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Comptonisation of Radiation Below the Cyclotron Frequency in a Strong Magnetic Field

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Abstract

We present new equations describing the Thomson scattering of the o-mode and z-mode in a strongly magnetised plasma, valid below the fundamental cyclotron frequency. Scattering by nonrelativistic thermal electrons leads to a frequency diffusion equation for the more strongly scattered o-mode and this equation is effectively the Kompaneets equation with cross section 2/15th of the Thomson cross section. Transfer of the photons tends to be dominated by the less strongly scattered z-mode; an o-mode photon is scattered occasionally into a z-mode photon, which then diffuses rapidly due to its large mean free path before being scattered back into an o-mode photon. Our results should have applications in X-ray pulsars and γ-ray burst sources, as well as magnetic white dwarfs occurring in cataclysmic variables.

1. Introduction

The modification to the spectrum of a source of radiation due to multiple Compton scattering in a medium is known as Comptonisation of the radiation. Comptonisation is of interest in astrophysical applications as one mechanism for generating thermal and non-thermal power-law spectra (Felten and Rees 1972; Katz 1976; Sunyaev and Titarchuk 1980). Some areas of interest include active galactic nuclei (AGN) and quasars (Brinkmann and Trumper 1984). Comptonisation of soft photons, along with synchrotron self-Compton scattering has been used to explain the X-ray emission from these sources which have characteristic power-law spectra (Rothschild *et al.* 1983; Rees 1980 and references therein). Katz (1976) proposed the now widely accepted mechanism of thermal Comptonisation for both the infrared and optical spectra of AGN. The X-ray emission from Cyg X-1 is also accepted as being due to Comptonisation, in this case of low-frequency photons by hot electrons in an accretion disc orbiting a black hole of several solar masses (Shapiro *et al.* 1976; Eardley *et al.* 1978). Sunyaev and Titarchuk (1980) obtained agreement between the observed spectrum of Cyg X-1 and a Comptonised spectrum of soft photons in a plasma of temperature 27 keV and Thomson optical depth $\tau_0 = 5$. Comptonisation is also likely to be important in other compact X-ray sources such as X-ray and gamma-ray bursters, and X-ray pulsars, all of which are thought to be accreting neutron stars (e.g. Lewin and Van der Heuvel 1983). In particular, the effect of Comptonisation on the cyclotron resonance detected in three X-ray pulsars, e.g. Her X-1 and a large number of gamma-ray bursters, is important in determining whether the line is an absorption or emission feature.

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There is extensive literature on the interaction of radiation with electrons in a strong magnetic field. The cross sections and probabilities of elementary processes, such as Thomson scattering and free–free scattering, can be greatly modified by the presence of the magnetic field; see e.g. Canuto *et al.* (1971), Melrose and Sy (1972), Gnedin and Sunyaev (1974), Lodenquai *et al.* (1974), Börner and Meszaros (1978, 1979), Ventura (1979), Kirk and Meszaros (1980). Some authors have emphasised applications of these results to radio pulsars (e.g. Blandford and Scharlemann 1976) and others to X-ray pulsars (Basko and Sunyaev 1975; Kanno 1980; Meszaros *et al.* 1980; Meszaros and Bonazzola 1981; Nagel 1981 amongst others). However, little attention has been given to the question of Comptonisation in a magnetic field, although recently Lyubarsky (1985, 1986) has included such Comptonisation in radiative transfer equations, with a view to explaining the radiation spectrum of X-ray pulsars.

Our purpose in this paper is to extend the theory for Comptonisation of radiation to include the effect of a strong magnetic field. The effects of such a field are important below about the cyclotron frequency, which is in the hard X-ray range for Her X-1 and other X-ray sources with observed cyclotron features. Below the cyclotron frequency the birefringence of the electron gas cannot be ignored. The two wave modes of the medium correspond to the o-mode and z-mode of magnetoionic theory, and we adopt these names here. Scatterings can be classified as o–o, z–z, o–z and z–o depending on the modes of the initial and final photons. The scatterings of photons in the two modes occur at markedly different rates and this has interesting consequences. For example, the shape of the spectrum tends to be determined by the more strongly scattered mode, and the transfer of the radiation by the less strongly scattered mode.

In the absence of a magnetic field Comptonisation is usually described using the Kompaneets (1956) equation

$$\frac{\partial N}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x}x^4\left(N+N^2+\frac{\partial N}{\partial x}\right), \tag{1}$$

where N is the photon occupation number, $x = \hbar\omega/kT_e$ is the photon energy in units of the thermal energy of the electrons and $y = (kT_e/m_e c^2)\sigma_T n_e ct$ is a parametrised time characterising the number of collisions experienced by a photon, with σ_T the Thomson cross section and n_e the electron density. The terms N, N^2 and $\partial N/\partial x$ are identified as the quantum recoil, induced or stimulated scattering and Doppler terms respectively. Analytic solutions to (1) have been obtained in limiting cases, the most important being the limit where the induced scattering term is neglected; see e.g. Kompaneets (1956), Zeldovich and Sunyaev (1969), Illarionov and Sunyaev (1972), Arons (1972) and Sunyaev and Titarchuk (1980).

Our primary objective here is to derive a pair of coupled equations which replace the Kompaneets equation (1) for photons below about the cyclotron frequency. These equations are for the occupation numbers N_o and N_z for the two modes; N_o evolves due to o–o and z–o scatterings and N_z evolves due to o–z and z–z scatterings.

In deriving these equations we assume that the electrons have an isotropic pitch angle distribution and a nonrelativistic Maxwellian velocity distribution. Although these assumptions are unlikely to be justified in the magnetosphere of a radio pulsar, they may be satisfied in the accretion column of a neutron star or white dwarf.

Additionally, we ignore thermal corrections to the wave properties; this may not be justified for frequencies sufficiently close to the cyclotron frequency.

Having derived these equations, the only application of them we discuss is that of spatial diffusion. Spatial diffusion is relevant only in an inhomogeneous system and the left-hand side of (1) needs to be modified to take account of the inhomogeneity. We show in detail how this may be done in the unmagnetised case in Appendix 3, when we re-derive, and generalise slightly, a result obtained by Blandford and Payne (1981).

In outline, our paper is as follows. In Section 2 the basic equations which describe the scattering are written down in semi-classical form. The semi-classical formalism allows one to include the first quantum corrections, which are important in (1); intrinsically quantum-field effects appear only to second order in Planck's constant and are not important here. In Section 3 the relevant wave properties are described. The nonrelativistic approximation is made in Section 4 and the particular assumption of a Maxwellian electron distribution is made in Section 5. The Kompaneets-like equations are derived in Section 6. Spatial diffusion is discussed in Section 7.

2. Kinetic Equations

In this section we write down the kinetic equations describing the evolution of the photon distribution of a particular mode. Our approach is based on the semi-classical formalism (cf. Melrose 1980*a*; Ch. 5).

Firstly we consider the scattering of waves from mode σ' into mode σ say. Let us introduce the electron and photon distribution functions $f(\boldsymbol{p})$ and $N_\sigma(\boldsymbol{k})$, respectively, where $\boldsymbol{k}$ is the photon wavevector. Also we denote the basic scattering probability per unit time by $w_{\sigma\sigma'}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{k}')$, where $\boldsymbol{p}$ is the momentum of the scattering electron and $\boldsymbol{k}'$ the wavevector of the scattered photon. The rate of transitions $\sigma' \rightarrow \sigma$ is proportional to

$$w_{\sigma\sigma'}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{k}')\, f(\boldsymbol{p})\, N_{\sigma'}(\boldsymbol{k}')\{1+N_\sigma(\boldsymbol{k})\}\,,$$

while the rate of transitions in the opposite direction $\sigma \rightarrow \sigma'$ is proportional to

$$w_{\sigma\sigma'}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{k}')\, f(\boldsymbol{p}+\hbar\boldsymbol{k}'-\hbar\boldsymbol{k})\, N_\sigma(\boldsymbol{k})\{1+N_{\sigma'}(\boldsymbol{k}')\}\,.$$

The probabilities of the direct and inverse processes are the same, as may be argued by appealing to the Einstein relations. The kinetic equations may be written by subtracting the rate of transitions $\sigma \rightarrow \sigma'$ from the rate of transitions $\sigma' \rightarrow \sigma$ and then integrating over all values of the electron momentum and scattered photon wavevector. We find in the classical limit

$$\begin{aligned}\frac{\mathrm{d}N_\sigma(\boldsymbol{k})}{\mathrm{d}t} = \sum_s \int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \int \mathrm{d}^3 p\big(w_{\sigma\sigma'}(s, \boldsymbol{p}, \boldsymbol{k}, \boldsymbol{k}')[f(\boldsymbol{p})\{N_{\sigma'}(\boldsymbol{k}')-N_\sigma(\boldsymbol{k})\} \\ + N_{\sigma'}(\boldsymbol{k}')\, N_\sigma(\boldsymbol{k})\, D_s f(\boldsymbol{p})] + w_{\sigma\sigma}(s, \boldsymbol{p}, \boldsymbol{k}, \boldsymbol{k}')[f(\boldsymbol{p})\{N_\sigma(\boldsymbol{k}')-N_\sigma(\boldsymbol{k})\} \\ + N_\sigma(\boldsymbol{k}')\, N_\sigma(\boldsymbol{k})\, D_s f(\boldsymbol{p})]\big)\,, \qquad (2)\end{aligned}$$

where the operator D_s is defined in (A7) in Appendix 1. Similarly we can write down a kinetic equation for waves in mode σ':

$$\frac{\mathrm{d}N_{\sigma'}(\boldsymbol{k})}{\mathrm{d}t} = -\sum_s \int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \int \mathrm{d}^3 p \big(w_{\sigma\sigma'}(s, \boldsymbol{p}, \boldsymbol{k}, \boldsymbol{k}')[f(\boldsymbol{p})\{N_{\sigma'}(\boldsymbol{k}) - N_{\sigma}(\boldsymbol{k}')\}$$

$$+ N_{\sigma'}(\boldsymbol{k})\, N_{\sigma}(\boldsymbol{k}')\, D_s f(\boldsymbol{p})] + w_{\sigma'\sigma'}(s, \boldsymbol{p}, \boldsymbol{k}, \boldsymbol{k}')$$

$$\times [f(\boldsymbol{p})\{N_{\sigma'}(\boldsymbol{k}) - N_{\sigma'}(\boldsymbol{k}')\} + N_{\sigma'}(\boldsymbol{k})\, N_{\sigma'}(\boldsymbol{k}')\, D_s f(\boldsymbol{p})]\big). \tag{3}$$

The terms linear in the photon distribution $N_\sigma(\boldsymbol{k})$ describe the effect of spontaneous scattering while those quadratic in $N_\sigma(\boldsymbol{k})$ represent the effect of induced scattering.

The probability for Thomson scattering undergoing spiralling motion in an ambient magnetic field was derived by Melrose and Sy (1972). Also, we include the effect of quantum recoil of the electron in the scattering probability. The quantum recoil is equivalent to finding the first quantum correction to the classical energy conservation condition and is discussed more fully in Appendix 1.

