Aust. J. Phys., 1981, **34**, 303–16

Corotation and Poloidal Flow in Pulsar Magnetospheres

R. R. Burman

Department of Physics, University of Western Australia, Nedlands, W.A. 6009.

Abstract

In recent years, authors have recognized that it is essential to allow for the non-corotational part $-\nabla\Phi$ of the electric field in pulsar magnetospheres, but have continued to neglect it in zones of corotation. It is shown here that, in order to understand the flow dynamics and avoid absurdities, it is essential to use the correct functional form for Φ everywhere outside the star—without exception. Implications include a potential difference between zones of corotating electrons and ions, resulting in a large local $\nabla\Phi$ field in vortical boundary layers separating those zones, directed so as to force a mixing of electrons and ions there. This result clashes sharply with a well-known conclusion of Holloway, based on use of the zero-inertia approximation with Φ piecewise constant on magnetic field lines: he found the electron and ion zones to be separated by evacuated regions. A further consequence of the present analysis is a possible mirroring of electrons between the northern and southern boundary layers, while they drift toward the star because of dissipative effects. These findings are not restricted to the usual axisymmetric case, but are valid for the oblique rotator, with non-aligned magnetic and rotation axes, so long as zones of corotation exist.

1. Introduction

In a recent paper (Burman 1980), I have given a careful analysis of flow dynamics in steadily rotating neutron star magnetospheres, using the dynamical equations expressing balance between the Lorentz force and relativistic inertia, neglecting dissipative forces. Inaccuracies and obscurities in the work of other authors were avoided by using the exact equations with inertial effects fully incorporated. The essential features of the flow dynamics were thus elucidated. The paper was based on the relativistic equations of continuity and motion of pressure-free charged fluids, together with the source-free members of Maxwell's set of equations, namely Faraday's law and $\nabla . \boldsymbol{B} = 0$, under the steady-rotation constraint. The electrodynamic equations containing source terms, namely the Ampère–Maxwell and Gauss laws, were not used. Nor was any particular spatial symmetry invoked, so the results are valid for the general oblique rotator, not just for the usual axisymmetric case.

The purpose of the previous work was not to propose any particular model but to reach an understanding of the essential features of the dissipation-free flow dynamics for the general oblique rotator. It is now time to begin deducing implications of those results for model building. In this paper, I shall concentrate on magnetospheric regions in which the particles have azimuthal velocity components that are close to the local speed of corotation.

2. Basic Formalism

The canonical pulsar model consists of a rotating neutron star with the magnetic axis inclined to the rotation axis. Although the electrical resistivity of the stellar material may bring about decay of the star's magnetic field over millions of years (Manchester and Taylor 1977), the star can be regarded as a perfect conductor for the purpose of studying magnetospheric structure. Let ϖ, ϕ and z be cylindrical polar coordinates with the positive z axis as the rotation axis. The system is steadily rotating at angular frequency Ω. Hence, it follows from Faraday's law and $\nabla . \boldsymbol{B} = 0$ that the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are connected by (Mestel 1971)

$$\boldsymbol{E} + c^{-1}\Omega\varpi\boldsymbol{t} \times \boldsymbol{B} = -\nabla\Phi,$$

where c is the vacuum speed of light, $\boldsymbol{t}$ is the unit toroidal vector and the gauge-invariant quantity Φ is related to the familiar scalar and vector potentials ϕ and $\boldsymbol{A}$ by $\Phi \equiv \phi - (\Omega\varpi/c)A_\phi$ (Endean 1972*a*); Φ is the potential for the non-corotational part $-\nabla\Phi$ of the electric field.

The magnetospheric plasma is taken to be cold and non-dissipative, with inertia the only non-electromagnetic term in the equation of motion of each species. Endean (1972*a*, 1972*b*) derived from the steady-rotation constraint an integral Ψ_k of the motion for particles of species k. Using the Endean integral, together with a fluxoid conservation theorem (Buckingham *et al.* 1972, 1973), Burman and Mestel (1978) reduced the equation of motion of each species to a simple form. They went on to point out that if all particles of species k are nonrelativistic when arbitrarily near the star, then the fixed value taken by Ψ_k over the stellar surface is propagated along the streamlines of species k throughout whatever portion of the magnetosphere contains particles of that species: Ψ_k is constant throughout all space occupied by species k, instead of having separate constant values on different streamlines of the species. (The term 'streamlines' refers here to lines of the reduced flow velocity $\boldsymbol{u}_k$, defined as $\boldsymbol{v}_k - \Omega\varpi\boldsymbol{t}$ where $\boldsymbol{v}_k$ is the fluid velocity of species k.) The following integral of the motion for species k results (Burman and Mestel 1978):

$$1 - \frac{e_k \Phi}{m_k c^2} = \gamma_k\left(1 - \frac{\Omega\varpi}{c}\frac{v_{k\phi}}{c}\right), \tag{1}$$

where e_k, m_k and γ_k represent the charge, rest mass and Lorentz factor of the particles of that species.

For equation (1) to be valid, the particles concerned must, if leaving the star, be emitted with nonrelativistic speeds; if returning to the star, or accreted by it, they must be decelerated so as to be nonrelativistic on impact. But use of the integral (1) does not prejudge the question of whether emitted particles become relativistic only near the light cylinder $\Omega\varpi/c = 1$, or whether the component of the electric field parallel to the magnetic field accelerates them to high Lorentz factors near the star: it is necessary for the particles to be nonrelativistic in only an arbitrarily thin neighbourhood of the stellar surface. This restriction may be less significant than might appear at first sight: I know of no reason to expect emitted particles to emerge from the stellar surface with relativistic speeds and, as discussed by Mestel *et al.* (1979), circulating particles may be decelerated to nonrelativistic speeds before returning to the star. Even accreting particles, if there are any, will be strongly centrifugally decelerated (Burman 1981*b*) and could be nonrelativistic on impact.

