*Aust. J. Phys.*, 1981, **34**, 261–5

# Spatially Homogeneous and Anisotropic Cosmological Models in Brans–Dicke Theory

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*Abstract*

Spatially homogeneous and anisotropic cosmological models corresponding to Bianchi type I solutions of Brans–Dicke theory are investigated. The physical and geometrical properties of the models are discussed and compared with the corresponding relativistic models.

## 1. Introduction

In recent years, the effect of neutrino viscosity in the primordial fire ball (Doroshkevich *et al.* 1967; Misner 1967) has stimulated theoretical interest in the study of anisotropic cosmological models, and a large number of solutions of Einstein's theory have been obtained by Ellis and MacCallum (1969), Matzner (1969), Johri (1972), Johri and Lal (1975), Liang (1976), Bandyopadhyay (1977), Roy and Singh (1977), and others. In this context we obtain spatially homogeneous and anisotropic cosmological solutions of the Brans–Dicke theory. The Bianchi type I metric is considered and the energy–momentum tensor is taken to be that of a perfect fluid. Using the equation of state $p = (\gamma-1)\rho$, we can find the exact solutions corresponding to $\gamma = 1$ and $\gamma = \frac{4}{3}$, i.e. for dust-filled and radiation-dominated stages of the universe. These solutions generalize the isotropic solutions obtained by Brans and Dicke (1961). Various physical and geometrical properties of the models are then discussed.

## 2. Brans–Dicke Field Equations

We start with the Bianchi type I metric

$$ds^2 = dt^2 - A^2\,dx^2 - B^2\,dy^2 - C^2\,dz^2\,, \tag{1}$$

where we take $A$, $B$ and $C$ to be functions of the time $t$ only. The Brans–Dicke field equations are

$$R_{ij} - \tfrac{1}{2}Rg_{ij} = -8\pi\phi^{-1}T_{ij} - w\phi^{-2}(\phi_{,i}\phi_{,j} - \tfrac{1}{2}g_{ij}\phi_{,k}\phi^{k}) - \phi^{-1}(\phi_{i;j} - g_{ij}\phi^{k}{}_{;k})\,, \tag{2a}$$

$$\phi^{k}{}_{;k} = 8\pi T/(2w+3)\,, \tag{2b}$$

where we take $\phi$ to be a function of $t$.

The energy–momentum tensor for a perfect fluid is

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}, \tag{3}$$

with the equation of state

$$p = (\gamma - 1)\rho. \tag{4}$$

The contracted Bianchi identity is

$$T^{ij}_{;j} = 0. \tag{5}$$

The field equations (2) in terms of the metric (1) are

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} = -\frac{8\pi p}{\phi} - \frac{w}{2}\frac{\phi_4^2}{\phi^2} + \frac{A_4 \phi_4}{A\phi} - \frac{\phi^k_{;k}}{\phi}, \tag{6a}$$

$$\frac{C_{44}}{C} + \frac{A_{44}}{A} + \frac{A_4 C_4}{AC} = -\frac{8\pi p}{\phi} - \frac{w}{2}\frac{\phi_4^2}{\phi^2} + \frac{B_4 \phi_4}{B\phi} - \frac{\phi^k_{;k}}{\phi}, \tag{6b}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} = -\frac{8\pi p}{\phi} - \frac{w}{2}\frac{\phi_4^2}{\phi^2} + \frac{C_4 \phi_4}{C\phi} - \frac{\phi^k_{;k}}{\phi}, \tag{6c}$$

$$\frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{C_4 A_4}{CA} = \frac{8\pi\rho}{\phi} + \frac{w}{2}\frac{\phi_4^2}{\phi^2} - \frac{(ABC)_4 \phi_4}{ABC\,\phi}, \tag{6d}$$

$$\frac{\phi_{44}}{\phi} + \frac{(ABC)_4 \phi_4}{ABC\,\phi} = \frac{8\pi(\rho - 3p)}{(2w+3)\phi}. \tag{6e}$$

Equation (5) gives

$$\frac{\rho_4}{\rho} + \gamma\frac{(ABC)_4}{ABC} = 0, \tag{7}$$

and integrating this equation, we get

$$\rho s^\gamma = \rho_0 s_0^\gamma, \tag{8}$$

where $s = ABC$, and $\rho_0$ and $s_0$ are the density and volume element at a given time $t_0$.

