 1.0 \end{pmatrix}^{0-}$	0		
$\begin{pmatrix} 21 & 21 & 21 \\ \frac{1}{2}.1 & \frac{1}{2}.-1 & 0.0 \end{pmatrix}^{1+}$	$-\frac{1}{4}\mathrm{i}$	$\begin{pmatrix} 21 & 21 & 21 \\ \frac{1}{2}.1 & \frac{1}{2}.-1 & 0.0 \end{pmatrix}^{0-}$	$-\frac{1}{4}\sqrt{\frac{1}{5}}$		
$\begin{pmatrix} 21 & 21 & 21 \\ \frac{1}{2}.1 & \frac{1}{2}.-1 & 1.0 \end{pmatrix}^{1-}$	$\frac{1}{4}$	$\begin{pmatrix} 21 & 21 & 21 \\ \frac{1}{2}.1 & \frac{1}{2}.-1 & 1.0 \end{pmatrix}^{0+}$	$\frac{3}{4}\mathrm{i}\sqrt{\frac{1}{5}}$		

judicious use of equation (18), arriving at the important result that the phases of the $3jm$ factors

$$\begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1^3.0 & 1^3.0 & 21^4.0 \end{pmatrix}^{+} \quad \text{and} \quad \begin{pmatrix} 1^6 & 1^6 & 21^6 \\ 1.1 & 1^5.1^2 & 21^4.0 \end{pmatrix}^{+}$$

must be chosen to be of opposite sign. Such a result would not be implied by simple use of the orthogonality relations (16).

With the primitive $3jm$ factors evaluated, it is a comparatively simple task to calculate the non-primitive $3jm$ factors by use of equation (17). Each non-primitive $3jm$ factor is calculated separately and then the resulting sets of $3jm$ factors are checked by demanding that they form orthonormal sets.

The $3jm$ factors evaluated for $E_7 \supset SU_6 \times SU_3$ are given in Table 4*a* while those for $SU_6 \supset SU_2 \times SU_3$ and $SU_3 \supset SU_2^I \times U_1^Y$ are given in Tables 4*b* and 4*c* respectively. These tables suffice to calculate the matrix elements of all the generators of E_7 within the fundamental and adjoint irreps of E_7.

8. Matrix Elements of E_7 Group Generators

With the $2jm$ and $3jm$ factors determined it is a comparatively simple task to use the Wigner–Eckart theorem (Butler 1975) to calculate the matrix elements of the group generators. The symmetry classification of the group generators has already been considered in Section 3. The group generators will be necessarily diagonal in the E_7 irreps but will not be so in the subgroup irreps.

If Q_i^λ is a tensor operator belonging to a tensorial set $\mathbf{Q}^\lambda$, where λ labels an irrep of the group G, and i the components of the irrep, then it follows from the Wigner–Eckart theorem that

$$\langle x_1 \lambda_1 i_1 | Q_i^\lambda | x_2 \lambda_2 i_2 \rangle = \sum_r (\lambda_1)_{i_1 i_1^*} \begin{pmatrix} \lambda_1^* & \lambda & \lambda_2 \\ i_1^* & i & i_2 \end{pmatrix}^r \langle x_1 \lambda_1 \| Q^{\lambda r} \| x_2 \lambda_2 \rangle, \tag{20}$$

where $\langle x_1 \lambda_1 \| Q^{\lambda r} \| x_2 \lambda_2 \rangle$ is a reduced matrix.

Table 5. Reduced kets

E_7 irrep	$SU_6^{\mathrm{fl}} \times SU_3^{\mathrm{C}}$ irrep	$SU_2^H \times SU_3^{\mathrm{fl}}$ irrep	Reduced ket	E_7 irrep	$SU_6^{\mathrm{fl}} \times SU_3^{\mathrm{C}}$ irrep	$SU_2^H \times SU_3^{\mathrm{fl}}$ irrep	Reduced ket
1^6	$1^3.0$	^{40}	$\lvert L^40\rangle$	21^6	$21^4.0$	^{30}	$\lvert VB^30\rangle$
		^{221}	$\lvert L^221\rangle$			^{321}	$\lvert VB^321\rangle$
						^{121}	$\lvert VB^121\rangle$
	1.1	^{21}	$\lvert Q^21\rangle$		0.21	^{10}	$\lvert G^10\rangle$
	$1^5.1^2$	$^21^2$	$\lvert \bar{Q}^21^2\rangle$		$1^4.1$	^{31}	$\lvert \overline{LQ}^31\rangle$
						$^12^2$	$\lvert \overline{LQ}^12^2\rangle$
					$1^2.1^2$	$^31^2$	$\lvert LQ^31^2\rangle$
						^{12}	$\lvert LQ^12\rangle$