3. Corotation and Poloidal Flow

For convenience, let x denote the dimensionless cylindrical polar radial coordinate $\Omega\varpi/c$. Let $v_{k\mathrm{p}}$ denote the poloidal flow speed of species k. Suppose that, in some region inside the light cylinder, $v_{k\phi} \approx \Omega\varpi$ and $v_{k\mathrm{p}}^2/c^2 \ll \min(x^2, 1-x^2)$ for some species; since the azimuthal speed is close to the corotational value and the poloidal speed is considerably smaller, this species is essentially just corotating with the star. The definition of the Lorentz factor becomes $\gamma_k \approx (1-x^2)^{-\frac{1}{2}}$ and the integral (1) of the motion becomes $\gamma_k \approx (1 - e_k\Phi/m_k c^2)/(1-x^2)$. Eliminating γ_k shows that (Burman 1980)

$$e_k\Phi/m_k c^2 \approx 1-(1-x^2)^{\frac{1}{2}}, \tag{2}$$

for an essentially corotating species. Note that Φ is proportional to the mass to charge ratio of species k: at least for conditions under which the integral (1) is valid, there can, at each point, be species of only a single charge to mass ratio (in sign as well as magnitude) essentially just corotating with the star.

Equation (2) shows that

$$\frac{e_k}{m_k c^2}\nabla\Phi \approx \frac{\Omega}{c}\frac{x}{(1-x^2)^{\frac{1}{2}}}\boldsymbol{i}, \tag{3}$$

where $\boldsymbol{i}$ denotes a unit vector in the ϖ direction. For a corotating species, the non-corotational electric force $-e_k\nabla\Phi$ on each particle acts in the $-\boldsymbol{i}$ direction, balancing the centrifugal effect; this occurs for either sign of charge on the particles. At any point, balance can obtain for species of only a single mass to charge ratio and, for balance, Φ must be proportional to that quantity. It follows from equation (3) that

$$\nabla^2\Phi \approx \frac{m_k\Omega^2}{e_k}\frac{2-x^2}{(1-x^2)^{3/2}}, \tag{4}$$

which will be used in Section 4 below.

Consider now a zone inside the light cylinder in which $v_{k\phi} \approx \Omega\varpi$ for some species but $v_{k\mathrm{p}}$ may be of similar order to $v_{k\phi}$, so that the zone is not necessarily one of corotation but can have a substantial poloidal flow. The definition of the Lorentz factor becomes $\gamma_k \approx (1-x^2-v_{k\mathrm{p}}^2/c^2)^{-\frac{1}{2}}$ and the integral (1) of the motion takes the same form as for corotation, but with the new γ_k. Eliminating γ_k shows that (Burman 1980)

$$\frac{e_k\Phi}{m_k c^2} \approx 1-\frac{1-x^2}{(1-x^2-v_{k\mathrm{p}}^2/c^2)^{\frac{1}{2}}}. \tag{5}$$

Since its right-hand side is not independent of the species, equation (5), unlike equation (2), does not imply that Φ depends on the mass to charge ratio of species k. Rather, equation (5) can be read as giving, when Φ is known, the poloidal flow speeds of species that are not in balance with respect to the centrifugal and non-corotational electric forces. If there is a species present that is essentially just corotating with the star, then Φ has to be given by equation (2) with the mass to charge ratio of that species; equation (5) then gives immediately the poloidal speeds of other species there with azimuthal speeds close to $\Omega\varpi$.

Equation (5) shows that

$$\gamma_k^{-3}\left(\frac{e_k}{m_k c^2}\right)\tilde{\nabla}\Phi \approx x\left(\gamma_k^{-2} - \frac{v_{kp}^2}{c^2}\right)\boldsymbol{i} - \tfrac{1}{2}(1-x^2)\,\tilde{\nabla}\left(\frac{v_{kp}^2}{c^2}\right), \tag{6}$$

where $\tilde{\nabla}$ denotes a dimensionless del operator $(c/\Omega)\nabla$. Increase of v_{kp} with x corresponds to reduced ability of the non-corotational electric force to oppose the centrifugal effect; in fact, equation (6) shows that the two effects assist each other when

$$\tfrac{1}{2}(1-x^2)\frac{\partial}{\partial x}\left(\frac{v_{kp}^2}{c^2}\right) > x\left(\gamma_k^{-2} - \frac{v_{kp}^2}{c^2}\right), \tag{7}$$

which, for $x^2 \ll 1$, reduces to

$$\tfrac{1}{2}\,\partial_x(v_{kp}^2/c^2) > x\,. \tag{8}$$

It follows from equation (6) that

$$\gamma_k^{-3}\left(\frac{e_k}{m_k c^2}\right)\tilde{\nabla}^2\Phi \approx 1+\gamma_k^{-2} - 3\gamma_k^2\left(1-\frac{v_{kp}^2}{c^2}\right)\frac{v_{kp}^2}{c^2}$$

$$-\left(1+3\gamma_k^2\frac{v_{kp}^2}{c^2}\right)x\frac{\partial}{\partial x}\left(\frac{v_{kp}^2}{c^2}\right) - \tfrac{3}{4}(1-x^2)\left\{\gamma_k\,\tilde{\nabla}\left(\frac{v_{kp}^2}{c^2}\right)\right\}^2 - \tfrac{1}{2}(1-x^2)\,\tilde{\nabla}^2\left(\frac{v_{kp}^2}{c^2}\right), \tag{9}$$

which will be used in Section 4.

4. The Axisymmetric Rotator

So far, no particular spatial symmetry has been invoked, and the results are valid for the general oblique rotator with an arbitrary angle between the magnetic and rotation axes. Axisymmetric 'pulsars', for which these axes are either parallel or antiparallel, will be discussed in this section. The results for these two cases swap over on interchanging positive and negative charges; of course, complete models for the two cases will differ substantially because of the great difference in mass between protons and electrons, but the ideas developed here are interchangeable.

