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Solenoidal Excitation of Eigenmodes in Cylindrical Magnetized Plasma with Two Ion Species

T. D. Kieu[A,B] *and W. N-C. Sy*[A]

[A] Department of Theoretical Physics, Research School of Physical Sciences, Australian National University, P.O. Box 4, Canberra, A.C.T. 2600.
[B] Permanent address: Department of Physics, University of Queensland, St. Lucia, Qld 4067.

Abstract

The conditions for the experimental observation of new types of eigenmodes in a typical laboratory magnetized plasma with two ion species are discussed. It is shown that the most favourable condition occurs during the current carrying phase of the discharge, with an appropriately chosen mixture of ions.

1. Introduction

Magnetoacoustic oscillations in a cylindrical magnetized plasma waveguide with one ion species have been extensively studied both theoretically and experimentally for the purposes of plasma diagnostics and heating. By contrast, the case with two ion species has been considered by relatively few authors (Baird and Swanson 1969; Schlüter and Schürger 1975; Jessup and McCarthy 1978), using largely numerical methods for rather restricted ranges of parameters. Although the situations considered pertain to certain experimental set-ups, they are not necessarily the best ones for observing new physical phenomena associated with the presence of another ion species. To find the optimal conditions, it is necessary to obtain functional relationships between certain parameters; this task is time-consuming and expensive by numerical computation. Janzen (1980, 1981) has done parametric studies which extend Buchsbaum's (1960) dispersion analysis to include collisional effects and more than two ion species, but only for plane waves in an infinite uniform plasma. In the present paper, a combination of analytical and numerical approaches is adopted to consider the problem of eigenmode excitation in a non-uniform cylindrical plasma.

In the next section, we describe the fundamentally new phenomenon to be expected from the addition of another ion species to an ideal plasma and we discuss the optimal ionic mixtures which facilitate the observation of the new effects. In Section 3, a discussion is presented on the modifications to the ideal picture arising from non-ideal effects such as dissipation, density non-uniformity and the presence of an axial current. The final section concludes with some remarks on the conditions for observation of the new phenomenon in relation to present day experimental situations.

2. Basic Phenomenon in Ideal Plasmas

It is known (Buchsbaum 1960) that the dispersive properties of a plasma with two ion species change dramatically near the ion–ion hybrid resonance frequency,

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which lies between the two ion cyclotron frequencies. To describe waves at this range of frequencies in a bounded plasma, the governing equation may be written as (Sy and Cotsaftis 1979)

$$\nabla \times (\nabla \times \boldsymbol{E}) = \mathbf{T}\boldsymbol{E}, \tag{1}$$

where $\mathbf{T}$ accounts for the microscopic properties of the medium. In an ideal plasma, without dissipation, we have $\mathbf{T} = -(\omega^2/c^2)\mathbf{K}$, where $\mathbf{K}$ is the 'cold' plasma dielectric tensor (Stix 1960).

For excitation by means of a long solenoid wound around a cylindrical magnetized plasma, equation (1) reduces to a simple equation in the azimuthal component of the perturbed electric field E_θ:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rE_\theta)\right) + k_r^2 E_\theta = 0, \tag{2}$$

where the radially dependent k_r is defined by

$$k_r^2 \equiv \frac{\omega^2}{c^2}\frac{S^2 - D^2}{S}, \tag{3}$$

with S and D as defined by Stix (1960). We consider a plasma with two ion species having number density n_i, charge Z_i, atomic number A_i and cyclotron frequency Ω_i $(i = 1, 2)$. In a well-mixed plasma, the ion charge concentration defined by

$$\eta_i = n_i Z_i/(n_1 Z_1 + n_2 Z_2) \tag{4}$$

is a constant, independent of total density variations. On writing the electron density variation in the form $n_e(r) = n_0\,\rho(r)$, where n_0 is the average density, one may rewrite (3) more explicitly as

$$k_r^2 = K^2 \Lambda(\omega)\,\rho(r), \tag{5}$$

where the characteristic wavenumber K (in units of m^{-1}) is given by

$$K^2 = 193\, n_0 (Z_1 Z_2/A_1 A_2)^{\frac{1}{2}}, \tag{6}$$

when n_0 is expressed in units of $10^{19}\ \mathrm{m}^{-3}$. The frequency dependent quantity $\Lambda(\omega)$ is defined by