Consider now the case where we have an operator $Q^{\lambda a\sigma}$ that is a tensor operator with respect to the group G and its subgroup H. Applying the Wigner–Eckart theorem to both groups (Butler 1975), we obtain

$$\langle x_1 \lambda_1 a_1 \sigma_1 \| Q^{\lambda a \sigma s} \| x_2 \lambda_2 a_2 \sigma_2 \rangle$$
$$= \sum_r (\lambda_1)_{a_1\sigma_1, a_1'\sigma_1^*} \begin{pmatrix} \lambda_1^* & \lambda & \lambda_2 \\ a_1' \sigma_1^* & a\sigma & a_2 \sigma_2 \end{pmatrix}_s^r \langle x_1 \lambda_1 \| Q^{\lambda r} \| x_2 \lambda_2 \rangle. \tag{21}$$

This result can be used along an entire group chain. The dependence of the matrix elements on the various subgroup irreps is fully contained in the relevant group–subgroup $2jm$ and $3jm$ factors.

There are 133 group generators for E_7 and clearly a table of the matrix elements of these generators even for just the 1^6 and 21^6 irreps would be very large. In order to restrict the size of the tabulation we shall assume that the Wigner–Eckart theorem has been used to factor off the dependence on the $SU_2^I \times U_1^Y$ subgroups of the SU_3^{fl} and SU_3^{C} groups and the U_1 subgroup of the SU_2^H group. The present calculation has thus been reduced to the calculation of the $SU_2^H \times SU_3^{\mathrm{fl}} \times SU_3^{\mathrm{C}}$ reduced matrix elements, for which we introduce a set of reduced kets to describe the ket states associated with the group–subgroup chain

$$E_7 \supset SU_6^{\mathrm{fl}} \times SU_3^{\mathrm{C}} \supset SU_2^H \times SU_3^{\mathrm{fl}} \times SU_3^{\mathrm{C}}. \tag{22}$$

These reduced kets are fully specified in the fermion sector (i.e. in the (1^6) irrep of E_7) and the boson sector (i.e. in the (21^6) irrep of E_7) by specifying the appropriate $SU_2^H \times SU_3^{\mathrm{fl}}$ irrep together with a descriptive label indicating whether the ket corresponds to a lepton (L), quark (Q), vector boson (VB), gluon (G) or leptoquark

(LQ), as shown in Table 5. The reduced ket labels serve equally as well to designate the reduced operators corresponding to the group generators of E_7. An example of a typical $SU_2^H \times SU_3^{\mathrm{fl}} \times SU_3^{\mathrm{C}}$ reduced matrix element would be

$$\langle \mathrm{L}^2 21 \,\|\, \mathrm{VB}^1 21 \,\|\, \mathrm{L}^2 21 \rangle_1 , \tag{23}$$

where here the subscript 1 is a product multiplicity index for the SU_3^{fl} product $21 \times 21 \times 21$. An expanded description in accord with the breakdown (22) would be

$$\langle 1^6 1^3 . 0^2 21 \,\|\, 21^6 21^4 . 0^1 21 \,\|\, 1^6 1^3 . 0^2 21 \rangle_1 \; (SU_2^H \times SU_3^{\mathrm{fl}} \times SU_3^{\mathrm{C}}) . \tag{24}$$

(Note that in both descriptions (23) and (24) the SU_3^{C} label at the $SU_2^H \times SU_3^{\mathrm{fl}} \times SU_3^{\mathrm{C}}$ level has been suppressed because all states are colour singlets.)

To obtain the actual matrix elements of the generators of E_7 in the fermion or boson sectors it is necessary to determine their dependence on the quantum numbers (I, Y, I_z) for SU_3^{fl}, $(I^{\mathrm{C}}, Y^{\mathrm{C}}, I_z^{\mathrm{C}})$ for SU_3^{C}, and H_z for SU_2^H.

The dependences of the matrix elements on the azimuthal quantum numbers I_z, I_z^{C} and H_z all follow by noting that the matrix elements of a tensor operator kq in the angular momentum basis $|\alpha JM\rangle$ is given by (Judd 1963)

$$\langle \alpha_1 J_1 M_1 | kq | \alpha_2 J_2 M_2 \rangle = (-1)^{J_1 - M_1} \begin{pmatrix} J_1 & k & J_2 \\ -M_1 & q & M_2 \end{pmatrix} \langle \alpha_1 J_1 \,\|\, k \,\|\, \alpha_2 J_2 \rangle . \tag{25}$$

The $3jm$ factor can be readily obtained from tables (e.g. Rotenberg *et al.* 1959).