The generalisation of Thomson scattering to include a background magnetic field leads to

$$w_{\sigma\sigma'}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{k}') = \frac{2\pi q^4}{\epsilon_0^2 m^2} \frac{R_\sigma(\boldsymbol{k})\, R_{\sigma'}(\boldsymbol{k}')}{w_\sigma(\boldsymbol{k})\, \omega_{\sigma'}(\boldsymbol{k}')} \sum_{n=-\infty}^{\infty} |a_{\sigma\sigma'}(n, \boldsymbol{k}, \boldsymbol{k}'; \boldsymbol{v})|^2$$

$$\times \delta\Big(\omega_\sigma(\boldsymbol{k}) - \omega_{\sigma'}(\boldsymbol{k}') - n\Omega - (k_\parallel - k'_\parallel) v_\parallel$$

$$+ \frac{\hbar}{2\epsilon}[\{\omega_\sigma(\boldsymbol{k}) - \omega_{\sigma'}(\boldsymbol{k}')\}^2 - (k_\parallel - k'_\parallel)^2 c^2]\Big), \tag{4}$$

with

$$a_{\sigma\sigma'}(n, \boldsymbol{k}, \boldsymbol{k}'; \boldsymbol{v}) = \sum_{t=-\infty}^{\infty} e^*_{\sigma i}(\boldsymbol{k})\, e_{\sigma' j}(\boldsymbol{k}')\, a_{ij}(n-t, \boldsymbol{k}; t, \boldsymbol{k}'; \boldsymbol{v}), \tag{5}$$

$$a_{ij}(s, \boldsymbol{k}; s', \boldsymbol{k}'; \boldsymbol{v}) = \frac{\mathrm{e}^{\mathrm{i}\eta(s'\psi' - s\psi)}}{\gamma \omega_s \omega'_{s'}} [\omega_s \omega'_{s'}\, J_s(z)\, J_{s'}(z')\, \tau_{ij}(\omega_s)$$

$$+ \omega_s J_s(z)\, \tau_{im}(\omega_s)\, k_m\, V^*_j(s', \boldsymbol{p}, \boldsymbol{k}') + \omega'_{s'} J_{s'}(z')\, V_i(s, \boldsymbol{p}, \boldsymbol{k}) k'_m \tau_{mj}(\omega_s)$$

$$+ \{k_l k'_m \tau_{lm}(\omega_s) - \omega\omega'/c^2\}\, V_i(s, \boldsymbol{p}, \boldsymbol{k})\, V^*_j(s', \boldsymbol{p}, \boldsymbol{k}')], \tag{6}$$

and where $\eta = q/|q|$, γ is the particle's Lorentz factor, $\Omega = |q| B/m$ is the gyrofrequency, and $\boldsymbol{v}$ is the particle velocity with components $v_\perp$ and $v_\parallel$ perpendicular and parallel to the magnetic field $\boldsymbol{B}$. The components of $\boldsymbol{k}$ and $\boldsymbol{k}'$ are written as

$$\boldsymbol{k} = (k_\perp \cos\psi,\ k_\perp \sin\psi,\ k_\parallel), \qquad \boldsymbol{k}' = (k'_\perp \cos\psi',\ k'_\perp \sin\psi',\ k'_\parallel), \tag{7}$$

and so define the angles ψ and ψ'. Also, we have

$$\omega_s = \omega - s\Omega - k_\parallel v_\parallel, \qquad \omega'_{s'} = \omega' - s'\Omega - k'_\parallel v_\parallel, \tag{8}$$

and where $z = k_\perp v_\perp/\Omega$, J_s is a Bessel function of order s, $R_\sigma(\boldsymbol{k})$ is the ratio of electric to total energy in waves in mode σ, $e_\sigma(k)$ is the polarisation vector of a wave in mode σ, $\omega_\sigma(k)$ is the frequency dispersion of waves in mode σ, and

$$V(s,p,k) = [\tfrac{1}{2} v_\perp\{e^{i\eta\psi} J_{s-1}(z)+e^{-i\eta\psi} J_{s+1}(z)\},$$

$$-\tfrac{1}{2} i\,\eta v_\perp\{e^{i\eta\psi} J_{s-1}(z)-e^{-i\eta\psi} J_{s+1}(z)\},\; v_\parallel\, J_s(z)]\,. \tag{9}$$

Finally, the quantity $\tau_{ij}(\omega)$ is generally given by

$$\tau_{ij}(\omega) = \frac{\omega^2}{\omega^2-\Omega^2}\left(\delta_{ij} - \frac{\Omega^2}{\omega^2}\, b_i\, b_j + \frac{i\eta\Omega}{\omega}\,\epsilon_{ijk}\, b_k\right), \tag{10}$$

where $b = B/|B|$ is a unit vector along B. For the case of the magnetic field along the z-axis, so that $b = (0,0,1)$, we can write

$$\tau_{ij}(\omega) = \begin{bmatrix} \dfrac{\omega^2}{\omega^2-\Omega^2} & \dfrac{i\eta\omega\Omega}{\omega^2-\Omega^2} & 0 \\ -\dfrac{i\eta\omega\Omega}{\omega^2-\Omega^2} & \dfrac{\omega^2}{\omega^2-\Omega^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

One can show (Appendix 1) that the effect of the quantum recoil in (4) is to introduce terms of the type $\frac{1}{2}\{N_{\sigma'}(k')+N_\sigma(k)\}D_s f(p)$ into the kinetic equations. Hence, we find

$$\begin{aligned}\frac{dN_\sigma(k)}{dt} &= \sum_s \int \frac{d^3k'}{(2\pi)^3}\int d^3p\big(w_{\sigma\sigma'}(s,p,k,k')[f(p)\{N_{\sigma'}(k')-N_\sigma(k)\} \\ &\quad +\{N_{\sigma'}(k')\,N_\sigma(k)+\tfrac{1}{2}N_{\sigma'}(k')+\tfrac{1}{2}N_\sigma(k)\}D_s f(p)] \\ &\quad + w_{\sigma\sigma}(s,p,k,k')[f(p)\{N_\sigma(k')-N_\sigma(k)\}+\{N_\sigma(k')\,N_\sigma(k) \\ &\quad +\tfrac{1}{2}N_\sigma(k')+\tfrac{1}{2}N_\sigma(k)\}D_s f(p)]\big),\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{dN_{\sigma'}(k)}{dt} &= -\sum_s \int \frac{d^3k'}{(2\pi)^3}\int d^3p\big(w_{\sigma\sigma'}(s,p,k,k')[f(p)\{N_{\sigma'}(k)-N_\sigma(k')\} \\ &\quad +\{N_{\sigma'}(k)\,N_\sigma(k')+\tfrac{1}{2}N_{\sigma'}(k)+\tfrac{1}{2}N_\sigma(k')\}D_s f(p)] \\ &\quad + w_{\sigma'\sigma'}(s,p,k,k')[f(p)\{N_{\sigma'}(k)-N_{\sigma'}(k')\} \\ &\quad +\{N_{\sigma'}(k)\,N_{\sigma'}(k')+\tfrac{1}{2}N_{\sigma'}(k)+\tfrac{1}{2}N_{\sigma'}(k')\}D_s f(p)]\big),\end{aligned} \tag{13}$$

The probability $w_{\sigma\sigma'}(s,p,k,k')$ in (12) and (13) is now of the classical form without quantum corrections, i.e. the delta function of (4) is replaced by $\delta\{\omega-\omega'-s\Omega-(k_\parallel-k'_\parallel)v_\parallel\}$.

3. Properties of Magnetoionic Waves

In this section we briefly present details of the magnetoionic wave modes of a cold magnetised plasma; see e.g. Stix (1962) and Melrose (1980*b*). There are four branches of real magnetoionic modes divided into two classes; the extraordinary and ordinary modes. The higher and lower frequency branches of the ordinary mode are called the o-mode and whistler mode respectively, while the higher and lower frequency branches of the extraordinary mode are called the x-mode and z-mode respectively.

The frequency cutoffs for the o, x, and z-modes are $\omega = \omega_p$, $\omega = \omega_x$ or $\omega = \omega_z$, with ω_p the plasma frequency and with

$$\omega_x = \tfrac{1}{2}\Omega + \tfrac{1}{2}(\Omega^2 + 4\omega_p^2)^{\frac{1}{2}}, \qquad \omega_z = \omega_x - \Omega. \tag{14a, b}$$

The o-mode exists at frequencies $\omega > \omega_p$, the x-mode at frequencies $\omega > \omega_x$ and the z-mode in the range $\omega_z < \omega \lesssim (\Omega_e^2 + \omega_p^2)^{\frac{1}{2}}$.

We find it convenient to introduce two magnetoionic parameters which incorporate the plasma frequency and gyrofrequency:

$$X = \omega_p^2/\omega^2, \qquad Y = \Omega/\omega. \tag{15}$$

Since we are interested in the strongly magnetised limit in which $\omega < \Omega$ with $\Omega \gg \omega_p$ then we can see from (14a) that the cutoff for the x-mode becomes $\omega_x \gtrsim \Omega$. So, in the range of interest $\omega_p \ll \omega < \Omega$ or $X \ll 1$, $Y > 1$, then the only modes of interest will be the o-mode and z-mode; the whistler mode is of no interest.

With the wave modes denoted $\sigma = \mathrm{o, z}$, the wave properties are given by (Melrose (1980*b*, Ch. 12)

$$\mu_\sigma^2 = 1 - \frac{X T_\sigma}{T_\sigma - Y\cos\theta}, \tag{16}$$

$$\boldsymbol{e}_\sigma(\boldsymbol{k}) = \frac{(a_\sigma \cos\psi - \mathrm{i}\sin\psi,\ a_\sigma \sin\psi + \mathrm{i}\cos\psi,\ b_\sigma)}{(K_\sigma^2 + T_\sigma^2 + 1)^{\frac{1}{2}}}, \tag{17}$$

$$R_\sigma(\boldsymbol{k}) = \frac{1 + K_\sigma^2 + T_\sigma^2}{2(1 + T_\sigma^2)\mu_\sigma\, \partial(\omega\mu_\sigma)/\partial\omega}, \tag{18}$$

with

$$a_\sigma = K_\sigma \sin\theta + T_\sigma \cos\theta, \qquad b_\sigma = K_\sigma \cos\theta - T_\sigma \sin\theta, \tag{19}$$

$$K_\sigma = -\frac{X Y T_\sigma \sin\theta}{(1-X)(T_\sigma - Y\cos\theta)}, \qquad T_\sigma = \frac{Y(1-X)\cos\theta}{\frac{1}{2}Y^2\sin^2\theta \mp \Delta}, \tag{20, 21}$$

$$\Delta^2 = \tfrac{1}{4}Y^4\sin^4\theta + (1+X)^2 Y^2\cos^2\theta, \tag{22}$$

where the upper (lower) sign refers to $\sigma = \mathrm{o}$ ($\sigma = \mathrm{z}$), μ_σ is the refractive index, θ the angle between the wavevector $\boldsymbol{k}$ and the magnetic field $\boldsymbol{B}$, while K_σ and T_σ

correspond to the longitudinal and transverse part of the polarisation vector $\boldsymbol{e}_\sigma$, and T_σ is the axial ratio of the polarisation ellipse.

4. Thomson Scattering by Nonrelativistic Electrons

If the velocity of the Thomson scattering is nonrelativistic the scattering probability is greatly simplified. One finds that only $n = 0$ (zeroth harmonic scattering) contributes, i.e. $s = s'$. Strictly if $\boldsymbol{v} \neq 0$ there is a contribution to the scattering probability from $n = \pm 1$; however, the contribution from these levels is of order $\max(v_\perp^2/c^2,\, v_\parallel^2/c^2)$ smaller than that of $n = 0$. Hence, on expanding in powers of $\beta = v/c$, only $s = s' = 0$ contributes to the order of relevance here. Thus, we are only concerned with $a_{ij}(0, \boldsymbol{k}; 0, \boldsymbol{k}', \boldsymbol{v})$ which from (6) becomes

$$a_{ij}(0, \boldsymbol{k}; 0, \boldsymbol{k}'; \boldsymbol{v}) = \frac{1}{\gamma}\left\{ J_0(z)\, J_0(z')\, \tau_{ij}(\omega_0) + \frac{J_0(z)}{\omega_0}\, \tau_{im}(\omega_0)\, k_m\, V_j^*(0, \boldsymbol{p}, \boldsymbol{k}') \right. \tag{23}$$

$$\left. + \frac{J_0(z')}{\omega_0}\, V_i(0, \boldsymbol{p}, \boldsymbol{k})\, k'_m\, \tau_{mj}(\omega_0) + \left(k_r\, k'_s \tau_{rs}(\omega_0) - \frac{\omega\omega'}{c^2} \right) \frac{V_i(0, \boldsymbol{p}, \boldsymbol{k})\, V_j^*(0, \boldsymbol{p}, \boldsymbol{k}')}{\omega_0^2} \right\}.$$

The classical Doppler condition (A1) implies that $\omega_s = \omega'_{s'}$ so that $\omega_0 = \omega'_0$.