If only a single species is present in a region, or if all species present have the same azimuthal velocity component v_ϕ, then the azimuthal component of the electric current density in that region is $\rho^e v_\phi$, where ρ^e denotes the electric charge density. Then, in the axisymmetric case, the Gauss and Ampère laws combine to give (Mestel *et al.* 1979)

$$\nabla^2\Phi + (2\Omega/c)B_z = -4\pi\rho^e(1 - xv_\phi/c)\,. \tag{10}$$

Substituting equation (4) for $\nabla^2\Phi$, together with $v_\phi \approx \Omega\varpi$, into equation (10) shows that

$$\rho^e \approx -\frac{\Omega B_z/2\pi c}{1-x^2} - \frac{m_k\Omega^2}{4\pi e_k}\frac{2-x^2}{(1-x^2)^{5/2}}, \tag{11}$$

in a zone of the axisymmetric model in which species k is essentially just corotating with the star. The first term on the right-hand side is the well-known Goldreich–Julian (1969) charge density. The second term is an inertial effect, as is indicated by the appearance of the particle rest mass in the numerator, and has arisen from allowing Φ to be nonzero. Since $\gamma_k \approx (1-x^2)^{-\frac{1}{2}}$, equation (11) can be written

$$\rho^{\mathrm{e}} \approx -\frac{\gamma_k^2}{4\pi}\left(\frac{2\Omega B_z}{c}+\frac{m_k \Omega^2}{e_k}(\gamma_k^3+\gamma_k)\right), \tag{11a}$$

which is the same as equation (3.9) of Mestel *et al.* (1979). Hence

$$\gamma_k \lesssim \{(-e_k B_z/m_k c)\Omega^{-1}\}^{\frac{1}{3}} \tag{12}$$

with approximate equality holding near the boundary of the zone (Wang 1978; Mestel *et al.* 1979).

In the Goldreich–Julian (1969) aligned model, the corotating region is, from their charge density expression, positively charged where $B_z < 0$ and negatively charged where $B_z > 0$. Goldreich and Julian found that outflowing positive ions pass through a negatively charged region. Instability and dissipation are to be expected here. Okamoto (1974) showed that the zero-inertia approximation leads to an unacceptable magnetic field structure in the charge-separated case: the magnetic field line separating the regions of positive and negative corotating charge must be a straight line parallel to the equator.

In the Goldreich–Julian theory, the $\rho^{\mathrm{e}} = 0$ surfaces in the corotating region coincide with the $B_z = 0$ surfaces. Equation (11) for ρ^{e} shows that inclusion of particle inertia through the non-corotational electric potential alters the $\rho^{\mathrm{e}} = 0$ surfaces to

$$-\Omega^{-1} e_k B_z/m_k c \approx \tfrac{1}{2}(2-x^2)/(1-x^2)^{3/2} \tag{13a}$$

$$\approx \tfrac{1}{2}(\gamma_k^3+\gamma_k) \tag{13b}$$

(cf. equation 3.10 of Mestel *et al.* 1979). The left-hand side is minus the z component of the nonrelativistic vector gyrofrequency of species k, divided by the rotation rate of the pulsar; it is of large magnitude everywhere inside the light cylinder, save for the immediate neighbourhood of the $B_z = 0$ surfaces and for large values of $|z|$.

For convenience, the regions $z > 0$ and $z < 0$ will be referred to as 'northern' and 'southern' respectively. Equations (13) represent an 'inner' $\rho^{\mathrm{e}} = 0$ surface, with k corresponding to the species that is corotating at lower latitudes, together with two 'outer' $\rho^{\mathrm{e}} = 0$ surfaces, with k corresponding to the species that is corotating at higher latitudes. Except near the light cylinder, these surfaces are close to the $B_z = 0$ surfaces. In the northern part of the magnetosphere, the inner $\rho^{\mathrm{e}} = 0$ surface is just south of the $B_z = 0$ surface from the star until close to the light cylinder; there it drops sharply south until it approaches the southern $B_z = 0$ surface where it swings sharply northward and goes to the star, remaining just north of that $B_z = 0$ surface. The bulk of the magnetosphere enclosed by the inner $\rho^{\mathrm{e}} = 0$ surface and the surface of the star contains a corotating species and has $\rho^{\mathrm{e}} > 0$ or $\rho^{\mathrm{e}} < 0$ according as the stellar magnetic and rotation axes are parallel or antiparallel.

The northern outer $\rho^{\mathrm{e}} = 0$ surface is just north of the $B_z = 0$ surface from the star until close to the light cylinder where it heads further north. The southern outer $\rho^{\mathrm{e}} = 0$ surface is just south of the $B_z = 0$ surface from the star until close to the light cylinder where it heads further south. The parts of the magnetosphere to the north of the northern outer $\rho^{\mathrm{e}} = 0$ surface and to the south of the southern $\rho^{\mathrm{e}} = 0$

surface that contain a corotating species have $\rho^{e} < 0$ or $\rho^{e} > 0$ according as the stellar magnetic and rotation axes are parallel or antiparallel.

It follows from the equations of motion and continuity of species k that (Burman 1981*a*)

$$\boldsymbol{u}_k \propto n_k^{-1}\{\boldsymbol{B}+(cm_k/e_k)\nabla\times(\gamma_k \boldsymbol{v}_k)\} \tag{14}$$

as position varies along any $\boldsymbol{u}_k$ line; n_k denotes the number density of this species. The term in the expression for $\boldsymbol{u}_k$ that involves the relativistic vorticity $\nabla\times(\gamma_k \boldsymbol{v}_k)$ represents an 'inertial drift' (Mestel *et al.* 1979; Burman and Mestel 1979). If inertial drifts are neglected, then the relation (14) becomes $\boldsymbol{u}_k \propto \boldsymbol{B}/n_k$ as position varies along the common lines of $\boldsymbol{u}_k$ and $\boldsymbol{B}$. In the corotating region of the Goldreich–Julian (1969) model with only a single species present at each point, $n_k \propto B_z/(1-x^2)$: as discussed by Mestel *et al.* (1979) and Mestel (1980), the poloidal flow must be sharply accelerated in the vicinity of the $B_z = 0$ surfaces of this model. As I have pointed out previously (Burman 1981*a*), the relation (14) shows that, much more generally, rapid poloidal accelerations, corresponding to development of vorticity, will occur as n_k becomes small. Regions of inertial development of vorticity form natural boundary layers to domains in which a species is concentrated (Burman 1981*a*). In particular, the $\rho^{e} = 0$ surfaces just discussed will be enclosed in these boundary layers.