The dependence of the matrix elements on I and Y follows by noting that if λ_i labels irreps of the appropriate SU_3 group then we have from equation (21)

$$\begin{aligned} &\langle \alpha_1 \lambda_1 I_1 Y_1 \,\|\, \alpha\lambda I Y \,\|\, \alpha_2 \lambda_2 I_2 Y_2 \rangle \\ &\quad = \sum_r (\lambda_1)_{I_1 Y_1, I_1 - Y_1} \begin{pmatrix} \lambda_1^* & \lambda & \lambda_2 \\ I_1 . -Y_1 & I . Y & I_2 . Y_2 \end{pmatrix}^r \langle \alpha_1 \lambda_1 \,\|\, \alpha\lambda r \,\|\, \alpha_2 \lambda_2 \rangle . \end{aligned} \tag{26}$$

The $3jm$ factors this time may be found in Table 4*c*. Thus we have given all the information required to calculate the matrix elements of all the generators of E_7 in the fermion and boson sectors.

For illustration let us calculate the $SU_2^H \times SU_3^{\mathrm{fl}} \times SU_3^{\mathrm{C}}$ reduced matrix element (23). To do this we are required to fix the normalization of the group operators, which we do by choosing below the corresponding E_7 reduced matrix elements $\langle 1^6 \,\|\, 21^6 \,\|\, 1^6 \rangle$. Consider the E_7 generator I_z. Since I_z is a generator of SU_6^{fl}, SU_3^{fl}, SU_2^I and U_1^I, it transforms as the adjoint irrep of each of these groups. I_z is scalar under the other groups SU_2^H, SU_3^{C} U_1^Y, and their various subgroups, and thus transforms like the ket

$$\left| \begin{array}{ccccc} & & 21^6(E_7) & & \\ & 21^4(SU_6^{\mathrm{fl}}) & & 0(SU_3^{\mathrm{C}}) & \\ 0(SU_2^H) & & 21(SU_3^{\mathrm{fl}}) & & \\ 0(U_1^{H_z}) & 1(SU_2^I) & 0(U_1^Y) & 0(SU_2^{I^{\mathrm{C}}}) & 0(U_1^{Y^{\mathrm{C}}}) \\ & 0(U_1^{I_z}) & & 0(U_1^{I_z{}^{\mathrm{C}}}) & \end{array} \right\rangle$$

From the list of reduced kets in Table 5 we see that I_z transforms as one of the set of partners $|\,\mathrm{VB}^1 21\rangle$.

The action of the operator I_z on any ket is known, its eigenvalue being the value of I_z of the ket. Take as an example a particular 1 of the $|\,\mathrm{L}^2 21\rangle$ set. We have

$$1 = \left\langle \begin{array}{l|l|l} 1^6 & 21^6 & 1^6 \\ 1^3\ .\ 0 & 21^4\ .\ 0 & 1^3\ .\ 0 \\ \frac{1}{2}\ .21 & 0\ .21 & \frac{1}{2}\ .21 \\ \frac{1}{2}.1.0.0.0 & 0.1.0.0.0 & \frac{1}{2}.1.0.0.0 \\ 1\quad 0 & 0\quad 0 & 1\quad 0 \end{array} \right\rangle$$

$$= \sum_r (1^6)_{1^3.0,1^3.0}\,(1^3)_{{}^221.{}^221}\,(0)_{0.0,0.0}\,(\tfrac{1}{2})_{\frac{1}{2},-\frac{1}{2}}\,(21)_{1.0,1.0}\,(1)_{1,-1}$$

$$\times\langle 1^6 \,\|\, 21^6 \,\|\, 1^6\rangle \begin{pmatrix} 1^6 & 21^6 & 1^6 \\ 1^3.0 & 21^4.0 & 1^3.0 \end{pmatrix} \begin{pmatrix} 1^3 & 21^4 & 1^3 \\ {}^221 & {}^121 & {}^221 \end{pmatrix}_r$$

$$\times \begin{pmatrix} 0 & 0 & 0 \\ 0.0 & 0.0 & 0.0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 21 & 21 & 21 \\ 1.0 & 1.0 & 1.0 \end{pmatrix}^r \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= -\frac{1}{21\sqrt{(3.133)}}\langle 1^6 \,\|\, 21^6 \,\|\, 1^6\rangle, \tag{27a}$$

that is, we have

$$\langle 1^6 \,\|\, 21^6 \,\|\, 1^6\rangle = -\sqrt{(3.133)}. \tag{27b}$$

In obtaining the value of this reduced matrix element we could equally have employed the irrep tensor operators H_z of SU_2^H or I_Z^C of SU_2^{IC} and arrived at different numerical values. However, the renormalization of tensor operators is completely arbitrary and the above value may be chosen; a proviso being that we adhere to this choice in subsequent calculations. In a completely analogous fashion the E_7 reduced matrix element $\langle 21^6 \,\|\, 21^6 \,\|\, 21^6\rangle$ can be determined as $3\sqrt{(6.133)}$.