For the present we retain contributions from $s \neq s' \neq 0$ and write an explicit expression for $a_{\sigma\sigma'}(n, \boldsymbol{k}, \boldsymbol{k}'; \boldsymbol{v})$ defined in (5). Let us introduce the quantity

$$\xi_\sigma = (1 + T_\sigma^2 + K_\sigma^2)^{-\frac{1}{2}}. \tag{24}$$

Now we need to evaluate the various elements of $e^*_{\sigma i}(\boldsymbol{k})\, e_{\sigma' j}(\boldsymbol{k}')\, a_{ij}(s, \boldsymbol{k}; s', \boldsymbol{k}'; \boldsymbol{v})$. If we assume for simplicity the magnetic field is along the z-direction and set $\eta = -1$, i.e. our particles are electrons, then from (16) and (17) we find that

$$e^*_{\sigma i}\, \tau_{ij}(\omega_s)\, e'_{\sigma j} = \xi_\sigma\, \xi'_{\sigma'} \left(\frac{\omega_s^2}{\omega_s^2 - \Omega_e^2} \{ (a_\sigma\, a'_{\sigma'} + 1) \cos(\psi - \psi') + \mathrm{i}(a_\sigma + a'_{\sigma'}) \sin(\psi - \psi') \} \right.$$

$$\left. + \frac{\omega_s\, \Omega_e}{\omega_s^2 - \Omega_e^2} \{ \mathrm{i}(a_\sigma a'_{\sigma'} + 1) \sin(\psi - \psi') + (a_\sigma + a'_{\sigma'}) \cos(\psi - \psi') \} + b_\sigma\, b'_{\sigma'} \right), \tag{25}$$

$$e^*_{\sigma i}\, \tau_{im}(\omega_s) k_m = \xi_\sigma \left(\frac{k_\perp\, \omega_s}{\omega_s^2 - \Omega_e^2} (a_\sigma\, \omega_s + \Omega_e) + k_\parallel\, b_\sigma \right). \tag{26}$$

Next using (9) we have

$$e^*_{\sigma i}\, V_i(s, \boldsymbol{p}, \boldsymbol{k}) = \xi_\sigma [v_\perp \{ a_\sigma\, s z^{-1} J_s(z) + J'_s(z) \} + v_\parallel\, b_\sigma\, J_s(z)], \tag{27}$$

and the final result is

$$k_r\, k'_s\, \tau_{rs}(\omega_s) = \frac{k_\perp\, k'_\perp\, \omega_s^2}{\omega_s^2 - \Omega_e^2} \cos(\psi - \psi') + \frac{\mathrm{i} k_\perp\, k'_\perp\, \omega_s \Omega_e}{\omega_s^2 - \Omega_e^2} \sin(\psi - \psi') + k_\parallel\, k'_\parallel\,. \tag{28}$$

Collecting the results gives

$$
\begin{aligned}
a_{\sigma\sigma'} = {} & \frac{\mathrm{e}^{\mathrm{i}(s\psi - s'\psi')}}{\gamma}\, \xi_\sigma\, \xi'_{\sigma'} \Bigg[\cos(\psi-\psi')\Bigg(J_s(z)\, J_{s'}(z')\, \frac{\omega_s}{\omega_s^2-\Omega_\mathrm{e}^2} \\
& \times \{\omega_s(a_\sigma\, a'_{\sigma'}+1)+\Omega_\mathrm{e}(a_\sigma+a'_{\sigma'})\} + \frac{k_\perp\, k'_\perp}{\omega_s^2-\Omega_\mathrm{e}^2}[v_\perp\{a_\sigma\, sz^{-1} J_s(z) + J'_s(z)\} \\
& + v_\parallel\, b_\sigma\, J_s(z)][v_\perp\{a'_{\sigma'}\, s'z'^{-1} J_{s'}(z') + J'_{s'}(z')\} + v_\parallel\, b'_{\sigma'}\, J_{s'}(z')]\Bigg) \\
& + \mathrm{i}\,\sin(\psi-\psi')\Bigg(J_s(z)\, J_{s'}(z')\, \frac{\omega_s}{\omega_s^2-\Omega_\mathrm{e}^2}\{\omega_s(a_\sigma+a'_{\sigma'})+\Omega_\mathrm{e}(a_\sigma\, a'_{\sigma'}+1)\} \\
& + \frac{\Omega_\mathrm{e}}{\omega_s}\, \frac{k_\perp\, k'_\perp}{\omega_s^2-\Omega_\mathrm{e}^2}[v_\perp\{a_\sigma\, sz^{-1} J_s(z) + J'_s(z)\} + v_\parallel\, b_\sigma\, J_s(z)] \\
& \times [v_\perp\{a'_{\sigma'}\, s'z'^{-1} J_{s'}(z') + J'_{s'}(z')\} + v_\parallel\, b'_{\sigma'}\, J'_{s'}(z')]\Bigg) \\
& + \Bigg\{ J_s(z)\, J_{s'}(z')\, b_\sigma\, b'_{\sigma'} + \frac{J_s(z)}{\omega_s}\Bigg(\frac{k_\perp\, \omega_s}{\omega_s^2-\Omega_\mathrm{e}^2}(a_\sigma\, \omega_s+\Omega_\mathrm{e}) + k_\parallel\, b_\sigma\Bigg) \\
& \times [v_\perp\{a'_{\sigma'}\, s'z'^{-1} J_{s'}(z') + J'_{s'}(z')\} + v_\parallel\, b'_{\sigma'}\, J_{s'}(z')] \\
& + \frac{J_{s'}(z')}{\omega_s}\Bigg(\frac{k'_\perp\, \omega_s}{\omega_s^2-\Omega_\mathrm{e}^2}(a'_{\sigma'}\, \omega_s+\Omega_\mathrm{e}) + k'_\parallel\, b'_{\sigma}\Bigg)[v_\perp\{a_\sigma sz^{-1} J_s(z) + J'_s(z)\} \\
& + v_\parallel\, b_\sigma\, J_s(z)] + \frac{k'_\parallel\, k_\parallel - \omega\omega'/c^2}{\omega_s^2}[v_\perp\{a_\sigma\, sz^{-1} J_s(z) + J'_s(z)\} \\
& + v_\parallel\, b_\sigma\, J_s(z)][v_\perp\{a'_{\sigma'}\, s'z'^{-1} J_{s'}(z') + J'_{s'}(z')\} + v_\parallel\, b'_{\sigma'}\, J_{s'}(z')]\Bigg\}\Bigg].
\end{aligned}
\tag{29}
$$

As usual $J'_s(z)$ denotes $\mathrm{d}J_s(z)/\mathrm{d}z$. Now we can express (29) in the form

$$
\begin{aligned}
a_{\sigma\sigma'} = {} & \mathrm{e}^{\mathrm{i}(s\psi - s'\psi')}\xi_\sigma\, \xi'_{\sigma'}\{A_{ss'}\cos(\psi-\psi') \\
& + \mathrm{i}\, B_{ss'}\sin(\psi-\psi') + C_{ss'}\},
\end{aligned}
\tag{30}
$$

which defines $A_{ss'}$, $B_{ss'}$ and $C_{ss'}$.

Suppose that we have a distribution of photons axisymmetric with respect to the magnetic field, and average $|a_{\sigma\sigma'}|^2$ over the azimuthal angle ψ:

$$
\overline{|a_{\sigma\sigma'}|^2} = \xi_\sigma^2\, \xi'^2_{\sigma'}(\tfrac{1}{2}A_{ss'}^2 + \tfrac{1}{2}B_{ss'}^2 + C_{ss'}^2).
\tag{31}
$$

If we now set $s' = s = 0$, as discussed previously, then we have

$$A_{00} = \frac{1}{\gamma}\left(J_0(z)\, J_0(z') \frac{\omega_0}{\omega_0^2 - \omega_e^2} \{ \omega_0(a_\sigma\, a'_{\sigma'} + 1) + \Omega_e(a_\sigma + a'_{\sigma'}) \} \right.$$

$$\left. \times \frac{k_\perp\, k'_\perp}{\omega_0^2 - \Omega_e^2} \{ v_\perp\, J'_0(z) + v_\parallel\, b_\sigma\, J_0(z) \} \{ v_\perp\, J'_0(z') + v_\parallel\, b'_{\sigma'}\, J_0(z') \} \right), \tag{32}$$

$$B_{00} = \frac{1}{\gamma}\left(J_0(z)\, J_0(z') \frac{\omega_0}{\omega_0^2 - \Omega_e^2} \{ \omega_0(a_\sigma + a'_{\sigma'}) + \Omega_e(a_\sigma\, a'_{\sigma'} + 1) \} \right.$$

$$\left. + \frac{\Omega_e}{\omega_0} \frac{k_\perp\, k'_\perp}{\omega_0^2 - \Omega_e^2} \{ v_\perp\, J'_0(z) + v_\parallel\, b_\sigma\, J_0(z) \} \{ v_\perp\, J'_0(z') + v_\parallel\, b'_{\sigma'}\, J_0(z') \} \right), \tag{33}$$

$$C_{00} = \frac{1}{\gamma}\left\{ J_0(z)\, J_0(z')\, b_\sigma\, b'_{\sigma'} + \frac{J_0(z)}{\omega_0}\left(\frac{k_\perp\, \omega_0}{\omega_0^2 - \Omega_e^2}(a_\sigma\, \omega_0 + \Omega_e) + k_\parallel\, b_\sigma \right) \right.$$

$$\times \{ v_\perp\, J'_0(z') + v_\parallel\, b'_{\sigma'}\, J_0(z') \} + \frac{J_0(z')}{\omega_0}\left(\frac{k'_\perp\, \omega_0}{\omega_0^2 - \Omega_e^2}(a'_{\sigma'}\, \omega_0 + \Omega_e) + k'_\parallel\, b'_{\sigma'} \right)$$

$$\times \{ v_\perp\, J'_0(z) + v_\parallel\, b_\sigma\, J_0(z) \} + \frac{k_\parallel\, k'_\parallel - \omega\omega'/c^2}{\omega_0^2} \{ v_\perp\, J'_0(z) + v_\parallel\, b_\sigma\, J_0(z) \}$$

$$\left. \times \{ v_\perp\, J'_0(z') + v_\parallel\, b'_{\sigma'}\, J_0(z') \} \right\}. \tag{34}$$

5. Maxwellian Electron Distribution

To proceed any further with the kinetic equations we need to specify the form of the electron distribution function $f(\boldsymbol{p})$. Here, we assume a nonrelativistic Maxwellian distribution of electrons, i.e.

$$f(\boldsymbol{p}) = \frac{n_e}{(2\pi)^{3/2}\, m_e^3\, V_e^3} \exp(-v^2/2\, V_e^2), \tag{35}$$

with $V_e = (k_b\, T_e/m_e)^{\frac{1}{2}}$ and T_e is the electron temperature. The normalisation is $\int d^3p\, f(\boldsymbol{p}) = n_e$. From (35) and (A7) we have

$$D_n f(\boldsymbol{p}) = \frac{n\Omega_e + (k_\parallel - k'_\parallel)v_\parallel}{T_e} \hbar f(\boldsymbol{p}),$$

where we have now set the Boltzmann constant to unity. Now using the delta function of the unmodified probability, which implies

$$n\Omega_e + (k_\parallel - k'_\parallel)v_\parallel = \omega'_{\sigma'} - \omega_\sigma,$$

we have

$$D_n f(p) = (\omega'_{\sigma'} - \omega_\sigma)\hbar f(p)/T_e. \tag{36}$$

Denoting the average over the distribution function by angle brackets and noting that

$$k_\parallel = k\cos\theta = (\mu_\sigma\,\omega_\sigma/c)\cos\theta\,, \tag{37}$$

the kinetic equation (12) can be written as

$$\begin{aligned}
\frac{\mathrm{d}N_\sigma(k)}{\mathrm{d}t} = 4(2\pi)^3 r_0^2\, n_e\, c^4 \int \frac{\mathrm{d}^3 k'}{(2\pi)^3}\Bigg\{ &\frac{R_\sigma(k)\,R_{\sigma'}(k')}{\omega_\sigma\,\omega'_{\sigma'}} \langle \delta\{\omega_\sigma(1-\mu_\sigma\beta_\parallel\cos\theta) \\
&-\omega'_{\sigma'}(1-\mu'_{\sigma'}\beta_\parallel\cos\theta')\}\,\overline{|a_{\sigma\sigma'}|^2}\rangle \Bigg(N_{\sigma'}(k') - N_\sigma(k) \\
&+\frac{\hbar(\omega'_{\sigma'}-\omega_\sigma)}{T_e}\{N_{\sigma'}(k')\,N_\sigma(k) + \tfrac{1}{2}N_{\sigma'}(k') + \tfrac{1}{2}N_\sigma(k)\}\Bigg) \\
&+\frac{R_\sigma(k)\,R_\sigma(k')}{\omega_\sigma\,\omega'_\sigma}\langle\delta\{\omega_\sigma(1-\mu_\sigma\beta_\parallel\cos\theta) - \omega'_\sigma(1-\mu'_\sigma\beta_\parallel\cos\theta')\}\,\overline{|a_{\sigma\sigma}|^2}\rangle \\
&\times\Bigg(N_\sigma(k') - N_\sigma(k) + \frac{\hbar(\omega'_\sigma-\omega_\sigma)}{T_e}\{N_\sigma(k')\,N_\sigma(k) + \tfrac{1}{2}N_\sigma(k') + \tfrac{1}{2}N_\sigma(k)\}\Bigg)\Bigg\}.
\end{aligned} \tag{38}$$

A similar form applies for $\mathrm{d}N_{\sigma'}(k)/\mathrm{d}t$.

In order to generalise the Kompaneets equation we need to expand $\overline{|a_{\sigma\sigma'}|^2}$ in powers of $\beta_\parallel = v_\parallel/c$, up to first order. To this end we note

$$\omega_0 = \omega(1-\mu\beta_\parallel\cos\theta)\,, \tag{39}$$

so that

$$\begin{aligned}
\frac{1}{\omega_0^2 - \Omega_e^2} &= \frac{1}{\omega^2(1-\mu\beta_\parallel\cos\theta)^2 - \Omega_e^2} \approx \frac{1}{\omega^2 - \Omega_e^2 - 2\mu\omega^2\beta_\parallel\cos\theta} \\
&\approx \frac{1}{\omega^2-\Omega_e^2}\left(1 + \frac{2\mu\omega^2\beta_\parallel\cos\theta}{\omega^2-\Omega_e^2} + \dots\right),
\end{aligned} \tag{40}$$

where we have dropped the mode label.