$$\Lambda(\omega) \equiv \frac{\nu_{\mathrm{HB}}^2}{\nu_{\mathrm{C}}}\frac{\nu^2(\nu^2 - \nu_{\mathrm{C}}^2)}{\nu^2 - \nu_{\mathrm{HB}}^2}, \tag{7}$$

where the wave frequency ω is normalized with respect to $\varpi = (\Omega_1 \Omega_2)^{\frac{1}{2}}$ and $\nu = \omega/\varpi$, while the normalized ion–ion hybrid frequency ν_{HB} and its associated cutoff frequency ν_{C} are given by

$$\nu_{\mathrm{HB}}^2 = (\eta_1 \Omega_2 + \eta_2 \Omega_1)/(\eta_1 \Omega_1 + \eta_2 \Omega_2), \qquad \nu_{\mathrm{C}} = \varpi^{-1}(\eta_1 \Omega_2 + \eta_2 \Omega_1). \tag{8a, b}$$

The appropriate boundary condition for (3) may be conveniently expressed in terms of the perturbed axial magnetic field given by

$$B_z(r) = r^{-1}\,\mathrm{d}(rE_\theta)/\mathrm{d}r. \tag{9}$$

The condition is $|B_z(a)| = B_0$, where B_0 is the field amplitude driven by the solenoid and a is the radius of the plasma.

The essentially new physical phenomenon is associated with the structure of the function $\Lambda(\omega)$, which reduces to the simple form $\Lambda(\omega) = \nu_C \nu^2$ in the case of a plasma with only one ion species. That is, the presence of an additional ion species introduces not only an ion–ion hybrid resonance frequency ν_{HB} but also an associated cutoff frequency ν_C in the frequency spectrum. To see the consequence of this on the excitation of global eigenmodes, i.e. on the discrete spectrum, it is simplest to consider a uniform plasma where $\rho(r) = 1$ in (5). In this case, the solution can be expressed in terms of a Bessel function:

$$B_z(r) = B_0 \mathrm{J}_0(k_r r)/\mathrm{J}_0(k_r a), \tag{10}$$

from which the eigenmode excitation condition may be deduced as

$$k_r^2 a^2 = K^2 a^2 \Lambda(\omega) = \zeta_n^2 ; \qquad n = 1, 2, 3, \dots, \tag{11}$$

where ζ_n is a zero of J_0. From the foregoing remarks on the structure of $\Lambda(\omega)$, it is evident that for each value of n in equation (11), there is only one value of ω^2 which satisfies (11) for a plasma with one ion species, whereas there are in general two values for a plasma with two ion species. This is graphically illustrated by curves A and B in Fig. 1. That is, the presence of another ion species causes the usual dispersion curve to bifurcate into two branches, each of which leads to a sequence of eigenmodes.

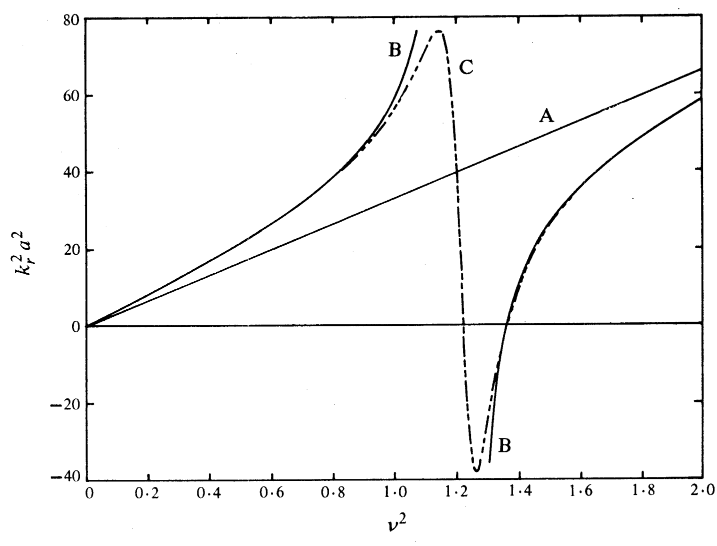

Fig. 1. Dispersion curves of $k_r^2 a^2$ against the square of the dimensionless frequency: curve A, the one ion species case; curve B, the two ion species case without dissipation and with $\Omega_2/\Omega_1 = \frac{1}{2}$ and $\eta_1 = 0\cdot35$; curve C, as for curve B but with the dissipation $\varepsilon_1 = 0\cdot1$.