For a zone of an axisymmetric model in which $v_{k\phi} \approx \Omega\varpi$ and $v_{k\mathrm{p}}$ may be of similar order to $v_{k\phi}$, substitution of equation (9) for $\nabla^2\Phi$ into equation (10) shows that

$$\rho^{e} \approx \rho^{\mathrm{GJ}} + \rho^{\Phi\mathrm{P}} + \rho^{\mathrm{PA}}, \tag{15}$$

where ρ^{GJ} is the Goldreich–Julian charge density while

$$\rho^{\Phi\mathrm{P}} = -\frac{m_k\Omega^2}{4\pi e_k}\frac{1}{1-x^2}\left\{\gamma_k^3+\gamma_k-3\gamma_k^5\left(1-\frac{v_{k\mathrm{p}}^2}{c^2}\right)\frac{v_{k\mathrm{p}}^2}{c^2}\right\} \tag{16a}$$

and

$$\rho^{\mathrm{PA}} = \frac{m_k\Omega^2}{4\pi e_k}\frac{\gamma_k^3}{1-x^2}\left[\left(1+3\gamma_k^2\frac{v_{k\mathrm{p}}^2}{c^2}\right)x\frac{\partial}{\partial x}\left(\frac{v_{k\mathrm{p}}^2}{c^2}\right)+\tfrac{3}{4}(1-x^2)\left\{\gamma_k\tilde{\nabla}\left(\frac{v_{k\mathrm{p}}^2}{c^2}\right)\right\}^2\right.$$
$$\left.+\tfrac{1}{2}(1-x^2)\tilde{\nabla}^2\left(\frac{v_{k\mathrm{p}}^2}{c^2}\right)\right]. \tag{16b}$$

The quantity $\rho^{\Phi\mathrm{P}}$ is the contribution to the charge density introduced by allowing for the non-corotational electric field $-\nabla\Phi$ in a zone with substantial poloidal flow of species k. When $v_{k\mathrm{p}}^2/c^2 \ll \max(x^2, 1-x^2)$, meaning that the poloidal flow of this species is negligible, $\rho^{\Phi\mathrm{P}}$ reduces to $\rho^{\Phi\mathrm{C}}$ where

$$\rho^{\Phi\mathrm{C}} = -(m_k\Omega^2/4\pi e_k)(2-x^2)/(1-x^2)^{5/2}; \tag{17}$$

this is the second term on the right-hand side of equation (11) above, and represents the contribution to the charge density in the zone of corotation arising from the inclusion of inertial effects through the potential Φ. The quantity ρ^{PA} is a contribution to the charge density introduced by allowing for the acceleration of the poloidal flow, which will be important where $\rho^{\mathrm{GJ}}+\rho^{\Phi\mathrm{C}}$ is small. For $x^2 \ll 1$, the Goldreich–Julian charge density reduces to $-\Omega B_z/2\pi c$ (Goldreich 1969) while $\rho^{\Phi P}$ and $\rho^{\Phi C}$ reduce to $-m_k\Omega^2/2\pi e_k$, which is negligible. At least for poloidal flow that is being accelerated away from the star, or decelerated as it approaches the star, the contribution ρ^{PA} to the charge density has the same sign as that of the particles in the flow, provided $\nabla^2(v_{kp}^2/c^2)$ is not a large negative quantity.

The above considerations suggest a picture of the axisymmetric pulsar magnetosphere which has features that may be described as follows. There is a zone containing corotating particles bounded within a surface, defined by equations (13), which is enclosed within the $B_z = 0$ surfaces. In the bulk of this zone, ρ^e is given by the Goldreich–Julian (1969) formula; towards the boundary, the inertial contribution $\rho^{\Phi C}$ becomes important. At higher latitudes, $\rho^{GJ} + \rho^{\Phi C}$ changes sign, corresponding to zones containing corotating particles of opposite sign to those within the $B_z = 0$ surfaces.

5. Implications for Model Building

I shall now treat the dynamics of poloidally flowing particles streaming through zones in which a different species is corotating. Most of this discussion is not restricted to the axisymmetric case but is also valid for any oblique rotator for which zones of corotation exist.

In a zone of corotation, the centrifugal effect on the corotating particles is balanced by the non-corotational electric force $-e_k\nabla\Phi$, given by equation (3), which is directed toward the rotation axis. Consider a zone of corotating positively charged particles. Any electron finding itself in this zone will experience a non-corotational electric force acting in the $+\boldsymbol{i}$ direction, which is assisted by the centrifugal effect. Any positively charged particle having a different charge to mass ratio from that of the corotating species will experience a non-corotational electric force acting toward the axis of symmetry, but not of the right magnitude to balance the centrifugal effect: the net force will be either toward the rotation axis or away from it according as the charge to mass ratio is greater or less than that of the corotating species.