Adding equations (6a)–(6c) to three times (6d), we get

$$\frac{s_{44}}{s} = \frac{12\pi(\rho - p)}{\phi} - \frac{3\phi_{44}}{2\phi} - \frac{5(ABC)_4 \phi_4}{2ABC\,\phi}. \tag{9}$$

Now we use the coordinate transformation

$$\mathrm{d}t \to \phi^{-1/2}\,\mathrm{d}\tau, \qquad s \to \phi^{-3/2}\bar{s}, \tag{10}$$

so that equations (9) and (6e) are transformed to

$$\frac{\bar{s}_{\tau\tau}}{\bar{s}} = \frac{12\pi(\rho - p)}{\phi^2}, \qquad \left(\frac{\phi_\tau}{\phi}\right)_\tau + \frac{\bar{s}_\tau \phi_\tau}{\bar{s}\phi} = \frac{8\pi(\rho - 3p)}{(2w+3)\phi^2}, \tag{11}$$

where differentiation with respect to $\tau$ is denoted by $\bar{s}_\tau$ etc.

## 3. Dust-filled Universe

For this case we have $\gamma = 1$, that is, $p = 0$. Equations (8) and (11) give the relations

$$s/s_0 = (\phi/\phi_0)^{3(w+1)}, \qquad \rho/\rho_0 = (\phi/\phi_0)^{-3(w+1)}. \tag{12}$$

Now integrating equation (6e) with the help of (12) we get

$$\phi/\phi_0 = \{r(t^2+2at+b)\}^{1/(3w+4)}, \tag{13}$$

where $r = (4\pi\rho_0/\phi_0)\{(3w+4)/(2w+3)\}$ and $a$ and $b$ are arbitrary constants. From the point of view of Mach's principle, we have $\phi \to 0$ as $t \to 0$, and therefore $b = 0$, which gives

$$\phi/\phi_0 = \{r(t^2+2at)\}^{1/(3w+4)}, \tag{14a}$$

$$\rho/\rho_0 = \{r(t^2+2at)\}^{-3(w+1)/(3w+4)}, \tag{14b}$$

$$s/s_0 = \{r(t^2+2at)\}^{3(w+1)/(3w+4)}. \tag{14c}$$

The metric coefficients $A$, $B$ and $C$ may be evaluated with the help of equations (6a)–(6c) and (14) to give

$$A = A_0\{t/(t+2a)\}^{P_1}s^{1/3}, \tag{15a}$$

$$B = B_0\{t/(t+2a)\}^{P_2}s^{1/3}, \tag{15b}$$

$$C = C_0\{t/(t+2a)\}^{P_3}s^{1/3}. \tag{15c}$$

where $A_0$, $B_0$, $C_0$, $P_1$, $P_2$ and $P_3$ are arbitrary constants satisfying

$$A_0 B_0 C_0 = 1, \qquad P_1+P_2+P_3 = 0. \tag{16}$$

The model has the following properties:

(1) It reduces to (*a*) the isotropic model with $A = B = C$ if $P_1 = P_2 = P_3$; (*b*) the locally rotational symmetric model with $B = C$ if $P_2 = P_3$; and (*c*) a special class of the Heckmann–Schucking model with $A^2 = BC$ if $2P_1 = P_2+P_3$.