In the present case, it is clear from Fig. 1 that the higher order eigenmodes in the lower branch accumulate at the ion–ion hybrid frequency, which is therefore a cluster point of the frequency spectrum. From numerical computations using a different model, Jessup and McCarthy (1978) showed that there are two eigenfrequencies for the first radial mode. Here, we have shown that for the ideal case, there are two eigenfrequencies for any radial mode. It is shown in the next section, however, that the absence of higher order radial modes near the cluster point could be related to dissipative effects near the ion–ion hybrid resonance frequency.

It is clear from the expressions (8) that if the number density of one of the ion species is too low then the resonance and cutoff frequencies are very close together, making observation of the new physical phenomenon difficult. Moreover, not all plasmas with two different ion species will exhibit the bifurcated dispersion structure seen in Fig. 1. For example, a plasma with any composition of the two ions D^+ and $^4He^{++}$ would behave like a plasma with only one ion species. Previous authors (Baird and Swanson 1969; Jessup and McCarthy 1978) have paid rather limited attention to this aspect of the problem. As the effect of varying the plasma composition is largely unrelated to the non-ideal effects to be considered in the next section, it is convenient to present a discussion here.

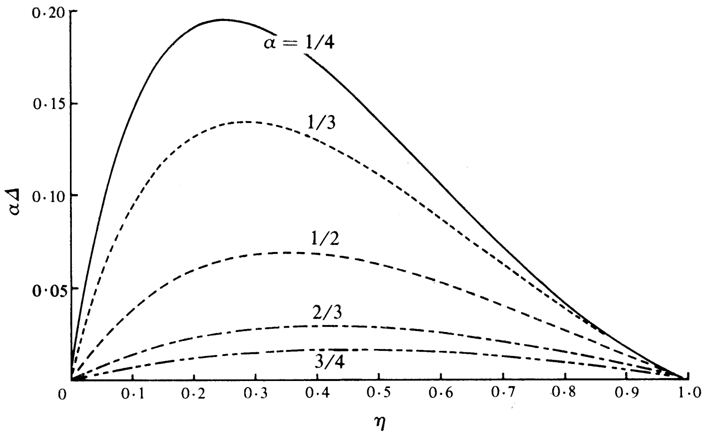

Fig. 2. Normalized separation of the characteristic frequencies $\alpha\Delta$ as a function of the light ion concentration η for various values of α.

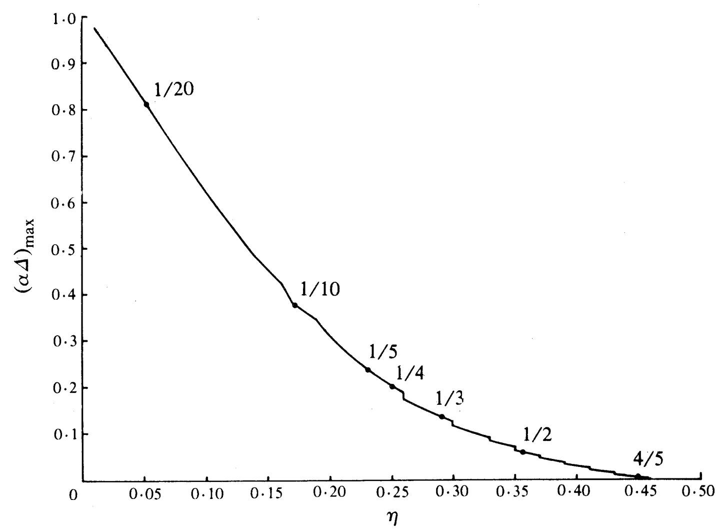

Fig. 3. Maximum separation of the characteristic frequencies $(\alpha\Delta)_{\max}$ as a function of the light ion concentration η for a range of values of α.

The optimal condition for experimental observation occurs when the two sequences of eigenfrequencies are as different as possible. This generally requires the resonance and cutoff frequencies be as different as possible. Introducing the parameter $\alpha \equiv \Omega_2/\Omega_1 \leqslant 1$, we write the normalized separation of the characteristic frequencies as

$$\Delta \equiv \nu_C^2 - \nu_{HB}^2 = \eta(1-\eta)(1-\alpha)^2(1-\eta+\alpha\eta)/\{\eta+\alpha(1-\eta)\}, \tag{12}$$