A net force $F_n\boldsymbol{i}$ per particle of species n will produce two effects. Since it has a nonzero component along the poloidal part $\boldsymbol{B}_P$ of the magnetic field, except where $B_\varpi = 0$, the unbalanced force will drive an accelerated poloidal motion of this species approximately along the $\boldsymbol{B}_P$ lines. The other effect is a toroidal drift of species n with velocity $\boldsymbol{q}_n$ given by

$$\boldsymbol{q}_n/c = (F_n/e_n B^2)\boldsymbol{i}\times\boldsymbol{B}_P. \tag{18}$$

Imagine that there is an electron in a zone of corotating positively charged ions at a point where the corotating ions are of atomic number Z and mass number A, and that the electron is, instantaneously, essentially just corotating with the star. Let $-e$ denote the electronic charge and let m_e and m_p denote the rest masses of the electron and proton. The non-corotational electric force $-Ze\nabla\Phi$ on each ion is balanced by the centrifugal effect. For the electron, the non-corotational electric force $e\nabla\Phi$ is much greater than the centrifugal 'force', which is $ZA^{-1}(m_e/m_p)e\nabla\Phi$. Thus, the unbalanced force $\boldsymbol{F}_e$ on the electron is very close to $e\nabla\Phi$, with $\nabla\Phi$ given by equation (3) with parameters appropriate to the ion zone; thus

$$\boldsymbol{F}_e \approx (A/Z)\gamma m_p\Omega^2\varpi\boldsymbol{i}, \tag{19}$$

where γ denotes the corotational Lorentz factor $(1-x^2)^{-\frac{1}{2}}$. Equation (18) shows that the resulting toroidal drift velocity, relative to the motion of corotation, is $\boldsymbol{w}_e$ where

$$\frac{\boldsymbol{w}_e}{\Omega\varpi} \approx \frac{A}{Z}\frac{\Omega}{eB/m_p c}\gamma\frac{\boldsymbol{B}_P \times \boldsymbol{i}}{B}. \tag{20}$$

From the limit (12) on γ_k, the corotational Lorentz factor in an ion corotation zone of an axisymmetric rotator is less than about $(Ze\,|B_z|/Am_p c\Omega)^{\frac{1}{3}}$. Hence

$$\frac{w_e}{\Omega\varpi} \lesssim \left(\frac{\Omega}{eB/m_p c}\right)^{2/3}. \tag{21}$$

The factor $\Omega \div (eB/m_p c)$ represents the ratio of the pulsar rotation frequency to the nonrelativistic proton gyrofrequency, and is always extremely small in this zone. Hence, throughout the ion corotation zone, the drift speed w_e is always very small compared with the speed of corotation. Thus, the only important effect of the unbalanced force on the electron is its driving of an accelerated poloidal motion approximately along the poloidal magnetic field lines.

Just outside the zone of corotating positively charged particles there is a boundary layer, defined in the axisymmetric case by $\rho^{GJ} + \rho^{\Phi C} \approx 0$, in which the process of inertial development of vorticity occurs. This boundary layer represents a skin of vortical current flow surrounding the positive-ion corotation zone. The inertial development of vorticity might involve departure of $v_{k\phi}$ from the value $\Omega\varpi$; if so, then the charge density for the axisymmetric case will not there be given accurately by equation (15) with equations (16).

Consider a zone of corotating electrons. Here the centrifugal effect on the electrons is balanced by the non-corotational electric force $e\nabla\Phi$, given by equation (3), which is directed toward the rotation axis. Any positively charged particle in this zone will experience a non-corotational electric force acting in the $+\boldsymbol{i}$ direction and so assisting the centrifugal effect. Such zones will end in boundary layers in which the electron number density tends to zero, so that acceleration of the electron poloidal flow must occur.

Imagine that there is a positive ion of atomic number Z and mass number A in a zone of corotating electrons, and that the ion is, instantaneously, essentially just corotating with the star. The non-corotational electric force $e\nabla\Phi$ on the electrons is balanced by the centrifugal effect. For the ion, the non-corotational electric force $-Ze\nabla\Phi$ is small compared with the centrifugal force, which is $-A(m_p/m_e)e\nabla\Phi$. Thus, the unbalanced force $\boldsymbol{F}_i$ on the ion is very close to $-A(m_p/m_e)e\nabla\Phi$, with $\nabla\Phi$ given by equation (3) with parameters appropriate to the electron zone; thus

$$\boldsymbol{F}_i \approx A\gamma m_p \Omega^2 \varpi \boldsymbol{i}. \tag{22}$$

The toroidal drift velocity produced by $\boldsymbol{F}_i$, relative to the motion of corotation, is $\boldsymbol{w}_i$ where

$$\frac{\boldsymbol{w}_i}{\Omega\varpi} \approx \frac{A}{Z}\frac{\Omega}{eB/m_p c}\gamma\frac{\boldsymbol{i} \times \boldsymbol{B}_P}{B}. \tag{23}$$

From the relation (12), the corotational Lorentz factor in an electron zone of an axisymmetric rotator is less than about $(eB_z/m_e c\Omega)^{\frac{1}{3}}$. Hence

$$\frac{w_i}{\Omega\varpi} \lesssim 10\left(\frac{\Omega}{eB/m_p c}\right)^{\frac{2}{3}}, \tag{24}$$

showing that, throughout the electron zone, the drift speed w_i is always small compared with the speed of corotation. Thus, the only important effect of this unbalanced force is its driving of an accelerated poloidal flow of positive ions approximately along the poloidal magnetic field lines.

In a zone of corotating electrons, Φ is a negative quantity proportional to the electron mass to charge ratio. In the zone of corotating positive ions, Φ is a positive quantity proportional to the ion mass to charge ratio: its magnitude exceeds that of Φ at similar values of x in the electron zone by a factor of a few thousand. Thus, there exists a gradient of Φ across the boundary layer separating these zones, directed from the electron zone to the ion zone.

Let Φ(i) and Φ(e) denote Φ at the same value of x in the zones of corotating ions and electrons respectively:

$$\Phi(\mathrm{i})/\Phi(\mathrm{e}) \approx -AZ^{-1}m_p/m_e, \tag{25}$$

where Z and A now refer to the corotating ions. The ratio $\Phi(\mathrm{i})/\Phi(\mathrm{e})$ is constant so long as the ion species does not change, and changes by only about a factor of 2 if the ion species changes from protons to heavier ions.

In the zones of corotation, the non-corotational electric force on the corotating particles is directed toward the rotation axis, balancing the centrifugal effect on those particles. But in the boundary layers separating those zones, the non-corotational electric field acts across the layer: the force it exerts there on positive ions is toward the electron zone while that on electrons is toward the ion zone. Thus charge mixing occurs in the boundary layers: rather than electron and ion zones terminating in electron and ion boundary layers, electrons and ions will mingle to form boundary layers of mixed plasma separating the zones.