Also to lowest order the Bessel function $J_0(z)$ and its derivative $J'_0(z)$ can be written as $J_0(z) \approx 1$, $J'_0(z) \approx -\frac{1}{2}z$. Hence, we have

$$\begin{aligned}
A_{00} = &\frac{\omega}{\omega^2-\Omega_e^2}(1-\mu_\sigma\beta_\parallel\cos\theta)\left(1 + \frac{2\mu_\sigma\,\omega^2\beta_\parallel\cos\theta}{\omega^2-\Omega_e^2}\right) \\
&\times\{\omega(1-\mu_\sigma\beta_\parallel\cos\theta)(a_\sigma\,a'_{\sigma'}+1) + \Omega_e(a_\sigma + a'_{\sigma'})\}\,,
\end{aligned}$$

and after some simple algebra we find to lowest order in $\beta_\parallel$ that

$$A_{00} \approx \frac{\omega}{\omega^2-\Omega_e^2}\left\{\omega(a_\sigma a'_{\sigma'}+1)+\Omega_e(a_\sigma+a'_{\sigma'})\right.$$

$$\left.+\mu_\sigma\beta_\parallel \cos\theta\left(\frac{2\Omega_e^2}{\omega^2-\Omega_e^2}\,\omega(a_\sigma a'_{\sigma'}+1)+\frac{\omega^2+\Omega_e^2}{\omega^2-\Omega_e^2}\,\Omega_e(a_\sigma+a'_{\sigma'})\right)\right\}. \tag{41}$$

Similarly we have

$$B_{00} = \frac{\omega}{\omega^2-\Omega_e^2}\left\{\omega(a_\sigma+a'_{\sigma'})+\Omega_e(a_\sigma a'_{\sigma'}+1)\right.$$

$$\left.+\mu_\sigma\beta_\parallel \cos\theta\left(\frac{2\Omega_e^2}{\omega^2-\Omega_e^2}\,\omega(a_\sigma+a'_{\sigma'})+\frac{\omega^2+\Omega_e^2}{\omega^2-\Omega_e^2}\,\Omega_e(a_\sigma a'_{\sigma'}+1)\right)\right\}. \tag{42}$$

Before we write C_{00} note that

$$\frac{1}{\omega_0}\left(\frac{k_\perp\omega_0}{\omega^2-\Omega_e^2}(a_\sigma\omega_0+\Omega_e)+k_\parallel b_\sigma\right)\{v_\perp J'_0(z')+v_\parallel b'_{\sigma'}J_0(z')\}$$

$$\approx \frac{1}{\omega}\left(\frac{\omega^2\mu_\sigma\sin\theta}{\omega^2-\Omega_e^2}(a_\sigma\omega+\Omega_e)+\omega\mu_\sigma\cos\theta\, b_\sigma\right)v_\parallel b'_{\sigma'}.$$

Hence, we find

$$C_{00} \approx b_\sigma b'_{\sigma'}+\left(\frac{\omega\mu\sin\theta(a_\sigma\omega+\Omega_e)}{\omega^2-\Omega_e^2}+\mu b_\sigma\cos\theta\right)b'_{\sigma'}\beta_\parallel$$

$$+\frac{\omega'}{\omega}\left(\frac{\omega\mu'\sin\theta'}{\omega^2-\Omega_e^2}(a'_{\sigma'}\omega+\Omega_e)+\mu' b'_{\sigma'}\cos\theta'\right)\beta_\parallel b_\sigma, \tag{43}$$

where $\mu=\mu_\sigma$ and $\mu'=\mu'_{\sigma'}$. Using (41), (42) and (43) we can write

$$\overline{|a_{\sigma\sigma'}|^2} \approx A^{(0)}_{\sigma\sigma'}+\beta_\parallel A^{(1)}_{\sigma\sigma'}, \tag{44}$$

with

$$A^{(0)}_{\sigma\sigma'} = \frac{\omega^2}{2(\omega^2-\Omega_e^2)^2}\Big[(\omega^2+\Omega_e^2)\{(a_\sigma a'_{\sigma'}+1)^2+(a_\sigma+a'_{\sigma'})^2\}$$

$$+4\omega\Omega_e(a_\sigma a'_{\sigma'}+1)(a_\sigma+a'_{\sigma'})\Big]+b_\sigma b_{\sigma'}, \tag{45}$$

$$A^{(1)}_{\sigma\sigma'} = 2b_\sigma b'_{\sigma'}\left\{\left(\frac{\omega\mu_\sigma\sin\theta}{\omega^2-\Omega_e^2}(a_\sigma\omega+\Omega_e)+\mu_\sigma\cos\theta\, b_\sigma\right)b'_{\sigma'}\right.$$

$$+\left(\frac{\omega\mu'_{\sigma'}\sin\theta'}{\omega^2-\Omega_e^2}(a'_{\sigma'}\omega+\Omega_e)+\mu'_{\sigma'}\cos\theta'\, b'_{\sigma'}\right)\frac{\omega' b_\sigma}{\omega}$$

$$\left.-\mu_\sigma\cos\theta\,\omega\frac{\partial A^{(0)}_{\sigma\sigma'}}{\partial\omega}\right\}. \tag{46}$$

Our next step is to expand $N(k')$, rewritten as $N(k',\theta')$, in a Taylor series in $k'-k$:

$$N(k',\theta') \approx \left(1+(k'-k)\frac{\partial}{\partial k}+\tfrac{1}{2}(k'-k)^2\frac{\partial^2}{\partial k^2}\right)N(k,\theta'), \tag{47}$$

where we use

$$k'-k = \frac{\mu'\omega'-\mu\omega}{c} = \frac{\mu'-\mu}{c}\,\omega\,\frac{\mu'}{c}(\omega'-\omega). \tag{48}$$

Also, the delta function implies that

$$\omega' = \omega\,\frac{1-\mu\beta_\parallel\cos\theta}{1-\mu'\beta_\parallel\cos\theta'},$$

and thus

$$\frac{\omega'-\omega}{\omega} \approx \beta_\parallel(\mu'\cos\theta'-\mu\cos\theta)(1+\mu'\beta_\parallel\cos\theta'). \tag{49}$$

So, in (48), we have

$$\frac{k'-k}{k} \approx \frac{\mu'-\mu}{\mu}+\frac{\beta_\parallel\,\mu'}{\mu}(\mu'\cos\theta'-\mu\cos\theta)(1+\mu'\beta_\parallel\cos\theta'). \tag{50}$$

Next we perform the integration over the delta function and the associated expansion in $\beta_\parallel$ and $(\omega'-\omega)/\omega$ in (38). Note that

$$\int\frac{d^3k'}{(2\pi)^3}\,\frac{R_{\sigma'}(k')}{\omega'}\,\delta\{\omega'(1-\mu'_{\sigma'}\beta_\parallel\cos\theta')-\omega(1-\mu_\sigma\beta_\parallel\cos\theta)\}$$

$$= \int\frac{d^2\Omega'}{(2\pi)^3}\int\frac{d\omega'}{c}\,\frac{\partial(\omega'\mu'_{\sigma'})}{\partial\omega'}\,\frac{(\omega'\mu'_{\sigma'})^2}{c^2\omega'}\,\frac{1}{\xi'^2_{\sigma'}(1+T'^2_{\sigma'})}$$

$$\times\frac{1}{2\mu'_{\sigma'}\,\partial(\omega'\mu'_{\sigma'})/\partial\omega'}\,\frac{\delta(\omega'-\bar\omega')}{|1-\{\partial(\omega'\mu'_{\sigma'})/\partial\omega'\}\beta_\parallel\cos\theta'|}, \tag{51}$$

where we have changed the variable of integration from k' to ω', used the result (18) for $R_{\sigma'}(k')$ and made use of

$$\delta\{g(y)-y_0\} = \frac{\delta(y-\bar y_0)}{dg(y)/dy},$$

where $\bar y_0$ is the value of y which satisfies $g(y) = y_0$. Hence, in (51) $\bar\omega'$ is the value of ω' which satisfies

$$\omega'(1-\mu'_{\sigma'}\beta_\parallel\cos\theta) = \omega(1-\mu_\sigma\beta_\parallel\cos\theta).$$

Performing the integral over ω' in (51) and replacing $\bar\omega'$ by ω' the right-hand side of (51) becomes

$$\int\frac{d^2\Omega'}{(2\pi)^3}\,\frac{\mu'_{\sigma'}\omega'}{2c^3}\,\frac{1}{\xi'^2_{\sigma'}(1+T'^2_{\sigma'})}\,\frac{1}{|1-\{\partial(\omega'\mu'_{\sigma'})/\partial\omega'\}\beta_\parallel\cos\theta'|}.$$

Then, using

$$\mu'_{\sigma'}\,\omega' = \mu_\sigma\,\omega\,\frac{\mu'_{\sigma'}\,\omega'}{\mu_\sigma\,\omega}$$

$$\approx \mu_\sigma\,\omega\left(1+\frac{\mu'_{\sigma'}-\mu_\sigma}{\mu_\sigma}\right)\{1+(\mu'_{\sigma'}\,\cos\theta'-\mu_\sigma\,\cos\theta)\beta_\parallel\}\,, \tag{52}$$

we obtain

$$\int\frac{\mathrm{d}^3k'}{(2\pi)^3}\,\frac{R_{\sigma'}(k')}{\omega'}\,\delta\{\omega'(1-\mu'_{\sigma'}\,\beta_\parallel\,\cos\theta')-\omega(1-\mu_\sigma\,\beta_\parallel\,\cos\theta)\}$$

$$\approx\int\frac{\mathrm{d}^2\Omega'}{(2\pi)^3}\,\frac{\mu_\sigma\,\omega}{2c^3}\,\frac{1}{\xi'^2_{\sigma'}(1+T'^2_{\sigma'})}\left(1+\frac{\mu'_{\sigma'}-\mu_\sigma}{\mu_\sigma}\right)$$

$$\times\left(1+(\mu'_{\sigma'}\,\cos\theta'-\mu_\sigma\,\cos\theta)\beta_\parallel+\frac{\partial(\omega'\mu'_{\sigma'})}{\partial\omega'}\,\beta_\parallel\,\cos\theta'\right), \tag{53}$$

where we ignore any change in $\xi'^2_{\sigma'}$ and $(1+T'^2_{\sigma'})$ over the range $\omega'-\omega$ in (53). Before we derive the frequency diffusion equation for o-mode and z-mode photons we need one more result, the Thomson scattering probability for magnetoionic waves by stationary electrons. For stationary electrons (14) implies

$$\overline{|a_{\sigma\sigma'}|^2} = A^{(0)}_{\sigma\sigma'}\,,$$

and then (4) and (45) imply

$$\overline{w_{\sigma\sigma'}(p{=}0,\,k,\,k')} = \frac{(2\pi)^3}{m_e^2\,\omega^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 g_{\sigma\sigma'}(\omega,\theta,\theta')\,\delta(\omega'-\omega)\,, \tag{54}$$

with

$$g_{\sigma\sigma'}(\omega,\theta,\theta') = \left((1+T_\sigma^2)(1+T'^2_{\sigma'})\mu'_{\sigma'}\,\frac{\partial(\omega'\mu'_{\sigma'})}{\partial\omega'}\,\mu_\sigma\,\frac{\partial(\omega\mu_\sigma)}{\partial\omega}\right)^{-1}$$

$$\times\left(\frac{\omega^2}{2(\omega^2-\Omega_e^2)^2}[(\omega^2+\Omega_e^2)\{(a_\sigma\,a'_{\sigma'}+1)^2+(a_\sigma+a'_{\sigma'})^2\}\right.$$

$$\left.+2\omega\Omega_e(a_\sigma\,a'_{\sigma'}+1)(a_\sigma+a'_{\sigma'})]+b_\sigma^2\,b'^2_{\sigma'}\right). \tag{55}$$

Here, primed quantities are evaluated at θ' rather than θ. For the frequency domain of interest, $\omega_p \ll \omega < \Omega_e$, the refractive indices are close to unity, as may be shown using (16), (20) and (21), so that $\mu_\sigma \approx 1$. Also, K_σ is negligible and we have

$$T_o = -\frac{1}{T_z} \approx -\frac{\Omega_e}{\omega}\,\frac{\sin^2\theta}{\cos\theta}\,. \tag{56}$$

So using (19) we find that

$$a_o \approx -Y\sin^2\theta\,,\qquad a_z \approx \cos^2\theta/Y\sin\theta\,, \tag{57}$$

$$b_o \approx \tan\theta\;a_o\,,\qquad b_z \approx -\tan\theta\;a_z\,. \tag{58}$$

We define

$$f_{\sigma\sigma'}(\omega,\theta,\theta') = \mu_{\sigma'}\,\frac{\partial(\omega'\mu'_{\sigma'})}{\partial\omega'}\,\mu_{\sigma}\,\frac{\partial(\omega\mu_{\sigma})}{\partial\omega}\,g_{\sigma\sigma'}(\omega,\theta,\theta'), \tag{59}$$

and expand it to lowest order in Y^{-1}:

$$f_{\mathrm{oo}'} \approx \sin^2\theta\,\sin^2\theta', \qquad f_{\mathrm{zz}'} \approx 1/2\,Y^2, \tag{60a, b}$$

$$f_{\mathrm{oz}'} \approx \frac{1}{2\,Y^2}\left(\cos^2\theta + \frac{2\,Y^2}{Y'^2}\,\frac{\sin^2\theta}{\tan^2\theta'}\right), \tag{60c}$$

$$f_{\mathrm{zo}'} \approx \frac{1}{2\,Y'^2}\left(\cos^2\theta' + \frac{2\,Y'^2}{Y^2}\,\frac{\sin^2\theta'}{\tan^2\theta}\right). \tag{60d}$$

These results indicate that o–o scattering is anisotropic and is of order $2\,Y^2$ faster than the scattering of z-mode photons and that of mode conversion z $\rightarrow$ o and o $\rightarrow$ z.