where the subscript 1 on η for ions with the higher cyclotron frequency has been dropped for convenience. The quantity $\alpha\Delta$ then represents the frequency separation function normalized to Ω_1^2, the square of the cyclotron frequency of the lighter ions. This quantity is plotted as a function of η for various values of α in Fig. 2. The value $\alpha = \frac{1}{2}$ corresponds to an H^+–D^+ or an H^+–$^4He^{++}$ plasma, while the value $\alpha = \frac{2}{3}$ corresponds to a D^+–T^+ or an H^+–$^3He^{++}$ plasma etc. It is seen that for given plasma constituents there exists an optimal relative concentration which maximizes the separation of the characteristic frequencies. In the case of an H^+–D^+ plasma, the optimal mixture turns out to be about 35% H^+ and 65% D^+. These results suggest a general trend where the largest frequency separations occur when α and η are small. This is seen to be indeed the case in Fig. 3, where $(\alpha\Delta)_{max}$ is plotted against η for various values of α. When α and η satisfy the condition

$$\alpha \ll \eta/(1-\eta) \ll 1, \tag{13}$$

the ion–ion hybrid frequency is just above Ω_2 and its associated cutoff frequency is just below Ω_1. Since in this case we have $\Omega_2/\Omega_1 \ll 1$, the separation of the characteristic frequencies is of the order of Ω_1.

In conclusion, a plasma composed of a high concentration of heavy ions (with a small charge/mass ratio) and a low concentration of light ions, satisfying condition (13), would be expected to show most clearly the new phenomenon associated with the presence of another ion species. It is beyond the scope of the paper to discuss more complicated situations where the ions may be ionized to various degrees. This aspect has been discussed in some detail elsewhere (Janzen 1980).

3. Non-ideal Effects

Non-ideal effects which exist in an actual plasma can modify considerably the simple picture for an ideal plasma described in the previous section, particularly when resonance phenomena are involved. Perhaps the most important of these are dissipative effects.

Dissipative Effects

Generally speaking, dissipative effects can remove the infinities in the refractive index when the resonances are related to plasma dispersion and they can also remove the infinities in the wave amplitudes when the resonances are eigenmode resonances related to the existence of a plasma boundary. However, a large number of dissipative mechanisms has been proposed, particularly for the dispersive resonances such as the ion–ion hybrid resonance. Apart from the usual ones of collisional absorption and Landau cyclotron damping, there are also possibilities of mode conversion (Perkins 1977), 'phase-mixing' due to geometric effects, and stochastic heating (Cotsaftis 1982). Except for the simplest situations, it is normally extremely difficult to determine the dominant dissipation mechanism (Stix and Swanson 1982). In the following, we introduce dissipation phenomenologically into the model by constants ε_1 and ε_2 in order to discuss the general effects of dissipation, whatever its origin.

In the presence of dissipation, the dispersive properties may be described by a modified form of (3):

$$k_r^2 = \frac{\omega^2}{c^2}\frac{S(S^2-D^2)}{S^2+\varepsilon_1^2}. \tag{14}$$

In this case, the dispersion curve B in Fig. 1 is changed into the curve C for $\varepsilon_1 = 0\cdot 1$. Curves similar to C have been obtained (Jessup and McCarthy 1978) from calculations based on a collisional model. Evidently, the fact that the curve C turns over near the ion–ion hybrid resonance implies that eigenmodes beyond a certain radial mode number are no longer accessible. The ion–ion hybrid frequency is no longer a cluster point of the discrete spectrum. Calculations show that the stronger the dissipation (and thus the larger the parameter ε_1) the lower the peak of curve C and hence the fewer the eigenmodes on the lower frequency branch. The results also show the existence of a central branch of the dispersion curve, which appears to allow the excitation of more eigenmodes. However, these modes have frequencies very close to the ion–ion hybrid frequency, where the dispersion is dominated by dissipation and hence their amplitudes would be very small.

It is well known from studies of magnetoacoustic oscillations that the main effect of dissipation on the geometric resonances is to limit the height and to broaden the width of the resonances in the frequency spectrum. This general picture is unchanged for plasmas with two ion species; further details are described on p. 498. In conclusion, situations where dissipation is weak are more favourable to the observation of new phenomena associated with plasmas with two ion species.