The $SU_2^H \times SU_3^{f1} \times SU_3^C$ reduced matrix elements for the generators of E_7 can be readily evaluated. An example would be:

$$\langle \mathrm{L}^2 21 \,\|\, \mathrm{VB}^1 21 \,\|\, \mathrm{L}^2 21\rangle_1 = \langle 1^6 1^3.0^2 21 \,\|\, 21^6 21^4.0^1 21 \,\|\, 1^6 1^3.0^2 21\rangle_1;$$

by equation (21), the right-hand side becomes

$$(1^3)_{{}^221,{}^221} \begin{pmatrix} 1^3 & 21^4 & 1^3 \\ {}^221 & {}^121 & {}^221 \end{pmatrix}_1 \langle 1^6 1^3.0 \,\|\, 21^6 21^4.0 \,\|\, 1^6 1^3.0\rangle,$$

and use of equation (11) and Table 4b gives this as

$$2\sqrt{\tfrac{2}{35}}\langle 1^6 1^3.0 \,\|\, 21^6 21^4.0 \,\|\, 1^6 1^3.0\rangle;$$

then by equation (21) again we have

$$2\sqrt{\tfrac{2}{35}}(1^6)_{1^3.0,1^3.0} \begin{pmatrix} 1^6 & 21^6 & 1^6 \\ 1^3.0 & 21^4.0 & 1^3.0 \end{pmatrix} \langle 1^6 \,\|\, 21^6 \,\|\, 1^6\rangle,$$

Table 6. Nonzero reduced matrix elements of E_7 generators

Where a product multiplicity r arises in the SU_3 subgroup, the value of r is given as a right subscript attached to the value of the matrix element

(*a*) *Fermion sector*

VB^30	$\lvert L^40\rangle$	$\lvert L^221\rangle$	$\lvert Q^21\rangle$	$\lvert \bar{Q}^21^2\rangle$
$\langle L^40\rvert$	$\sqrt{10}$			
$\langle L^221\rvert$		$-2\sqrt{2}$		
$\langle Q^21\rvert$			-3	
$\langle \bar{Q}^21^2\rvert$				-3

VB^321	$\lvert L^40\rangle$	$\lvert L^221\rangle$	$\lvert Q^21\rangle$	$\lvert \bar{Q}^21^2\rangle$
$\langle L^40\rvert$		$-4\sqrt{2}$		
$\langle L^221\rvert$	$-4\sqrt{2}$	$-4\sqrt{5}_0$		
$\langle Q^21\rvert$			$-6\sqrt{2}$	
$\langle \bar{Q}^21^2\rvert$				$-6\sqrt{2}$

VB^121	$\lvert L^40\rangle$	$\lvert L^221\rangle$	$\lvert Q^21\rangle$	$\lvert \bar{Q}^21^2\rangle$
$\langle L^40\rvert$				
$\langle L^221\rvert$		$4\sqrt{3}_1$		
$\langle Q^21\rvert$			$-2i\sqrt{6}$	
$\langle \bar{Q}^21^2\rvert$				$2i\sqrt{6}$

G^10	$\lvert L^40\rangle$	$\lvert L^221\rangle$	$\lvert Q^21\rangle$	$\lvert \bar{Q}^21^2\rangle$
$\langle L^40\rvert$				
$\langle L^221\rvert$				
$\langle Q^21\rvert$			$-4i\sqrt{3}$	
$\langle \bar{Q}^21^2\rvert$				$4i\sqrt{3}$

$\bar{L}\bar{Q}^31$	$\lvert L^40\rangle$	$\lvert L^221\rangle$	$\lvert Q^21\rangle$	$\lvert \bar{Q}^21^2\rangle$
$\langle L^40\rvert$				6
$\langle L^221\rvert$				$-6\sqrt{2}$
$\langle Q^21\rvert$	-6	$-6\sqrt{2}$		
$\langle \bar{Q}^21^2\rvert$			$-6\sqrt{3}$	