We consider the expansion (47) in (38), ignoring the difference of the refractive index from unity and the induced and quantum terms. Using the results (60) the leading terms in (38) give

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} N_{\mathrm{o}}(k,\theta) \\ N_{\mathrm{z}}(k,\theta)\end{pmatrix} = 2\pi r_0^2\, n_{\mathrm{e}}\, c\int_{-1}^{1}\mathrm{d}(\cos\theta')$$

$$\times\begin{pmatrix} f_{\mathrm{oo}'}\{N_{\mathrm{o}}(k,\theta')-N_{\mathrm{o}}(k,\theta)\}+f_{\mathrm{oz}'}\{N_{\mathrm{z}}(k,\theta')-N_{\mathrm{o}}(k,\theta)\} \\ f_{\mathrm{zo}'}\{N_{\mathrm{o}}(k,\theta')-N_{\mathrm{z}}(k,\theta)\}+f_{\mathrm{zz}'}\{N_{\mathrm{z}}(k,\theta')-N_{\mathrm{z}}(k,\theta)\}\end{pmatrix} \tag{61}$$

$$= 2\pi r_0^2\, n_{\mathrm{e}}\, c\int_{-1}^{1}\mathrm{d}(\cos\theta')\begin{pmatrix} f_{\mathrm{oo}'} & f_{\mathrm{oz}'} \\ f_{\mathrm{zo}'} & f_{\mathrm{zz}'}\end{pmatrix}\begin{pmatrix} N_{\mathrm{o}}(k,\theta') \\ N_{\mathrm{z}}(k,\theta')\end{pmatrix} - 2\pi r_0^2\, n_{\mathrm{e}}\, c\begin{pmatrix} F_{\mathrm{o}}\, N_{\mathrm{o}}(k,\theta) \\ F_{\mathrm{z}}\, N_{\mathrm{z}}(k,\theta)\end{pmatrix}, \tag{62}$$

with

$$\begin{pmatrix} F_{\mathrm{o}} \\ F_{\mathrm{z}}\end{pmatrix} = \int_{-1}^{1}\mathrm{d}(\cos\theta')\begin{pmatrix} f_{\mathrm{oo}'}+f_{\mathrm{oz}'} \\ f_{\mathrm{zo}'}+f_{\mathrm{zz}'}\end{pmatrix} = \frac{2}{3}\begin{pmatrix} \sin^2\theta \\ 1/\,Y^2\sin^2\theta\end{pmatrix}. \tag{63}$$

In (63) we have assumed that $\theta, \theta' \gg Y^{-\frac{1}{2}}$.

6. Frequency Diffusion Equation

Returning to the kinetic equation (38) let us re-express it in the form

$$\frac{\mathrm{d}N_{\sigma}(k)}{\mathrm{d}t} = \left[\frac{\mathrm{d}N_{\sigma}(k)}{\mathrm{d}t}\right]^{(\mathrm{S})} + \left[\frac{\mathrm{d}N_{\sigma}(k)}{\mathrm{d}t}\right]^{(\mathrm{D})} + \left[\frac{\mathrm{d}N_{\sigma}(k)}{\mathrm{d}t}\right]^{(\mathrm{Q})}$$

$$+ \left[\frac{\mathrm{d}N_{\sigma}(k)}{\mathrm{d}t}\right]^{(\mathrm{I})} + \left[\frac{\mathrm{d}N_{\sigma}(k)}{\mathrm{d}t}\right]^{(\mu)}, \tag{64}$$

where S refers to spatial scattering which is discussed in more detail in Section 7, D the effect of the Doppler shifts, Q the effect of quantum recoil, I the effect of induced scattering and μ the changes in the refractive index.

The μ-dependent terms include a term which is simply a correction in (53) and terms in (47) which rely on expanding $N(k')$ in $k'-k$ rather than $\omega'-\omega$. Since our studies are motivated by the theory of X-ray pulsars, we ignore these terms as they are small in such applications.

We first consider the Doppler terms. These arise from the expansion (47) for the term $N_{\sigma'}(k')$ in the kinetic equation, and thus we obtain

$$\begin{aligned}
\left[\frac{\mathrm{d}N_\sigma(\omega,\theta)}{\mathrm{d}t}\right]^{(\mathrm{D})} &= 4(2\pi)^3 r_0^2\, n_\mathrm{e}\, c^4 \int \frac{\mathrm{d}^3 k'}{(2\pi)^3}\left\{\frac{R_\sigma(k)\,R_{\sigma'}(k')}{\omega\omega'}\overline{\langle |a_{\sigma\sigma'}|^2}\right.\\
&\quad \times \delta\{\omega'(1-\mu'_{\sigma'}\beta_\parallel \cos\theta')-\omega(1-\mu_\sigma\beta_\parallel\cos\theta)\}\rangle\Bigg((k'-k)\frac{\partial N_{\sigma'}(k,\theta')}{\partial k}\\
&\quad +\tfrac{1}{2}(k'-k)^2\frac{\partial^2 N_{\sigma'}(k,\theta')}{\partial k^2}\Bigg)+\frac{R_\sigma(k)\,R_\sigma(k')}{\omega\omega'}\overline{\langle |a_{\sigma\sigma}|^2}\delta\{\omega'(1-\mu'_\sigma\beta_\parallel\cos\theta')\\
&\quad \left.-\omega(1-\mu_\sigma\beta_\parallel\cos\theta)\}\Bigg((k'-k)\frac{\partial N_\sigma(k,\theta')}{\partial k}+\tfrac{1}{2}(k'-k)^2\frac{\partial^2 N_\sigma(k,\theta')}{\partial k^2}\Bigg)\right\}\\
&= \frac{2\pi r_0^2\, n_\mathrm{e}\, c}{(1+T_\sigma^2)\,\partial(\omega\mu_\sigma)/\partial\omega}\int_{-1}^{1}\mathrm{d}(\cos\theta')\Bigg[\frac{\mu'_{\sigma'}}{\mu_\sigma}\,\frac{1}{1+T'^2_{\sigma'}}\\
&\quad \times(1+a_{\sigma\sigma'}\beta_\parallel)(A^{(0)}_{\sigma\sigma'}+\beta_\parallel A^{(1)}_{\sigma\sigma'})\\
&\quad \times\beta_\parallel\frac{\mu'_{\sigma'}}{\mu_\sigma}(\mu'_{\sigma'}\cos\theta'-\mu_\sigma\cos\theta)\Bigg((1+\mu'_{\sigma'}\beta_\parallel\cos\theta')k\frac{\partial N_{\sigma'}(k,\theta')}{\partial k}\\
&\quad +\tfrac{1}{2}\beta_\parallel\frac{\mu'_{\sigma'}}{\mu_\sigma}(\mu'_{\sigma'}\cos\theta'-\mu_\sigma\cos\theta)k^2\frac{\partial^2 N_{\sigma'}(k,\theta')}{\partial k^2}\Bigg)+\frac{\mu'_\sigma}{\mu_\sigma}\,\frac{1}{1+T'^2_\sigma}\\
&\quad \times(1+a_{\sigma\sigma}\beta_\parallel)(A^{(0)}_{\sigma\sigma}+\beta_\parallel A^{(1)}_{\sigma\sigma})\beta_\parallel\frac{\mu'_\sigma}{\mu_\sigma}\\
&\quad \times(\mu'_\sigma\cos\theta'-\mu_\sigma\cos\theta)\Bigg((1+\mu'_\sigma\beta_\parallel\cos\theta')k\\
&\quad \times\frac{\partial N_\sigma(k,\theta')}{\partial k}+\tfrac{1}{2}\beta_\parallel\frac{\mu'_\sigma}{\mu_\sigma}(\mu'_\sigma\cos\theta'-\mu_\sigma\cos\theta)k^2\frac{\partial^2 N_\sigma(k,\theta')}{\partial k^2}\Bigg)\Bigg],
\end{aligned}\tag{65}$$

where we have used (50) and (53) and neglected the term $(\mu'_{\sigma'}-\mu_\sigma)/\mu_\sigma$, and where

$$a_{\sigma\sigma'}=\mu'_{\sigma'}\cos\theta'-\mu_\sigma\cos\theta+\frac{\partial(\omega'\mu'_{\sigma'})}{\partial\omega}\cos\theta', \tag{66}$$

with $A^{(0)}_{\sigma\sigma}$ and $A^{(1)}_{\sigma\sigma'}$ defined in (45) and (46). Also the large square brackets implicitly include the average over the Maxwellian distribution.

In performing the average over the Maxwellian we use

$$\langle\beta_\parallel\rangle = 0, \qquad \langle\beta_\parallel^2\rangle = \tfrac{1}{3}\langle\beta^2\rangle = T_e/m_e c^2,$$

so that we need to keep terms up to $\beta_\parallel^2$ when multiplying out terms in (65). We find that

$$\begin{aligned}
\left[\frac{\mathrm{d}N_\sigma(\omega,\theta)}{\mathrm{d}t}\right]^{(\mathrm{D})} &= \frac{2\pi r_0^2\, n_e\, c}{(1+T_\sigma^2)\,\partial(\omega\mu_\sigma)/\partial\omega}\,\frac{T_e}{m_e c^2}\int_{-1}^{1}\mathrm{d}(\cos\theta')\left\{\left(\frac{\mu'_{\sigma'}}{\mu_\sigma}\right)^2\frac{1}{1+T'^2_{\sigma'}}\right.\\
&\quad\times(\mu'_{\sigma'}\cos\theta'-\mu_\sigma\cos\theta)\left((A^{(0)}_{\sigma\sigma'}\,a_{\sigma\sigma'}+A^{(1)}_{\sigma\sigma'})k\,\frac{\partial N_{\sigma'}(k,\theta')}{\partial k}\right.\\
&\quad+A^{(0)}_{\sigma\sigma'}\,\mu'_{\sigma'}\cos\theta'\,k\,\frac{\partial N_{\sigma'}(k,\theta')}{\partial k}\\
&\quad\left.+\tfrac{1}{2}A^{(0)}_{\sigma\sigma'}\,\frac{\mu'_{\sigma'}}{\mu_\sigma}(\mu'_{\sigma'}\cos\theta'-\mu_\sigma\cos\theta)k^2\,\frac{\partial^2 N_{\sigma'}(k,\theta')}{\partial k^2}\right)+\left(\frac{\mu'_\sigma}{\mu_\sigma}\right)^2\frac{1}{1+T'^2_\sigma}\\
&\quad\times(\mu'_\sigma\cos\theta'-\mu_\sigma\cos\theta)\left((A^{(0)}_{\sigma\sigma}\,a_{\sigma\sigma}+A^{(1)}_{\sigma\sigma})k\,\frac{\partial N_\sigma(k,\theta')}{\partial k}\right.\\
&\quad+A^{(0)}_{\sigma\sigma}\,\mu'_\sigma\cos\theta'\,k\,\frac{\partial N_\sigma(k,\theta')}{\partial k}\\
&\quad\left.\left.+\tfrac{1}{2}A^{(0)}_{\sigma\sigma}\,\frac{\mu'_\sigma}{\mu_\sigma}(\mu'_\sigma\cos\theta'-\mu_\sigma\cos\theta)k^2\,\frac{\partial^2 N_\sigma(k,\theta')}{\partial k^2}\right)\right\}.
\end{aligned} \tag{67}$$

Let us consider $A^{(0)}_{\sigma\sigma'}$ and $A^{(1)}_{\sigma\sigma'}$ in more detail. We may write from (45)

$$\begin{aligned}
A^{(0)}_{\sigma\sigma'} &= \frac{1}{2(1-Y^2)^2}[(1+Y^2)\{(a_\sigma a'_{\sigma'}+1)^2+(a_\sigma+a'_{\sigma'})^2\}\\
&\quad+4Y(a_\sigma a'_{\sigma'}+1)(a_\sigma a'_{\sigma'})]+b_\sigma^2\,b_{\sigma'}^2.
\end{aligned} \tag{68}$$