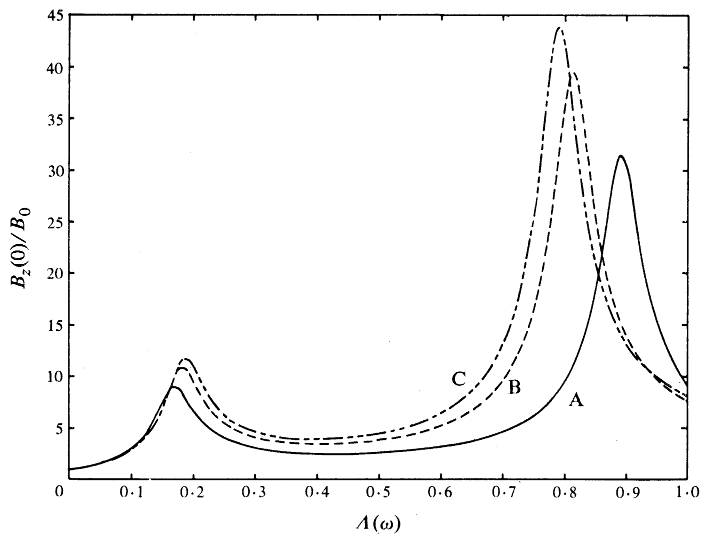

Fig. 4. Variation of the maximum field amplitude $|B_z(0)/B_0|$ as a function of $\Lambda(\omega)$ for $\varepsilon_2 = 1\cdot 0$, $\bar{a} = 0\cdot 9$ and $\Omega_2/\Omega_1 = \frac{1}{2}$: curve A, the uniform profile with $\beta = 0$; curve B, the parabolic profile with $\beta = 1$; curve C, $\beta = 2$.

Density Non-uniformities

Much effort has been expended in the literature on the study of the effect of plasma non-uniformities. In general, the dispersive properties of the eigenmodes are rather insensitive to density profile variations, but they are sensitive to the average density of plasma. The reason is that radial eigenmodes necessarily have a component of the wave vector in the direction of the density variation and the net effect is one of averaging from the wave sampling across the whole plasma. On the other hand, profiles of the perturbed field amplitudes are much more sensitive to local variations in the density; the profiles have a tendency to shift towards the higher density regions of the plasma. These general observations are confirmed here for the case with two ion species.

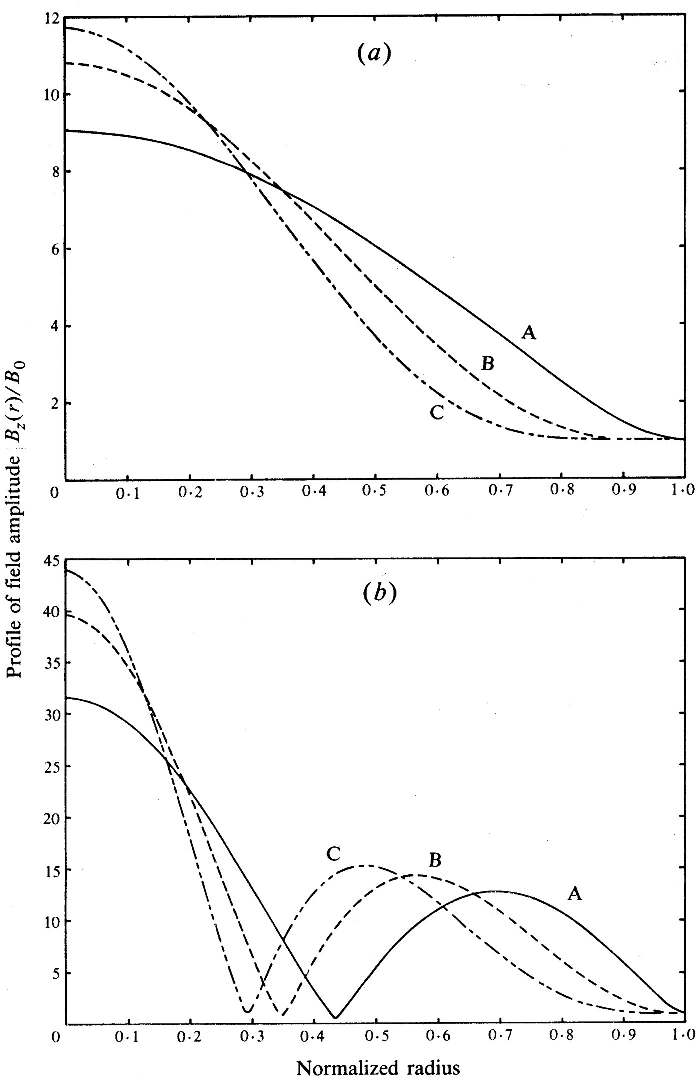

Fig. 5. Field amplitude profiles $|B_z(r)/B_0|$ corresponding to the parameters of Fig. 4 for (*a*) the fundamental radial modes and (*b*) the second radial modes.