(2) From the energy densities of the Brans–Dicke and corresponding relativistic model, we find

$$\rho_{\mathrm{BD}}/\rho_{\mathrm{rel}} \propto (t^2+2at)^{1/(3w+4)},$$

which indicates that the density decreases more rapidly with expansion in the relativistic model compared with the Brans–Dicke.

(3) The components of the expansion tensor are

$$\theta_{ii} = \frac{2(w+1)}{3w+4}\frac{t+a}{t^2+2at}+\frac{2aP_i}{t^2+2at}, \qquad i = 1,2,3,$$

and therefore the expansion scalar is given by

$$\theta = \tfrac{1}{3}(\theta_{11}+\theta_{22}+\theta_{33}) = \frac{2(w+1)}{3w+4}\frac{t+a}{t^2+2at}.$$

These expressions show that the singularities occurring in the anisotropic Brans–Dicke model are similar to those occurring in the corresponding relativistic model.

(4) The shear scalar is given by

$$3\sigma^2 = (\theta_{11}+\theta_{22}+\theta_{33})^2 - 3(\theta_{11}\theta_{22}+\theta_{22}\theta_{33}+\theta_{33}\theta_{11}),$$

or $\sigma = \sigma_0/(t^2+2at)$, where $\sigma_0$ is an arbitrary constant. This gives the following relation between the energy density and shear scalar:

$$\rho = k'\sigma^{3(w+1)/(3w+4)}.$$

(5) The relative anisotropy, given by

$$\sigma^2/\rho = k'(t^2+2at)^{-(3w+5)/(3w+4)},$$

is the ratio of the anisotropic energy (except for collisionless radiation) to the total energy of the universe. This expression suggests that in a dust-filled Brans–Dicke universe, the anisotropic energy decreases more rapidly with time in comparison with the total energy. A similar result also holds for the relativistic model.

(6) The rate of change of the gravitational constant is given by

$$\left(\frac{G_4}{G}\right)_0 = -\left(\frac{\phi_4}{\phi}\right)_0 = -\frac{2(t_0+a)}{(3w+4)(t_0+2at_0)} = -\frac{H_0}{(w+1)},$$

where $H_0$ is the present value of Hubble's constant. This expression shows that the gravitational constant decreases with time in this model.

## 4. Radiation-filled Universe

For this case we have $\gamma = \frac{4}{3}$, that is, $p = \frac{1}{3}\rho$. Proceeding exactly in the same way as in the previous section, we get the following solution for the radiation-filled universe:

$$\tau = \bar{s}^{1/3}u - M(3k)^{-3/2}\log\{(3ku)^{1/2}+3k\bar{s}^{1/3}\},$$

$$\phi = \phi_0\{(u-M^{1/2})/(u+M^{1/2})\}^{3L/2M^{1/2}}, \qquad \rho = k\phi^2/\bar{s}^{1/3},$$

$$A = A_0\{(u-M^{1/2})/(u+M^{1/2})\}^{P_1}\bar{s}^{1/3},$$

$$B = B_0\{(u-M^{1/2})/(u+M^{1/2})\}^{P_2}\bar{s}^{1/3},$$

$$C = C_0\{(u-M^{1/2})/(u+M^{1/2})\}^{P_3}\bar{s}^{1/3},$$

where $u = (3k\bar{s}^{2/3}+M)^{1/2}$, and $M$, $L$, $k$, $A_0$, $B_0$, $C_0$, $P_1$, $P_2$ and $P_3$ are arbitrary constants with

$$P_1+P_2+P_3 = 0, \qquad A_0B_0C_0 = 1.$$

The properties of the model are similar to those discussed in the previous section.

## 5. Conclusions

The Brans–Dicke solutions obtained here have a close correspondence with the relativistic solutions, and as such the singularities occurring in the solutions of the two theories follow the same pattern in the case of Bianchi type I models. In conclusion, we hope at least that our investigation will lead to deeper understanding of the Einstein and Brans–Dicke theories

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Manuscript received 22 April 1980, accepted 12 January 1981