$\bar{L}\bar{Q}^12^2$	$\lvert L^40\rangle$	$\lvert L^221\rangle$	$\lvert Q^21\rangle$	$\lvert \bar{Q}^21^2\rangle$
$\langle L^40\rvert$				
$\langle L^221\rvert$				$-6\sqrt{2}$
$\langle Q^21\rvert$		$-6\sqrt{2}$		
$\langle \bar{Q}^21^2\rvert$			$-6\sqrt{2}$	

LQ^31^2	$\lvert L^40\rangle$	$\lvert L^221\rangle$	$\lvert Q^21\rangle$	$\lvert \bar{Q}^21^2\rangle$
$\langle L^40\rvert$			6	
$\langle L^221\rvert$			$-6\sqrt{2}$	
$\langle Q^21\rvert$				$-6\sqrt{3}$
$\langle \bar{Q}^21^2\rvert$	-6	$-6\sqrt{2}$		

LQ^12	$\lvert L^40\rangle$	$\lvert L^221\rangle$	$\lvert Q^21\rangle$	$\lvert \bar{Q}^21^2\rangle$
$\langle L^40\rvert$				
$\langle L^221\rvert$			$-6\sqrt{2}$	
$\langle Q^21\rvert$				$-6\sqrt{2}$
$\langle \bar{Q}^21^2\rvert$		$-6\sqrt{2}$		

(*b*) *Boson sector*

VB^30	$\lvert VB^30\rangle$	$\lvert VB^321\rangle$	$\lvert VB^121\rangle$	$\lvert G^10\rangle$	$\lvert \bar{L}\bar{Q}^31\rangle$	$\lvert \bar{L}\bar{Q}^12^2\rangle$	$\lvert LQ^31^2\rangle$	$\lvert LQ^12\rangle$
$\langle VB^30\rvert$	$-2\sqrt{3}_0$							
$\langle VB^321\rvert$		$-4\sqrt{6}$						
$\langle VB^121\rvert$								
$\langle G^10\rvert$								
$\langle \bar{L}\bar{Q}^31\rvert$					$6\sqrt{2}$			
$\langle \bar{L}\bar{Q}^12^2\rvert$								
$\langle LQ^31^2\rvert$							$6\sqrt{2}$	
$\langle LQ^22\rvert$								

VB^321	$\lvert VB^30\rangle$	$\lvert VB^321\rangle$	$\lvert VB^121\rangle$	$\lvert G^10\rangle$	$\lvert \bar{L}\bar{Q}^31\rangle$	$\lvert \bar{L}\bar{Q}^12^2\rangle$	$\lvert LQ^31^2\rangle$	$\lvert LQ^12\rangle$
$\langle VB^30\rvert$		$-4\sqrt{6}$						
$\langle VB^321\rvert$	$-4\sqrt{6}$	$-4\sqrt{15}_0$	$6\sqrt{6}_1$					
$\langle VB^121\rvert$		$6\sqrt{6}_1$						
$\langle G^10\rvert$								
$\langle \bar{L}\bar{Q}^31\rvert$					$-6\sqrt{6}$	18		
$\langle \bar{L}\bar{Q}^12^2\rvert$					18			
$\langle LQ^31^2\rvert$							$-6\sqrt{6}$	18
$\langle LQ^12\rvert$							18	

Table 6*b* (*Continued*)

$\overline{LQ}^31$	$\|VB^30\rangle$	$\|VB^321\rangle$	$\|VB^121\rangle$	$\|G^10\rangle$	$\|\overline{LQ}^31\rangle$	$\|\overline{LQ}^12^2\rangle$	$\|LQ^31^2\rangle$	$\|LQ^12\rangle$
$\langle VB^30\|$							$6\sqrt{3}$	
$\langle VB^321\|$							$-6\sqrt{6}$	18
$\langle VB^121\|$							$6i\sqrt{3}$	
$\langle G^10\|$							$6\sqrt{6}$	
$\langle\overline{LQ}^31\|$	$6\sqrt{3}$	$-6\sqrt{6}$	$-6i\sqrt{3}$	$-6\sqrt{6}$				
$\langle\overline{LQ}^12^2\|$		18						
$\langle LQ^31^2\|$					18	-18		
$\langle LQ^12\|$					-18			

$\overline{LQ}^12^2$	$\|VB^30\rangle$	$\|VB^321\rangle$	$\|VB^121\rangle$	$\|G^10\rangle$	$\|\overline{LQ}^31\rangle$	$\|\overline{LQ}^12^2\rangle$	$\|LQ^31^2\rangle$	$\|LQ^12\rangle$
$\langle VB^30\|$								
$\langle VB^321\|$							18	
$\langle VB^121\|$								$-6i\sqrt{5}$
$\langle G^10\|$								12
$\langle\overline{LQ}^31\|$		18						
$\langle\overline{LQ}^12^2\|$			$6i\sqrt{5}$	-12				
$\langle LQ^31^2\|$					-18			
$\langle LQ^12\|$						18		