Electrons in the boundary layers can leak across into the positive ion zone, where the force $\boldsymbol{F}_e$, given by equation (19), will produce a poloidal acceleration approximately along the magnetic field lines. Thus, for aligned and quasi-aligned rotators, mirroring of electrons between the boundary layers in the two hemispheres seems to be quite possible. A full analysis of this effect would have to include a study of the process of reflection in the boundary layers, incorporating dissipation because of the relative motions of the species there, together with allowance for dissipative effects as the electrons stream through the corotating positive ions. It is very probable that the energy losses will cause the electrons to move to lower magnetic field lines until they eventually return to the star, but this remains to be checked in a more complete theory.

For aligned and quasi-aligned rotators, I visualize electrons spilling over from the corotating electron zones at high latitudes into the boundary layers and thence into the zone of corotating positive ions, forming a curtain of electrons raining down from high to low latitudes, mirroring between the northern and southern boundary layers, gradually losing energy and drizzling down onto the star.

The potential difference across the boundary layer is given by

$$\Phi(\mathrm{i})-\Phi(\mathrm{e}) \approx AZ^{-1}(m_{\mathrm{p}}c^2/e)\{1-(1-x^2)^{\frac{1}{2}}\}, \tag{26}$$

and so increases monotonically from zero on the stellar surface to about $m_{\mathrm{p}}c^2/e$ near the light cylinder. Over most of the boundary layer, the potential difference is not sufficient to accelerate ions to relativistic energies. But the potential difference is capable of accelerating electrons to Lorentz factors given, for $x^2 \ll 1$, by

$$\gamma_{\mathrm{e}} \approx 1 + AZ^{-1}(m_{\mathrm{p}}/m_{\mathrm{e}})\tfrac{1}{2}x^2\,; \tag{27}$$

that is, for example, to $\gamma_{\mathrm{e}} \approx 200$ at $x = 1/2$ and to $\gamma_{\mathrm{e}} \approx 10$ at $x = 1/10$. In fact, there is sufficient potential difference available to accelerate electrons to relativistic speeds from $x \approx 1/30$ outwards.

A fundamental question in pulsar magnetosphere studies is this: Do electrons emitted from a pulsar surface rapidly reach highly relativistic energies very near the surface, accelerated by a substantial local electric field component parallel to the magnetic field (e.g. Sturrock 1971; Michel 1974), or is the poloidal electron flow everywhere nonrelativistic, with the electrons reaching highly relativistic energies because of centrifugal acceleration near the light cylinder (Mestel *et al.* 1979; Mestel 1980)? The above considerations say nothing about the former possibility, but do show that the latter cannot be valid: acceleration of the electron poloidal flow to relativistic speeds occurs from $x \approx 1/30$ outwards. Dissipation in the boundary layer might somewhat reduce the effectiveness of this acceleration mechanism, but it is clear that electrons will be accelerated to relativistic energies well inside the light cylinder, thus settling a point that has been argued about for years.

6. Physical Basis of the Implications

The steady-rotation constraint $\boldsymbol{E}+c^{-1}\Omega\varpi\boldsymbol{t}\times\boldsymbol{B} = -\nabla\Phi$ on the electromagnetic field implies that the Lorentz force on a particle of species k can be written (Burman and Mestel 1978) as $e_k(-\nabla\Phi + c^{-1}\boldsymbol{u}_k\times\boldsymbol{B})$. Use of the Endean integral and the fluxoid conservation theorem enables the equation of motion, equating the relativistic inertial term to the Lorentz force, to be expressed in a simple form (Burman and Mestel 1978). For a species that exists and is nonrelativistic in a neighbourhood, however thin, of the stellar surface, on which $\Phi = 0$, the integral (1) of the motion results. Also, the equation of motion reduces further (Burman and Mestel 1978) to a form which states that the reduced flow velocity $\boldsymbol{u}_k$ is along the magnetoidal field $\boldsymbol{B}+(cm_k/e_k)\nabla\times(\gamma_k\boldsymbol{v}_k)$; this is a generalized isorotation law. Thus, the Lorentz force can be written as

$$-e_k\nabla\Phi - m_k\boldsymbol{u}_k\times\{\nabla\times(\gamma_k\boldsymbol{v}_k)\}, \tag{28}$$

expressing it as the sum of the non-corotational electric force and an inertial contribution which is the negative of a relativistic Coriolis-type force.

For particles that are corotating with the star, with negligible poloidal motions, so that $\boldsymbol{u}_k = \boldsymbol{0}$, the Lorentz force reduces to the non-corotational electric force $-e_k\nabla\Phi$. If this and the inertial term are the only significant terms in the equation of motion, then $-e_k\nabla\Phi$ is just the centripetal force providing the acceleration directed toward the rotation axis. Thus, Φ must be a function of only the distance ϖ from the axis of rotation, and is given by equation (2) above, as derived previously

(Burman 1980). The fictitious centrifugal 'force' is the negative of the inertial term: conventionally, one describes the circular motion as representing a balance of the true force by the centrifugal 'force'; more strictly, the true force—here the non-corotational electric force—is the centripetal force providing the acceleration.

If the zero-inertia approximation is used for species k in some zone, then, in the absence of non-electromagnetic forces, the relation $\boldsymbol{E}+c^{-1}\boldsymbol{v}_k\times\boldsymbol{B}\approx\boldsymbol{0}$ holds there. Hence $\boldsymbol{E}.\boldsymbol{B}\approx 0$ and so the steady-rotation constraint implies that Φ is constant on magnetic field lines in that zone. Also, the magnetoidal field reduces to $\boldsymbol{B}$, so the usual isorotation law, stating that the reduced flow velocity $\boldsymbol{u}_k$ is along the magnetic field, applies. In this approximation, the equation of motion of any particle with $\boldsymbol{u}_k$ along $\boldsymbol{B}$, including particles that are just corotating with the star, is satisfied identically, since both the inertial term and the Lorentz force vanish.