To investigate the effects of a non-uniform density, the electron density profiles chosen are of the form

$$\rho(r) \equiv \bar{\alpha}(\beta+1)(1-\bar{\alpha}r^2/a^2)^{\beta}/\{1-(1-\bar{\alpha})^{\beta+1}\}, \tag{15}$$

from which it may be verified that the average density is independent of the parameters $\bar{\alpha}$ and β. Finite eigenmode resonances may be obtained by including a small imaginary part due to dissipation in equation (5):

$$k_r^2 = K^2 \Lambda(\omega)\,\rho(r) + \mathrm{i}\,\varepsilon_2 . \tag{16}$$

On solving equation (2) with the appropriate boundary conditions, we plot the quantity $|B_z(0)/B_0|$ against $\Lambda(\omega)$ for three different density profiles in Fig. 4 for $\varepsilon_2 = 1 \cdot 0$. It should be noted that the second radial modes (curves B and C) have substantially higher resonance peaks than the fundamental mode (curve A) due to the simplifying assumption that the dissipation is frequency independent. In an actual situation, such as that of a resistive plasma, the quantity ε_2 increases rapidly with frequency and the relative sizes of the peaks would be significantly altered. However, the locations of the eigenfrequencies are not dependent on such variations in the models for dissipation. It is seen that the eigenfrequencies, particularly for the fundamental modes, are not greatly sensitive to density profile variations. The eigenfunctions, however, are much more sensitive; they are plotted in Figs 5*a* and 5*b* for the first and second eigenmodes respectively for corresponding density profiles. Such profile effects would need to be taken into account for accurate calculations of antenna loop impedance due to eigenmode excitation in the plasma.

Effect of an Axial Current

Many laboratory plasma discharges are formed by drawing an axial current along a gas-filled cylinder. In general, the plasma temperature is high during the current carrying phase and it drops progressively to lower values as the current decays. Since a high temperature plasma, being usually less dissipative, is expected to be more suitable for the study of the new eigenmodes associated with the presence of another ion species, it seems appropriate to evaluate the effect of an axial current on the excitation of the eigenmodes.

By considering the simple special case of a small uniform axial current, where $\gamma^2 \equiv B_{0\theta}^2(a)/B_{0z}^2 \ll 1$, one may show from a general set of equations (Sy 1983) that the eigenmode condition (11) is modified to read

$$k_r^2 a^2 = K^2 a^2 \Lambda(\omega) + 4\gamma^2 = \zeta_n^2 ; \qquad n = 1, 2, \ldots . \tag{17}$$

Evidently, an axial current has only a small effect provided

$$B_{0\theta}(a)/B_{0z} \ll \tfrac{1}{2}\zeta_n ; \qquad n = 1, 2, \ldots , \tag{18}$$

which is not a stringent condition for a plasma with a strong axial magnetic field. Expressed in terms of the axial current I (in kA), the axial magnetic field B_{0z} (in T) and the radius of the plasma a (in cm), the condition (18) becomes

$$I \ll 95 \cdot 8\, a B_{0z} , \tag{19}$$

which is satisfied, for example, for a typical plasma where $I = 20$ kA, $B_{0z} = 0 \cdot 5$ T and $a = 5$ cm.

4. Conclusions

It has been shown that a plasma with two ion species can have two sequences of radial global eigenmodes. The lower branch below the ion–ion hybrid resonance frequency may perhaps be called ion–ion hybrid global eigenmodes in analogy with those below the shear Alfvén resonance frequency (Appert *et al.* 1982). The existence of these eigenmodes may be masked by a poor choice of ionic mixture and by strong dissipative effects. But density non-uniformities and a small axial current do not significantly affect the possibility of observing these eigenmodes experimentally.

For any two given ion species, an optimal relative concentration has been shown to exist. The most favourable situation appears to be one where there is a large majority of heavy ions and a small minority of light ions satisfying condition (13). Dissipative effects, which could smear out the new eigenmodes, may be minimized in a higher temperature plasma during the current carrying phases of a typical laboratory discharge.

Finally, it should be mentioned that impurity ions exist in any realistic plasma. Their inclusion in a theoretical analysis leads to a plasma model with more than two ion species and a much more complicated picture could emerge (Janzen 1981). However, provided impurity ion concentrations are much smaller than either of the plasma ion concentrations, it may be shown that the conclusions reached here remain valid.

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