LQ^31^2	$\|VB^30\rangle$	$\|VB^321\rangle$	$\|VB^121\rangle$	$\|G^10\rangle$	$\|\overline{LQ}^31\rangle$	$\|\overline{LQ}^12^2\rangle$	$\|LQ^31^2\rangle$	$\|LQ^12\rangle$
$\langle VB^30\|$					$6\sqrt{3}$			
$\langle VB^321\|$					$-6\sqrt{6}$	18		
$\langle VB^121\|$					$-6i\sqrt{3}$			
$\langle G^10\|$					$-6\sqrt{6}$			
$\langle\overline{LQ}^31\|$							18	-18
$\langle\overline{LQ}^12^2\|$							-18	
$\langle LQ^31^2\|$	$6\sqrt{3}$	$-6\sqrt{6}$	$6i\sqrt{3}$	$6\sqrt{6}$				
$\langle LQ^12\|$		18						

LQ^12	$\|VB^30\rangle$	$\|VB^321\rangle$	$\|VB^121\rangle$	$\|G^10\rangle$	$\|\overline{LQ}^31\rangle$	$\|\overline{LQ}^12^2\rangle$	$\|LQ^31^2\rangle$	$\|LQ^12\rangle$
$\langle VB^30\|$								
$\langle VB^321\|$					18			
$\langle VB^121\|$						$6i\sqrt{5}$		
$\langle G^10\|$						-12		
$\langle\overline{LQ}^31\|$							-18	
$\langle\overline{LQ}^12^2\|$								18
$\langle LQ^31^2\|$		18						
$\langle LQ^12\|$			$-6i\sqrt{5}$	12				

VB^121	$\|VB^30\rangle$	$\|VB^321\rangle$	$\|VB^121\rangle$	$\|G^10\rangle$	$\|\overline{LQ}^31\rangle$	$\|\overline{LQ}^12^2\rangle$	$\|LQ^31^2\rangle$	$\|LQ^12\rangle$
$\langle VB^30\|$								
$\langle VB^321\|$		$6\sqrt{6}_1$						
$\langle VB^121\|$			$6\sqrt{2}_1$					
$\langle G^10\|$								
$\langle\overline{LQ}^31\|$					$6i\sqrt{3}$			
$\langle\overline{LQ}^12^2\|$						$-6i\sqrt{5}$		
$\langle LQ^31^2\|$							$-6i\sqrt{3}$	
$\langle LQ^12\|$								$6i\sqrt{5}$

Table 6b (*Continued*)

G^10	$\vert VB^30\rangle$	$\vert VB^321\rangle$	$\vert VB^121\rangle$	$\vert G^10\rangle$	$\vert\bar{L}\bar{Q}^31\rangle$	$\vert\bar{L}\bar{Q}^12^2\rangle$	$\vert LQ^31^2\rangle$	$\vert LQ^12\rangle$
$\langle VB^30\vert$								
$\langle VB^321\vert$								
$\langle VB^121\vert$								
$\langle G^10\vert$				12_1				
$\langle\bar{L}\bar{Q}^31\vert$					$6\sqrt{6}$			
$\langle\bar{L}\bar{Q}^12^2\vert$						$6\sqrt{6}$		
$\langle LQ^31^2\vert$							$-6\sqrt{6}$	
$\langle LQ^12\vert$								$-6\sqrt{6}$

whence we use equation (11), Table 4*a* and equation (27b) to finally obtain

$$\langle L^221 \,\|\, VB^121 \,\|\, L^221\rangle_1 = 4\sqrt{3}\,.$$

We attach to the numerical value of the matrix element the SU_3^{fl} product multiplicity as a right subscript. This is important in the further use of the reduced matrix elements of the group generators.

The reduced matrix elements of the generators of E_7 are given in Table 6*a* for the fermion sector and in Table 6*b* for the boson sector.

9. E_7 Symmetrized Operators

It is sometimes useful in developing models of symmetry breaking to construct operators that transform as tensor operators with respect to a group G and a chain of its subgroups. If these operators are constructed from products of the generators of G, they will have the property of preserving the irreps of G while at the same time coupling different irreps of the subgroups. Thus these operators will allow one to introduce a symmetry breaking in a given irrep of G without at the same time coupling the irreps of G.