After early attempts (e.g. Goldreich and Julian 1969) to apply the zero-inertia approximation throughout the pulsar magnetosphere gave unacceptable results, it became clear that the approximation must fail somewhere (e.g. Mestel 1973, 1974; Mestel *et al.* 1976), but it is still invariably used in zones of corotation. The considerations of this paper have shown that it is inadequate even there, for at least two reasons. One is that knowledge of Φ is needed in order to calculate the net force on any poloidally flowing species passing through a zone where another species is just corotating. The other is that the functional form of Φ demonstrates the existence and amount of a rapid change in its value across boundary layers separating electron and positive ion zones. It is clear that the behaviour of Φ must be properly accounted for even in zones of corotation.

Furthermore, not only is neglect of Φ inadequate: it can produce absurdities. Consider particles with negligible poloidal motions and try imposing the requirement that $\Phi=0$. The integral (1) of the motion leads to $v_{k\phi}=0$ or $2\Omega\varpi/(1+x^2)$, which also follow from my general solutions for $v_{k\phi}$ (Burman 1980) on putting $v_{kp}=0=\Phi$. But these two expressions for $v_{k\phi}$ are both absurd, for neither corresponds to particles that are corotating with the star: one describes particles that are actually stationary in the inertial frame; the other describes particles that have azimuthal speeds which, near the star, are twice the local speed of corotation, whereas particles just emitted from the star, or orbiting near it, should have azimuthal speeds close to the local speed of corotation. In fact, taking $e_k\Phi/m_k c^2$ to be approximately $\frac{1}{2}x^2$ gives (Burman 1980) the physically acceptable result that $v_{k\phi}\approx\Omega\varpi$ near the star for both solution branches. The point here, of course, is that $e_k\Phi/m_k c^2$ is of the same order of magnitude as $(v_{k\phi}/c)^2$ and one cannot neglect the former while including the latter.

I conclude that the functional form of Φ must be properly accounted for everywhere outside the star—without exception. Even where Φ is small, in the sense that $e_k\Phi/m_k c^2\ll 1$, neglect of Φ leads, locally, to absurdities and, globally, to the misunderstanding and even omission of essential physics.

7. Comparison with Holloway's Work

The deductions of this paper clash sharply with a well-known conclusion of Holloway (1973), who discussed charge-separated axisymmetric magnetospheres in the zero-inertia approximation, in which $\boldsymbol{E}$ is perpendicular to $\boldsymbol{B}$. Since $\boldsymbol{E}=-\nabla\phi$ for steadily rotating axisymmetric structures, the condition $\boldsymbol{E}.\boldsymbol{B}=0$ implies that the scalar potential ϕ is constant on magnetic field lines. As Holloway pointed out, although ϕ remains constant on any magnetic field line within an electron or ion zone, it can jump as the line crosses the zero-charge surfaces, which are also surfaces of zero particle density, between the zones. Because these boundaries can sustain a potential difference in one direction only, Holloway borrowed the term 'p–n junction' from solid state physics to describe them. In the aligned case, the electron-zone sections of the magnetic field lines connect to the star, and so the stellar potential is propagated throughout the electron zones. Lines that connect to the star at colatitudes exceeding $\arccos(1/\sqrt{3})$ do not pass through the electron zones: they thread a part of the ion zone that is at stellar potential. Other magnetic field lines have ion-zone sections that are isolated from the star; these sections have potentials that vary from line to line.

The steady-rotation constraint on $\boldsymbol{E}$ and $\boldsymbol{B}$, together with $\boldsymbol{E} = -\nabla\phi$, imply that $\boldsymbol{B}.\nabla(\Phi-\phi) = 0$: for steadily rotating axisymmetric structures, $\Phi-\phi$ is constant on the $\boldsymbol{B}$ lines. This is a purely electromagnetic field result—an implication of Faraday's law and $\nabla.\boldsymbol{B} = 0$—regardless of particle dynamics or even whether or not any particles are present. Thus, any change in ϕ along a $\boldsymbol{B}$ line is accompanied by an equal change in Φ.

Holloway deduced the existence of the potential jump by means of a thought experiment. He imagined the equilibrium state of an aligned rotator, with $\boldsymbol{E}.\boldsymbol{B} = 0$ at any point occupied by plasma, that would be attained following removal of the positive particles from a toroidal region bounded by two surfaces of the poloidal magnetic field together with the northern and southern zero-charge cones, followed by deposition of those particles onto the star. He noted that the potential on the isolated sections of magnetic field lines can only be more negative than on sections that connect to the star, and thus deduced that, surrounding the zero-charge cones, there must be finite evacuated regions in which $\boldsymbol{E}.\boldsymbol{B} \neq 0$ and the potential change occurs.

Leaving his thought experiment, Holloway went on to consider 'physical pulsar models'. He argued that, when their inertia is allowed for, particles near the light cylinder will be swept away by the centrifugal effect, thus causing what he called a 'partial depletion' of the ion zone near the light cylinder. He claimed that this 'partial depletion' will have similar effects to those caused by the imagined removal of ions in his thought experiment, including the existence of evacuated regions separating the electron and ion zones.

It seems to me that it is in drawing an analogy between the so-called 'partial depletion' of the ion zone and the imagined removal of ions that Holloway's analysis has gone wrong. The 'partial depletion' is only relative to that predicted by the Goldreich–Julian (1969) charge-density formula, which is based on neglecting the non-corotational electric force. When one wishes to take account of the centrifugal effect, one needs must also take account of the non-corotational electric force, which Holloway did not. Holloway's results for what he called 'physical pulsar models'—as distinct from his results for his thought experiment, which I am not challenging—are artefacts of his not allowing for the non-corotational electric potential. In Holloway's thought experiment, ϕ, and hence Φ, in much of the ion zone are below their values in the electron zones. In more physical models, Φ has the sign of the charge on the corotating particles: it is larger in ion zones than in electron zones. This accounts for the diametrically opposed results for Holloway's thought experiment and truly physical models.