There are a number of contributions to $A^{(1)}_{\sigma\sigma'}$, and one such arises from the Y dependence in $A^{(0)}_{\sigma\sigma'}$. This dependence stems from (39):

$$\Omega_e/\omega_0 \approx Y(1+\mu_\sigma\beta_\parallel\cos\theta).$$

Making the replacement $\omega\,\partial/\partial\omega \to -Y\,\partial/\partial Y$ gives

$$\begin{aligned}
\mu_\sigma\beta_\parallel\cos\theta\,Y\,\frac{\partial A^{(0)}_{\sigma\sigma'}}{\partial Y} &= \frac{\mu_\sigma\beta_\parallel\cos\theta}{(1-Y^2)^3}[Y^2(3+Y^2)\{(a_\sigma a'_{\sigma'}+1)^2\\
&\quad+(a_\sigma+a'_{\sigma'})^2\}+(1+3Y^2)2Y(a_\sigma a'_{\sigma'}+1)(a_\sigma+a'_{\sigma'})].
\end{aligned} \tag{69}$$

Another contribution arises from the first term in (43) which we express in the form

$$\Delta C_{00}^2 = 2b_\sigma b'_{\sigma'}\beta_\parallel\left(\frac{1}{1-Y^2}\{\mu_\sigma \sin\theta(a_\sigma+Y)b'_{\sigma'}\right.$$

$$\left.+\mu'_{\sigma'}\sin\theta'(a'_{\sigma'}+Y)b_\sigma\}+b_\sigma b'_{\sigma'}(\mu_\sigma\cos\theta+\mu'_{\sigma'}\cos\theta')\right). \qquad (70)$$

Finally, there is an additional term due to the fact that $T'_{\sigma'}$, $a'_{\sigma'}$ and $b'_{\sigma'}$ all depend on $\omega' = \omega+\Delta\omega$:

$$(1+T'^2_{\sigma'})(\omega'-\omega)\frac{\partial}{\partial\omega'}\left(\frac{A^{(0)}_{\sigma\sigma'}}{1+T'^2_{\sigma'}}\right). \qquad (71)$$

Thus, if we write (59) as

$$f_{\sigma\sigma'}(\omega,\omega',Y) = \frac{1}{(1+T^2_\sigma)(1+T'^2_{\sigma'})}\left(\frac{1}{2(1-Y^2)^2}[(1+Y^2)\{(a_\sigma a'_{\sigma'}+1)^2\right.$$

$$\left.+(a_\sigma+a'_{\sigma'})^2\}+4Y(a_\sigma a'_{\sigma'}+1)(a_\sigma+a'_{\sigma'})]+b^2_\sigma b'^2_{\sigma'}\right), \qquad (72)$$

then we have

$$f^{(0)}_{\sigma\sigma'} = f_{\sigma\sigma'}(\omega,\omega',Y),$$

$$\beta_\parallel f^{(1)}_{\sigma\sigma'} = \left(\mu_\sigma\beta_\parallel\cos\theta\, Y\frac{\partial}{\partial Y}+\beta_\parallel(\mu'_{\sigma'}\cos\theta'-\mu_\sigma\cos\theta)\omega\frac{\partial}{\partial\omega'}\right)$$

$$\times f^{(0)}_{\sigma\sigma'}(\omega,\omega',Y)+\frac{\Delta C^2_{00}}{(1+T^2_\sigma)(1+T'^2_{\sigma'})}, \qquad (73)$$

where use has been made of (49) with $1+\mu'_{\sigma'}\beta_\parallel\cos\theta' \approx 1$ in the second term of (73), which corresponds to (71). Also note that $\partial/\partial\omega'$ operates on $T'_{\sigma'}$, $a'_{\sigma'}$ and $b'_{\sigma'}$.

Consistent with our earlier neglect of the μ-dependence, we now set $\mu_\sigma = \mu'_{\sigma'} = 1$ so that (67) reduces to

$$\left[\frac{\mathrm{d}N_\sigma(\omega,\theta)}{\mathrm{d}t}\right]^{(\mathrm{D})} = \frac{2\pi r_0^2 n_\mathrm{e} c T_\mathrm{e}}{m_\mathrm{e}c^2}\int_{-1}^{1}\mathrm{d}(\cos\theta')(\cos\theta'-\cos\theta)$$

$$\times\left\{\left(\{(3\cos\theta'-\cos\theta)f^{(0)}_{\sigma\sigma'}\right.\right.$$

$$\left.+f^{(1)}_{\sigma\sigma'}\}\omega\frac{\partial N_{\sigma'}(\omega,\theta')}{\partial\omega}+\tfrac{1}{2}f^{(0)}_{\sigma\sigma'}(\cos\theta'-\cos\theta)\omega^2\frac{\partial^2 N_{\sigma'}(\omega,\theta')}{\partial\omega^2}\right)$$

$$+\left(\{(3\cos\theta'-\cos\theta)f^{(0)}_{\sigma\sigma}+f^{(1)}_{\sigma\sigma}\}\omega\frac{\partial N_\sigma(\omega,\theta')}{\partial\omega}\right.$$

$$\left.\left.+\tfrac{1}{2}f^{(0)}_{\sigma\sigma}(\cos\theta'-\cos\theta)\omega^2\frac{\partial^2 N_\sigma(\omega,\theta')}{\partial\omega^2}\right)\right\}. \qquad (74)$$

To simplify the remaining calculations we now assume that the distribution of photons is isotropic and write

$$\begin{pmatrix} F_{\sigma\sigma'} \\ F'_{\sigma\sigma'} \end{pmatrix} = \tfrac{1}{4}\int_{-1}^{1} \mathrm{d}(\cos\theta)\int_{-1}^{1} \mathrm{d}(\cos\theta')\begin{pmatrix} \cos^2\theta \\ \cos^2\theta' \end{pmatrix} f^{(0)}_{\sigma\sigma'}, \tag{75}$$

$$F^{(1)}_{\sigma\sigma'} = \tfrac{1}{4}\int_{-1}^{1} \mathrm{d}(\cos\theta)\int_{-1}^{1} \mathrm{d}(\cos\theta')(\cos\theta' - \cos\theta) f^{(1)}_{\sigma\sigma'}. \tag{76}$$

Then (74) yields after a short amount of algebra

$$\left[\frac{\mathrm{d}N_\sigma(\omega)}{\mathrm{d}t}\right]^{(\mathrm{D})} = \frac{4\pi r_0^2\, n_\mathrm{e}\, c\, T_\mathrm{e}}{m_\mathrm{e}\, c^2}\Bigg\{(3F'_{\sigma\sigma'} + F_{\sigma\sigma'} + F^{(1)}_{\sigma\sigma'})\omega\,\frac{\partial N_{\sigma'}(\omega)}{\partial\omega}$$
$$+\tfrac{1}{2}(F'_{\sigma\sigma'} + F_{\sigma\sigma'})\omega^2\,\frac{\partial^2 N_{\sigma'}(\omega)}{\partial\omega^2} + \Bigg((4F_{\sigma\sigma} + F^{(1)}_{\sigma\sigma})\omega\,\frac{\partial N_\sigma(\omega)}{\partial\omega}$$
$$+ F_{\sigma\sigma}\,\omega^2\,\frac{\partial^2 N(\omega)}{\partial\omega^2}\Bigg)\Bigg\}. \tag{77}$$

We have noted that $F'_{\sigma\sigma} = F_{\sigma\sigma}$ and the assumption of an isotropic distribution of photons allows one to remove terms $\partial N_\sigma/\partial\omega$ outside the integral.

The quantum recoil and induced scattering terms, which involve $N_\sigma(\boldsymbol{k})\, N_{\sigma'}(\boldsymbol{k}')$ and $\frac{1}{2}N_\sigma(\boldsymbol{k}) + \frac{1}{2}N_{\sigma'}(\boldsymbol{k}')$ respectively, may be treated in an analogous manner and reduce to a form similar to (77). All three terms combine into

$$\frac{\mathrm{d}N_\sigma(\omega)}{\mathrm{d}t} = \frac{4\pi r_0^2\, n_\mathrm{e}\, c\, T_\mathrm{e}}{m_\mathrm{e}\, c^2}\Bigg(\Bigg[(3F'_{\sigma\sigma'} + F_{\sigma\sigma'} + F^{(1)}_{\sigma\sigma'})\Bigg(\omega\,\frac{\partial N_{\sigma'}(\omega)}{\partial\omega}$$
$$+\frac{\hbar\omega}{T_\mathrm{e}}\{N_{\sigma'}(\omega)\, N_\sigma(\omega) + \tfrac{1}{2}N_{\sigma'}(\omega) + \tfrac{1}{2}N_\sigma(\omega)\Bigg) + \tfrac{1}{2}(F'_{\sigma\sigma'} + F_{\sigma\sigma'})$$
$$\times\left\{\omega^2\,\frac{\partial^2 N_{\sigma'}(\omega)}{\partial\omega^2} + \frac{\hbar\omega}{T_\mathrm{e}}\left(2N_\sigma(\omega)\omega\,\frac{\partial N_{\sigma'}(\omega)}{\partial\omega} + \omega\,\frac{\partial N_{\sigma'}(\omega)}{\partial\omega}\right)\right\}\Bigg]$$
$$+\Bigg\{(4F_{\sigma\sigma} + F^{(1)}_{\sigma\sigma})\Bigg(\omega\,\frac{\partial N_\sigma(\omega)}{\partial\omega} + \frac{\hbar\omega}{T_\mathrm{e}}\{N_\sigma^2(\omega) + N_\sigma(\omega)\}\Bigg)$$
$$+F_{\sigma\sigma}\Bigg(\omega^2\,\frac{\partial^2 N_\sigma(\omega)}{\partial\omega} + \frac{\hbar\omega^2}{T_\mathrm{e}}\,\frac{\partial}{\partial\omega}\{N_\sigma(\omega) + N_\sigma^2(\omega)\}\Bigg)\Bigg\}\Bigg). \tag{78}$$

In the evaluation of the $f^{(1)}_{\sigma\sigma}$ term we can see from (76) and (70) that the contribution from ΔC^2_{00} in (78) will be zero. Thus, inserting (73) and (76) and using (75) we find that

$$F^{(1)}_{\sigma\sigma} = -Y\,\frac{\partial F_{\sigma\sigma'}}{\partial Y} + \omega\,\frac{\partial}{\partial\omega'}(F'_{\sigma\sigma'} + F_{\sigma\sigma'}). \tag{79}$$

Recall that $\omega\partial/\partial\omega'$ involves derivatives with respect to $T'_{\sigma'}$, $a'_{\sigma'}$ and $b'_{\sigma'}$. In view of the fact that for $X \ll 1$, i.e. $\omega_p \ll \omega$, $F'_{\sigma\sigma'}$ and $F_{\sigma\sigma'}$ depend on ω' only through $Y' = \Omega_e/\omega'$, then we have

$$F^{(1)}_{\sigma\sigma'} \approx -Y\frac{\partial F_{\sigma\sigma'}}{\partial Y} - \frac{Y'^2}{Y}\frac{\partial}{\partial Y'}(F'_{\sigma\sigma'}+F_{\sigma\sigma'}). \tag{80}$$

As we have indicated in equations (60) the o–o′ scattering process is faster than that of z–z′ and o–z′ and z–o′ scattering by a factor $\approx 2Y^2$, and hence the leading term in (78) is that due to o–o′ scattering. In this case using (60) we have $f^{(0)}_{oo'} \approx \sin^2\theta\,\sin^2\theta'$, and hence $F'_{oo'} = F_{oo} = \frac{2}{3}\times\frac{2}{15}$ and $F^{(1)}_{oo} \approx 0$. Thus (78) implies

$$\frac{\mathrm{d}N^{o}(\omega)}{\mathrm{d}t} = \tfrac{2}{15}\sigma_T\, n_e\, c\frac{T_e}{m_e c^2}\,\omega\left(4+\frac{\partial}{\partial\omega}\right)$$

$$\times\left(\frac{\partial N^{o}(\omega)}{\partial\omega}+\frac{\hbar}{T_e}N^{o}(\omega)\{1+N^{o}(\omega)\}\right), \tag{81}$$

where the mode is now indicated by a superscript and σ_T the Thomson cross section is defined by $\sigma_T = \frac{8}{3}\pi r_0^2$. [Lyubarsky (1986) has derived an equation equivalent to (81) for the diffusion of o-mode photons in frequency.]

Notice that in dimensionless form the frequency diffusion equation is the same as the Kompaneets equation (1) except for the factor of 2/15, so effectively we have a smaller cross section $\sigma = (2/15)\sigma_T$ in the strongly magnetised limit.

Since, the scattering of o-mode photons is on a timescale $\approx 2Y^2$ faster than that of z-mode scattering it is of no use to derive a frequency diffusion equation for the scattering of z-mode photons. Also, such an equation would be rather tedious as we would have to include the contributions from z–z, z–o and o–z scattering.