In the case of E_7, the generators can be regarded as forming the 133 components of a tensor operator $\mathbf{T}^{(21^6)}$. The matrix elements of this tensor operator have already been evaluated. New tensor operators $[\mathbf{T}^{(21^6)}\mathbf{T}^{(21^6)}]^{(\lambda)}$ can be formally constructed from bilinear products of the group generators. These operators will be symmetric in the generators for $(\lambda) = (0)$ and (2^51^2). The operator $[\mathbf{T}^{(21^6)}\mathbf{T}^{(21^6)}]^{(0)}$ will have matrix elements proportional to those for the second-order Casimir invariant for E_7 (Wybourne 1974). The reduced matrix elements of these operators may be found by noting that (cf. Butler 1975, equation 19.5)

$$\begin{aligned}
&\langle x_1\lambda_1 \,\|\, [\mathbf{T}^{(21^6)}\mathbf{T}^{(21^6)}]^{(\lambda)r} \,\|\, x_2\lambda_2\rangle \\
&\quad = \delta_{x_1x_2}\delta_{\lambda_1\lambda_2}|\lambda|^{\frac{1}{2}}\phi_{\lambda_1}\{(12)\lambda_1 21^6\lambda_1\}_{r_2r_2'}\{(23)\lambda_1 21^6\lambda_1\}_{r_1r_1'} \\
&\qquad \times\{(123)21^621^6\lambda\}_{rr'}\begin{Bmatrix}21^6 & \lambda & 21^6\\ \lambda_1 & \lambda_1 & \lambda_1\end{Bmatrix}_{r_2'0r_1r'} \\
&\qquad \times\langle x_1\lambda_1 \,\|\, T^{(21^6)r_1} \,\|\, x_1\lambda_1\rangle\langle x_1\lambda_1 \,\|\, T^{(21^6)r_2} \,\|\, x_1\lambda_1\rangle\,,
\end{aligned} \tag{28}$$

where there is a summation over repeated product multiplicity indices. If λ_1 is identified with the fermion or boson irreps of E_7 then the reduced matrix elements on the right-hand side of equation (28) follow from equation (27b) and we find

$$\langle 1^6 \,\|\, [\mathbf{T}^{(21^6)}\mathbf{T}^{(21^6)}]^{(\lambda)} \|\, 1^6 \rangle = -1596\,|\lambda|^{\frac{1}{2}} \begin{Bmatrix} 21^6 & \lambda & 21^6 \\ 1^6 & 1^6 & 1^6 \end{Bmatrix} \tag{29}$$

for the fermion sector and

$$\langle 21^6 \,\|\, [\mathbf{T}^{(21^6)}\mathbf{T}^{(21^6)}]^{(\lambda)} \|\, 21^6 \rangle = 2394\,|\lambda|^{\frac{1}{2}} \begin{Bmatrix} 21^6 & \lambda & 21^6 \\ 21^6 & 21^6 & 21^6 \end{Bmatrix} \tag{30}$$

for the boson sector. The $6j$ symbols follow directly from the work of Butler *et al.* (1978).

The eigenvalues of the operator $[\mathbf{T}^{(21^6)}\mathbf{T}^{(21^6)}]^{(0)}$ may be placed into correspondence with those of the second-order Casimir operator I_2 by writing

$$I_2 = -\tfrac{1}{18}\sqrt{133}\,[\mathbf{T}^{(21^6)}\mathbf{T}^{(21^6)}]^{(0)}, \tag{31}$$

with (Wybourne 1974)

$$I_2 = (\Lambda, \Lambda + 2g), \tag{32}$$

where Λ is the highest weight of the E_7 irrep and $2g$ is the sum of the positive roots of the E_7 Lie algebra. The eigenvalues of I_2 may be read from Table 1 of Wybourne and Bowick (1977) by noting that the eigenvalues of their Dynkin index $B(\lambda)$ are related to those of I_2 by

$$I_2 = \tfrac{1}{3}\sqrt{133}\,B(\lambda)/N(\lambda), \tag{33}$$

where $N(\lambda)$ is the dimension of the E_7 irrep (λ).

The other symmetric bilinear operator $[\mathbf{T}^{(21^6)}\mathbf{T}^{(21^6)}]^{(2^51^2)}$ has couplings both within the fermion sector and in the boson sector. These matrix elements can be found by using equations (29) and (30) to calculate the E_7 reduced matrix elements together with the tables of $3jm$ factors. We note that the (2^51^2) irrep, also often designated as the *1539* irrep, has been used as a possible candidate for the Higgs field to give superheavy masses to the leptoquarks (Ramond 1977; Sikivie and Gürsey 1977).