8. Concluding Remarks

The findings presented in this paper have substantial implications for model building. The expression (2) for Φ in zones of corotation shows it to be positive in ion zones and negative in electron zones, with its magnitude smaller by a factor of several thousand in electron zones than in ion zones, at similar distances from the rotation axis. The resulting local non-corotational electric field between the electron and ion zones is directed so as to cause a merging of positive ions and electrons to form a boundary layer of mixed plasma. Consideration of the net forces acting on ions in electron zones and electrons in ion zones, as well as of forces acting in the boundary layers, suggests that, for aligned and quasi-aligned rotators, electrons can enter the ion zone and mirror between the northern and southern boundary layers while drifting toward the star because of dissipative effects.

A number of authors have discussed the possible existence of magnetospheres containing corotating particles only, and so, of course, confined within the light cylinder. In the axisymmetric case, the outer surface would be defined by

$$(1-x^2)^{-\frac{1}{2}} \approx \gamma_k \approx (-e_k B_z/m_k c\Omega)^{\frac{1}{3}},$$

as shown by Wang (1978), and it would be necessary to match the plasma electromagnetic field components with the corresponding vacuum components across this surface. To Mestel *et al.* (1979), it seemed probable that these conditions cannot be satisfied: they argued that it is likely that one would merely be constructing a model with the difficulty inherent in the vacuum model, namely a discontinuity in the component of $\boldsymbol{E}$ along $\boldsymbol{B}$ (Goldreich 1969; Goldreich and Julian 1969; Michel 1969), transferred from the stellar surface to the magnetospheric surface. However, if Φ and $\nabla\Phi$ are (as they should be) taken to be given by the formulae (2) and (3) above, then this difficulty might not arise: the particles are everywhere in balance between the centrifugal effect and the non-corotational electric force, and the vacuum solution and the position of the magnetospheric boundary would have to be chosen so that the vacuum values of Φ and $\nabla\Phi$ match those given by equations (2) and (3) on the surface.

In fact though, this model is oversimplified for another reason. As well as having the vortical boundary layers of mixed dissipative plasma between electron and ion zones, the magnetosphere must have another boundary layer, namely an outer skin of plasma in which the inertial development of vorticity occurs. It is well worth trying to construct a model along these lines, with electrons extracted from the star into the magnetospheric skin, and with a return current of electrons passing through the ion zone by the mirroring and drifting processes discussed above. It should lead to an upper limit on the pulsar period in order for the vacuum electric field at the polar caps to be sufficiently strong to extract electrons from the star.

Speaking more generally, I envisage a magnetospheric model with a cellular structure. The cell walls, or boundary layers, I expect to be dissipative. Within the cells proper, dissipation will probably be less important than in the cell walls; but it will certainly be necessary to approximate Φ accurately there. I believe that the problem of constructing a self-consistent pulsar magnetosphere model is, even in the axisymmetric case, much more subtle than has hitherto been thought. My program is to model each cell, and the cell walls, at least one step more accurately than has so far been done, paying particular attention to particle dynamics and the construction of Φ. The techniques being developed by Mestel and Wang (1979, preprint 1980) for magnetic field construction should be adaptable to this model.

Acknowledgment

I thank the referee for some suggestions for improving the presentation of this paper.

References

Buckingham, M. J., Byrne, J. C., and Burman, R. R. (1972). *Phys. Lett.* A **38**, 233.
Buckingham, M. J., Byrne, J. C., and Burman, R. R. (1973). *Plasma Phys.* **15**, 669.
Burman, R. R. (1980). *Aust. J. Phys.* **33**, 771.
Burman, R. R. (1981*a*). *Aust. J. Phys.* **34**, 91.
Burman, R. R. (1981*b*). *Aust. J. Phys.* **34**, 317.
Burman, R. R., and Mestel, L. (1978). *Aust. J. Phys.* **31**, 455.
Burman, R. R., and Mestel, L. (1979). *Aust. J. Phys.* **32**, 681.
Endean, V. G. (1972*a*). *Nature Phys. Sci.* **237**, 72.
Endean, V. G. (1972*b*). *Mon. Not. R. Astron. Soc.* **158**, 13.
Goldreich, P. (1969). *Proc. Astron. Soc. Aust.* **1**, 227.
Goldreich, P., and Julian, W. H. (1969). *Astrophys. J.* **157**, 869.
Holloway, N. J. (1973). *Nature Phys. Sci.* **246**, 6.
Manchester, R. N., and Taylor, J. H. (1977). 'Pulsars', see pp. 113, 164, 173–4, 235 (Freeman: San Francisco).
Mestel, L. (1971). *Nature Phys. Sci.* **233**, 149.
Mestel, L. (1973). *Astrophys. Space Sci.* **24**, 289.
Mestel, L. (1974). *In* 'Magnetohydrodynamics' (Eds L. Mestel and N. O. Weiss) Saas-Fee Lectures 1974 (Swiss Society of Astronomy and Astrophysics: Sauverny).
Mestel, L. (1980). *In* 'Gravitational Radiation, Collapsed Objects and Exact Solutions' (Ed. C. Edwards), Lecture Notes in Physics, Vol. 124, pp. 147–71 (Springer: Berlin).
Mestel, L., Phillips, P., and Wang, Y.-M. (1979). *Mon. Not. R. Astron. Soc.* **188**, 385.
Mestel, L., and Wang, Y.-M. (1979). *Mon. Not. R. Astron. Soc.* **188**, 799.
Mestel, L., Wright, G. A. E., and Westfold, K. C. (1976). *Mon. Not. R. Astron. Soc.* **175**, 257.
Michel, F. C. (1969). *Phys. Rev. Lett.* **23**, 247.
Michel, F. C. (1974). *Astrophys. J.* **192**, 713.
Okamoto, I. (1974). *Mon. Not. R. Astron. Soc.* **167**, 457.
Sturrock, P. A. (1971). *Astrophys. J.* **164**, 529.
Wang, Y.-M. (1978). *Mon. Not. R. Astron. Soc.* **182**, 157.

Manuscript received 8 August 1980, accepted 2 December 1980