7. Spatial Diffusion

In this section we consider spatial diffusion equations for the scattering of o-mode and z-mode photons. As discussed in Appendix 2 we assume that the distribution of o-mode photons is weakly inhomogeneous and that there is an associated weak anisotropy. We now denote the mode by a superscript (o or z) with subscripts $l = 0, 1, 2, \ldots$ available to describe terms in the Legendre expansion. We may now write for the o-mode in particular

$$\frac{\mathrm{d}N^{o}}{\mathrm{d}t} = \frac{\partial N^{o}}{\partial t} + c\boldsymbol{\kappa}.\frac{\partial N^{o}}{\partial \boldsymbol{x}}. \tag{82}$$

Here, we are assuming the refractive index of the plasma differs little from unity and that there is no bulk motion of the plasma [this could be easily incorporated by adding a term $\dot{\boldsymbol{p}}.\partial N^{o}/\partial\boldsymbol{p}$ to (82)]. The $\boldsymbol{\kappa}$ is a unit vector in the direction of $\boldsymbol{k}$. If we denote by $\bar{N}$ the isotropic part of N and let N_1 be a weakly anisotropic part associated with a gradient in the z-direction, then since we can write

$$N^{o} \approx \bar{N}^{o} + N^{o}_1\cos\theta + \ldots, \tag{83}$$

$[P_1(\cos\theta) = \cos\theta]$ equation (83) implies that

$$\frac{\partial \bar{N}^{\mathrm{o}}}{\partial t} + c\langle\cos^2\theta\rangle \frac{\partial N_1^{\mathrm{o}}}{\partial z} = 0, \tag{84}$$

$$c\langle\cos^2\theta\rangle \frac{\partial \bar{N}^{\mathrm{o}}}{\partial z} = -\sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c\langle\sin^2\theta\,\cos^2\theta\rangle\, N_1^{\mathrm{o}}, \tag{85}$$

where in both (84) and (85) we have used (A18), as well as assumed $N_2^{\mathrm{o}} \approx 0$ and neglected $\partial N_1^{\mathrm{o}}/\partial t$ in obtaining (85). The angle brackets denote integration over $\cos\theta$.

If on the other hand the anisotropy and gradient were along the x-axis then we would expand N^{o} in spherical harmonics rather than Legendre polynomials, and proceed in an analogous manner. Then, an average over angles, e.g. $\langle\cos^2\theta\rangle = 1/3$, gives the final result

$$\frac{\partial \bar{N}^{\mathrm{o}}}{\partial t} = \frac{5}{12}\frac{c}{\sigma_{\mathrm{T}}\, n_{\mathrm{e}}}\left(\frac{\partial^2 \bar{N}^{\mathrm{o}}}{\partial x^2} + \frac{\partial^2 \bar{N}^{\mathrm{o}}}{\partial y^2}\right) + \frac{5}{6}\frac{c}{\sigma_{\mathrm{T}}\, n_{\mathrm{e}}}\frac{\partial^2 \bar{N}^{\mathrm{o}}}{\partial z^2}, \tag{86}$$

or

$$\frac{\partial \bar{N}^{\mathrm{o}}}{\partial t} = D_{\perp}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{N}^{\mathrm{o}}}{\partial r}\right) + D_z\frac{\partial^2 \bar{N}^{\mathrm{o}}}{\partial z^2} \tag{87}$$

for the case of anisotropy and gradients in all directions, where

$$D_{\perp} = \frac{5}{12}\frac{c}{\sigma_{\mathrm{T}}\, n_{\mathrm{e}}}, \qquad D_z = \frac{5}{6}\frac{c}{\sigma_{\mathrm{T}}\, n_{\mathrm{e}}}. \tag{88}$$

In deriving (86) we have neglected the contribution from the z-mode. We find from (61) with (60) that, in fact, (84) has the non-vanishing right-hand side

$$\begin{aligned}\frac{\partial \bar{N}^{\mathrm{o}}}{\partial t} + \tfrac{1}{3}c\frac{\partial N_z^{\mathrm{o}}}{\partial z} &= 2\pi r_0^2\, n_{\mathrm{e}}\, c\int_{-1}^{1} \mathrm{d}(\cos\theta)\int_{-1}^{1}\mathrm{d}(\cos\theta')\tfrac{1}{2}\,\frac{1}{2\,Y^2}\\ &\quad\times\left(\cos^2\theta + \frac{2\sin^2\theta}{\tan^2\theta'}\right)(\bar{N}^{\mathrm{z}} - \bar{N}^{\mathrm{o}}) \\ &= 2\pi r_0^2\, n_{\mathrm{e}}\, c\,\frac{1}{2\,Y^2}\,\frac{2}{3}\int_{-1}^{1}\mathrm{d}(\cos\theta')\left(1 + \frac{4\cos^2\theta'}{\sin^2\theta'}\right)(\bar{N}^{\mathrm{z}} - \bar{N}^{\mathrm{o}}).\end{aligned} \tag{89}$$

We must cutoff the divergent integral at $|\cos\theta'| = 1 - 1/2\,Y$ which corresponds to a breakdown of the condition $Y\sin^2\theta' \gg 1$ (cf. equation 57). Thus, we obtain

$$\begin{aligned}\int_{-1+\Delta}^{1-\Delta}\mathrm{d}(\cos\theta')\,\frac{\cos^2\theta'}{\sin^2\theta'} &= \ln(2/\Delta - 1) - (2 - 2\Delta) \\ &\approx \ln 4Y - 2; \qquad \Delta = 1/2\,Y \ll 1,\end{aligned}$$

so that (89) becomes

$$\frac{\partial \bar{N}^{\mathrm{o}}}{\partial t}+\tfrac{1}{3}c\frac{\partial N^{\mathrm{o}}_{z}}{\partial t}=\tfrac{1}{4}\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,c\,\frac{1}{Y^{2}}(2\ln 4Y-3)(\bar{N}^{\mathrm{z}}-\bar{N}^{\mathrm{o}}). \tag{90}$$

When considering the diffusion of the z-mode photons there is an analogous term in $\partial\bar{N}^{\mathrm{z}}/\partial t$. So, let us write (90) and the analogous term for the z-mode as

$$\frac{\partial \bar{N}^{\mathrm{o}}}{\partial t}=\frac{\zeta\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,c}{Y^{2}}(\bar{N}^{\mathrm{z}}-\bar{N}^{\mathrm{o}}),\qquad \frac{\partial \bar{N}^{\mathrm{z}}}{\partial t}=\frac{\zeta\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,c}{Y^{2}}(\bar{N}^{\mathrm{o}}-\bar{N}^{\mathrm{z}}), \tag{91a, b}$$

where $\zeta=\tfrac{1}{4}(2\ln 4Y-3)$ is of order unity.

The diffusion of z-mode photons is isotropic with a cross section smaller than the Thomson cross section by a factor $\tfrac{1}{2}Y^{2}$ (cf. equation 60). Hence, we have

$$\frac{\partial \bar{N}^{\mathrm{z}}}{\partial t}=\frac{2Y^{2}c}{3\sigma_{\mathrm{T}}\,n_{\mathrm{e}}}\nabla^{2}\bar{N}^{\mathrm{z}}. \tag{92}$$

The diffusion approximation (92) applies for a source thickness L, such that

$$\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,L/Y^{2}\gg 1, \tag{93}$$

i.e. the source has to be optically thick to Thomson scattering of z-mode photons. If, however, the inequality is reversed, i.e. $\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,L/Y^{2}\ll 1$ but with $\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,L\gg 1$, so that the source is optically thin to Thomson scattering of z-mode photons but optically thick to Thomson scattering of o-mode photons, i.e. they will diffuse, then we need to supplement (89) and (91) by

$$\frac{\partial \bar{N}^{\mathrm{z}}}{\partial t}=-\frac{c\bar{N}^{\mathrm{z}}}{L}, \tag{94}$$

rather than (92), i.e. corresponding to free escape of z-mode photons. So, if we ignore diffusion in frequency, then a reasonable set of equations describing the spatial scattering of o-mode and z-mode photons is

$$\frac{\partial \bar{N}^{\mathrm{o}}}{\partial t}=\frac{c}{\sigma_{\mathrm{T}}\,n_{\mathrm{e}}}\left\{\frac{5}{12}\,\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial \bar{N}^{\mathrm{o}}}{\partial r}\right)+\frac{5}{6}\,\frac{\partial^{2}\bar{N}^{\mathrm{o}}}{\partial z^{2}}\right\}$$

$$+\frac{\zeta\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,c}{Y^{2}}(\bar{N}^{\mathrm{z}}-\bar{N}^{\mathrm{o}})-\frac{c\bar{N}^{\mathrm{o}}}{L}, \tag{95}$$

$$\frac{\partial \bar{N}^{\mathrm{z}}}{\partial t}=\frac{2Y^{2}c}{3\sigma_{\mathrm{T}}\,n_{\mathrm{e}}}\nabla^{2}\bar{N}^{\mathrm{z}}+\frac{\zeta\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,c}{Y^{2}}(\bar{N}^{\mathrm{o}}-\bar{N}^{\mathrm{z}})-\frac{c\bar{N}^{\mathrm{z}}}{L}. \tag{96}$$

This neglects photons in the range $\sin^{2}\theta\lesssim 1/Y$ where they are nearly circularly polarised.

Our final set of scattering equations for o-mode and z-mode photons, including both spatial and frequency diffusion, may then be written as

$$\begin{aligned}\frac{\partial \bar{N}^{\mathrm{o}}}{\partial t} = {} & \frac{2}{15}\frac{\sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c T_{\mathrm{e}}}{m_{\mathrm{e}}\, c^2}\,\omega\left(4+\frac{\partial}{\partial\omega}\right)\left(\frac{\partial \bar{N}^{\mathrm{o}}}{\partial\omega}+\frac{\hbar \bar{N}^{\mathrm{o}}}{T_{\mathrm{e}}}(1+\bar{N}^{\mathrm{o}})\right)\\ & +\frac{c}{\sigma_{\mathrm{T}}\, n_{\mathrm{e}}}\left\{\frac{5}{12}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{N}^{\mathrm{o}}}{\partial r}\right)+\frac{5}{6}\frac{\partial^2 \bar{N}^{\mathrm{o}}}{\partial z^2}\right\}\\ & +\frac{\zeta\sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c}{Y^2}(\bar{N}^{\mathrm{z}}-\bar{N}^{\mathrm{o}})-\frac{c\bar{N}^{\mathrm{o}}}{L},\end{aligned} \tag{97}$$

$$\frac{\partial \bar{N}^{\mathrm{z}}}{\partial t} = \frac{2\,Y^2 c}{3\sigma_{\mathrm{T}}\, n_{\mathrm{e}}}\nabla^2 \bar{N}^{\mathrm{z}}+\frac{\zeta\sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c}{Y^2}(\bar{N}^{\mathrm{o}}-\bar{N}^{\mathrm{z}})-\frac{c\bar{N}^{\mathrm{z}}}{L}. \tag{98}$$

8. Conclusions

In this paper we have presented new equations describing how the distribution of o-mode and z-mode photons evolve due to Thomson scattering in a strongly magnetised plasma. Our procedure is based on semi-classical kinetic equations describing the scattering process which include the first quantum correction, allowing for the effects of recoil of the electron due to emission of a photon.

We have considered two aspects of the scattering: diffusion in frequency due to scattering by nonrelativistic thermal electrons, and spatial diffusion. In analysing the diffusion in frequency we find that for nonrelativistic velocities only zeroth order harmonic scattering contributes. The contribution from first harmonic scattering is of order $\sim \max(v_\perp^2/c^2,\, v_\parallel^2/c^2)$ smaller, and this allows one to expand the scattering probability in powers of $\beta_\parallel$. It is found in the limit of stationary electrons that the scattering cross section for o–o scattering is anisotropic, with $w_{\sigma\sigma'} \propto \sin^2\theta\,\sin^2\theta'$, while z–z scattering is isotropic. Also o–o scattering occurs on a timescale $\sim 2(\Omega_{\mathrm{e}}/\omega)^2$ faster than that of z–z, z–o or o–z scattering. This means only a frequency diffusion equation for the o-mode is obtained, which turns out to be effectively the Kompaneets equation with cross section smaller than the classical Thomson cross section, i.e. $\sigma = 2\sigma_{\mathrm{T}}/15$. This in effect means a photon undergoes a smaller number of scatterings before escaping in a magnetised plasma than in the corresponding unmagnetised case.