An operator having eigenvalues proportional to those of the sixth-order Casimir invariant of E_7 can be constructed by first constructing the tensor operator

$$\mathbf{U}^{(2^6)} \equiv [[\mathbf{T}^{(21^6)}\mathbf{T}^{(21^6)}]^{(2^51^2)}\mathbf{T}^{(21^6)}]^{(2^6)} \tag{34}$$

and then the operator

$$[\mathbf{U}^{(2^6)}\mathbf{U}^{(2^6)}]^{(0)}. \tag{35}$$

The E_7 irrep (2^7) (often designated as the *912* irrep) has also been considered as a candidate for the Higgs field to give superheavy masses to the leptoquarks (Ramond 1977; Sikivie and Gürsey 1977). It is interesting to note that to construct a tensor operator transforming as (2^7) from the group generators of E_7 we must go to operators that are certainly higher than third order in the generators. Of course such an operator will necessarily be null in the fermion and boson sectors.

10. Concluding Remarks

The $3jm$ factors given here have been systematically evaluated, paying unusual care in the assignment of phases. The entire calculation has been made within a particular E_7 group chain, avoiding the need to resort to Gelfand basis states as is frequently done. The calculations required a knowledge of the character theory of the relevant group chain and little more, other than the dimensions of the group representations. There would be little difficulty in extending the tables to include other E_7 triads such as $\{1^6, 21^6, 2^6\}$ and $\{1^6, 21^6, 32^5 1\}$ or to obtain $3jm$ factors for the triads $\{\lambda_1 \lambda_2 \lambda_3\}$ where $\lambda_i = 1^6, 21^6, 2^6$ or $2^5 1^2$. The necessary character theory already exists.

The examples we have discussed expose most of the problems that arise in the evaluation of $3jm$ factors and encourage the view that it is comparatively simple to evaluate $6j$ symbols and $3jm$ factors directly in the physical group structure without transforming to nonphysical canonical group structures.

The $3jm$ factors given here are fully symmetrized and permit full use of the Wigner–Racah calculus to be made. These $3jm$ factors have been used to compute the matrix elements of the generators of E_7 in a particular basis for the fermion and boson sectors. It does not appear difficult to obtain the results for other bases. The calculations reported here will form the basis for a more detailed study of symmetry breaking in E_7 models.

References

Butler, P. H. (1975). *Philos. Trans. R. Soc. London* A **277**, 545.
Butler, P. H. (1979). *In* 'Recent Advances in Group Theory' (Ed. J. C. Donini) (Plenum: New York).
Butler, P. H., Haase, R. W., and Wybourne, B. G. (1978). *Aust. J. Phys.* **31**, 131.
Butler, P. H., and King, R. C. (1974). *Can. J. Math.* **26**, 328.
Butler, P. H., and Wybourne, B. G. (1976*a*). *Int. J. Quantum Chem.* **10**, 581.
Butler, P. H., and Wybourne, B. G. (1976*b*). *Int. J. Quantum Chem.* **10**, 615.
Cung, V. K., and Kim, C. W. (1977). *Phys. Lett.* B **69**, 359.
Gell-Mann, M., Ramond, P., and Slansky, R. (1978). *Rev. Mod. Phys.* **50**, 721.
Gürsey, F., Ramond, P., and Sikivie, P. (1975). *Phys. Rev. Lett.* **12**, 2166.
Gürsey, F., and Sikivie, P. (1976). *Phys. Rev. Lett.* **36**, 775.
Judd, B. R. (1963). 'Operator Techniques in Atomic Spectroscopy' (McGraw-Hill: New York).
Ramond, P. (1976). *Nucl. Phys.* B **110**, 214.
Ramond, P. (1977). *Nucl. Phys.* B **126**, 509.
Rotenberg, M., Bivins, R., Metropolis, N., and Wooten, J. K. (1959). 'The 3-*j* and 6-*j* Symbols' (M.I.T. Press: Boston, Mass.).
Saclioglu, C. (1977). *Phys. Rev.* D **15**, 2267.
Sikivie, P., and Gürsey, F. (1977). *Phys. Rev.* D **16**, 816.
Wybourne, B. G. (1970). 'Symmetry Principles and Atomic Spectroscopy' (with an Appendix of Tables by P. H. Butler) (Wiley–Interscience: New York).
Wybourne, B. G. (1973). *Int. J. Quantum Chem.* **7**, 1117.
Wybourne, B. G. (1974). 'Classical Groups for Physicists' (Wiley–Interscience: New York).
Wybourne, B. G. (1978). *J. Math. Phys. (New York)* **19**, 529.
Wybourne, B. G., and Bowick, M. J. (1977). *Aust. J. Phys.* **30**, 259.

Manuscript received 7 August 1978