In the presence of spatial gradients in the plasma both the o-mode and z-mode diffuse in space. Diffusion of the o-mode photons is anisotropic with $D_\perp = \frac{1}{2}D_z$, while diffusion of the z-mode photons is nearly isotropic. Our diffusion equations also include the contribution from mode conversion $\mathrm{z} \to \mathrm{o}$ and $\mathrm{o} \to \mathrm{z}$ as well as a term (cf. equation 94), allowing for the free escape of photons. One limitation is that we need to restrict photons to angles $\theta > (\omega/\Omega_{\mathrm{e}})^{\frac{1}{2}}$; photons with $\theta < (\omega/\Omega_{\mathrm{e}})^{\frac{1}{2}}$ are nearly circularly polarised. The mean free path for the z-mode photons is much greater than that for o-mode photons, which implies that the spatial transfer is dominated by z-mode photons.

In summary, in a strongly magnetised plasma, photons below the cyclotron frequency are affected by Compton scattering in three important ways. First, o-mode photons are scattered relatively efficiently, and their spectrum evolves in the same manner as in the unmagnetised case, i.e. as described by the Kompaneets equation with a reduced cross section. Second, less frequent scattering events can convert o-mode photons into z-mode photons and vice versa, and locally a secondary distribution of z-mode photons results from these less frequent scattering events. Third, because the z-mode photons are relatively weakly scattered, they can propagate relatively large distances between each scattering event. This allows transfer of radiant energy, and is the dominant transfer mechanism. Thus the spectrum of o-mode photons in one localised region is coupled to the spectrum of o-mode photons in another localised region by o–z scatterings plus spatial diffusion of the z-mode photons and then z–o scatterings.

We have also presented an extension of the Kompaneets equation in an unmagnetised plasma to include both anisotropies (again due to spatial gradients in the plasma) and bulk motions of the plasma which could be non-uniform, i.e. corresponding to converging fluid flow. Our result reproduces that by Blandford and Payne (1981) who employed a covariant, radiative transfer equation in their derivation.

As yet we have not used our result in any astrophysical applications; however, we were motivated by problems associated with the formation of spectra in the accretion columns of X-ray pulsars, where magnetic fields $\gtrsim 10^8$ T (i.e. $\gtrsim 10^{12}$ G) are encountered. This is also true of γ-ray burst sources and probably X-ray bursters. Even magnetic white dwarfs in cataclysmic variables have fields up to $\sim 10^4$–10^5 T. These should be the principal astrophysical applications, but any situation under which the condition $\omega_p \ll \omega < \Omega_e$ holds would be available to our results.

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Appendix 1: First Quantum Correction

Here we derive the first quantum correction appearing in the scattering probability (4) and indicate how it may be transferred from the delta function to the integrand of (2) and (3). The delta function appearing in (4) ensures conservation of energy in the scattering event. The purely classical energy conservation law for scattering of photons by spiralling electrons is

$$\omega-\omega'-s\Omega-(k_\parallel-k'_\parallel)v_\parallel=0;\qquad s=0,\pm1,\pm2,\dots,\tag{A1}$$

where $v_\parallel$ is the component of the electron velocity along the magnetic field and $k_\parallel$ is the component of the photon wavevector along the magnetic field. Condition (A1) is called the Doppler or gyro-resonance condition. To obtain the first quantum correction to (A1) we proceed as follows. An electron in a magnetic field has energy eigenvalues $\epsilon(p_\parallel, n)$ given by solving the Dirac equation in the relativistic quantum case: the eigenvalues are given by

$$\epsilon(p_\parallel, n)=(m_{\rm e}^2c^4+2ne\hbar B+p_\parallel^2c^2)^{\frac{1}{2}},\tag{A2}$$

where $p_\parallel = mv_\parallel$ and n is an integer quantum number labelling the Landau level of the electron. [We ignore the spin of the electron in (A2) since we ultimately take the classical limit.] Suppose an electron $(\epsilon, p_\parallel, n)$ is scattered by a photon $(\omega', \boldsymbol{k}')$ and emits a photon $(\omega, \boldsymbol{k})$ so that its final state is $(\epsilon', p'_\parallel, n')$. The final state must satisfy

$$\epsilon'(p'_\parallel, n')=(m_{\rm e}^2c^4+2n'e\hbar B+p_\parallel'^2c^2)^{\frac{1}{2}},\tag{A3}$$

with conservation of energy and momentum requiring that

$$\epsilon'=\epsilon+\hbar\omega'-\hbar\omega,\qquad p'_\parallel=p_\parallel-\hbar(k_\parallel-k'_\parallel),\qquad n'=n-s\geqslant 0.\tag{A4}$$

Only parallel momentum is conserved, and n' can only differ from n by an integer. For $|\hbar(k_\parallel-k'_\parallel)|\ll p_\parallel$ and $|s|\ll n$, then (A4) in (A3) along with (A2) leads to, after expanding in powers of s/n and $\hbar(k_\parallel-k'_\parallel)/p_\parallel$,

$$\epsilon'=(1-\hat{D}_s+\tfrac{1}{2}\hat{D}_s^2+\dots)\epsilon,\tag{A5}$$

with

$$\hat{D}_s = s\frac{\partial}{\partial n} + \hbar(k_\parallel - k'_\parallel)\frac{\partial}{\partial p_\parallel}\,. \tag{A6}$$

In the classical limit we have $\hbar \to 0$, $n \to \infty$ such that $\hbar n \to p_\perp^2/2eB$. Hence, (A6) becomes

$$D_s = \hbar\left(\frac{s\Omega}{v_\parallel}\frac{\partial}{\partial p_\perp} + (k_\parallel - k'_\parallel)\frac{\partial}{\partial p}\right). \tag{A7}$$

To lowest order in $\hbar$ equation (A5) implies the Doppler condition (A1) while the second derivative in (A5) leads to the first quantum correction

$$\frac{\hbar}{2\epsilon}\{(\omega-\omega')^2 - (k_\parallel - k'_\parallel)^2 c^2\}\,, \tag{A8}$$

which describes the effect of recoil of the electron due to emission of the photon.

Next we show that the first quantum correction leads to an extra term $\frac{1}{2}\{N(k')+N(k)\}D_s f(p)$ in the kinetic equations (2) and (3). For simplicity we derive the result for the unmagnetised case, but the derivation is the same in the magnetised limit. Let us write

$$w_{\sigma\sigma'}(p, k, k') = \tilde{w}_{\sigma\sigma'}(p, k, k')\delta(\epsilon - \epsilon' - \hbar\omega + \hbar\omega')\,, \tag{A9}$$

and consider

$$\begin{aligned}
&\int \mathrm{d}^3p\ \tilde{w}_{\sigma\sigma'}(p, k, k')\delta(\epsilon-\epsilon'-\hbar\omega+\hbar\omega')[f(p)\{1+N_\sigma(k)\}N_{\sigma'}(k') \\
&\qquad -f\{p-\hbar(k-k')\}N_\sigma(k)\{1+N_{\sigma'}(k')\}] \\
&= \int \mathrm{d}^3p\ \tilde{w}_{\sigma\sigma'}(p, k, k')\,\delta(\hbar\omega'-\hbar\omega+\hat{D}^2\epsilon) \\
&\qquad \times[f(p)\{1+N_\sigma(k)\}N_{\sigma'}(k') - (1-\hat{D})f(p)\,N_\sigma(k)\{1+N_{\sigma'}(k')\}]\,,
\end{aligned} \tag{A10}$$

where $\hat{D} = \hbar(k-k')\,.\,\partial/\partial p$ and in expanding $f(p+\hbar k'-\hbar k)$ we have only gone to first order. Next we make use of the identity

$$\begin{aligned}
\int \mathrm{d}x\ \delta(x-x_0-\Delta)\,F(x) &= \int \mathrm{d}x\ \delta(x-x_0)\,F(x+\Delta) \\
&= \int \mathrm{d}x\ \delta(x-x_0)\left(1+\Delta\frac{\partial}{\partial x}\right)F(x)
\end{aligned}$$

and, hence, in effect we have

$$\delta(x-x_0-\Delta) \to \delta(x-x_0)\left(1+\Delta\frac{\partial}{\partial x}\right). \tag{A11}$$

So, using this result in (A10) we have

$$\int \mathrm{d}^3p\ \tilde{w}_{\sigma\sigma'}(p,k,k')\,\delta(\hbar\omega'-\hbar\omega+\hat{D}\epsilon-\hat{D}\Delta\epsilon)[f(p)\{1+N_\sigma(k)\}N_{\sigma'}(k')$$
$$-(1-\hat{D})f(p)\,N_\sigma(k)\{1+N_{\sigma'}(k')\}]$$
$$=\int \mathrm{d}^3p\ w^0_{\sigma\sigma'}(p,k,k')(1+\tfrac{1}{2}\hat{D})[f(p)\{1+N_\sigma(k)\}N_{\sigma'}(k')$$
$$-(1-\hat{D})f(p)\,N_\sigma(k)\{1+N_{\sigma'}(k')\}]$$
$$=\int \mathrm{d}^3p\ w^0_{\sigma\sigma'}(p,k,k')[\{1+N_\sigma(k)\}N_{\sigma'}(k')(1+\tfrac{1}{2}\hat{D})f(p)$$
$$-(1+\tfrac{1}{2}\hat{D})f(p)\,N_\sigma(k)\{1+N_{\sigma'}(k')\}+\hat{D}f(p)\,N_\sigma(k)\{1+N_{\sigma'}(k')\}] \qquad \text{(A12)}$$
$$=\int \mathrm{d}^3p\ w^0_{\sigma\sigma'}(p,k,k')[\{N_{\sigma'}(k')-N_\sigma(k)\}f(p)+N_{\sigma'}(k')\,N_\sigma(k)\hat{D}f(p)$$
$$+\tfrac{1}{2}\{N_{\sigma'}(k')+N_\sigma(k)\}\,\hat{D}f(p)], \qquad \text{(A13)}$$

where

$$w^0_{\sigma\sigma'}(p,k,k') = \tilde{w}_{\sigma\sigma'}(p,k,k')\,\delta(\hbar\omega'-\hbar\omega+\hat{D}\epsilon) \qquad \text{(A14)}$$

is the scattering probability with the classical delta function and $\Delta\epsilon = \frac{1}{2}\hat{D}\epsilon$. So we can see from (A13) that the effect of the first quantum correction is to introduce the terms $\frac{1}{2}\{N_{\sigma'}(k')+N_\sigma(k)\}\,\hat{D}f(p)$ into the kinetic equation.

Appendix 2: Generalisation to Weakly Anisotropic Photon Distribution

We derive the kinetic equation for the case of a weakly anisotropic distribution of photons (though still possessing azimuthal symmetry) such as could arise if the scattering plasma undergoes bulk motions. Suppose that the photon distribution for o-mode photons is weakly dependent on angle, and let us expand in Legendre polynomials:

$$N^{\mathrm{o}}(k,\theta) = \sum_{l=0}^{\infty} N^{\mathrm{o}}_l(k)\,P_l(\cos\theta)\,. \qquad \text{(A15)}$$

This assumes a gradient in the z-direction. As far as the kinetic equation (38) is concerned we consider only the leading terms, which by virtue of (47) correspond to $N(k,\theta')-N(k,\theta)$, and ignore corrections of order $\beta_\parallel$. The generalisation to include all terms is straightforward but tedious, and has been performed in the unmagnetised case, the results of which are presented in Appendix 3.

Now using (60a) we may write

$$\frac{\mathrm{d}N^{\mathrm{o}}(k,\theta)}{\mathrm{d}t} \approx 2\pi r_0^2\,n_{\mathrm{e}}\,c\int_{-1}^{1}\mathrm{d}(\cos\theta')\,\sin^2\theta\,\sin^2\theta'\{N^{\mathrm{o}}(k,\theta')-N^{\mathrm{o}}(k,\theta)\}\,, \qquad \text{(A16)}$$

and using the results

$$\sin^2\theta = \tfrac{2}{3}\{1-P_2(\cos\theta)\}\,, \tag{A17a}$$

$$\int_{-1}^{1} \mathrm{d}(\cos\theta)\; P_l(\cos\theta)\; P_{l'}(\cos\theta) = \frac{2\delta_{ll'}}{2l+1}\,, \tag{A17b}$$

we may express (A16) as

$$\begin{aligned}\frac{\mathrm{d}N^{\mathrm{o}}(k,\theta)}{\mathrm{d}t} &\approx 2\pi r_0^2\, n_{\mathrm{e}}\, c \int_{-1}^{1} \mathrm{d}(\cos\theta')\; \tfrac{4}{9}\{1-P_2(\cos\theta)\}\{1-P_2(\cos\theta')\} \\ &\qquad\qquad \times \{N^{\mathrm{o}}(k,\theta')-N^{\mathrm{o}}(k,\theta)\} \\ &= 2\pi r_0^2\, n_{\mathrm{e}}\, c\tfrac{8}{9}\{1-P_2(\cos\theta)\}\{N_0^{\mathrm{o}}(k)-\tfrac{1}{5}N_2^{\mathrm{o}}(k)-N^{\mathrm{o}}(k,\theta)\}\,. \end{aligned} \tag{A18}$$

Next we make use of the recursion relation

$$\begin{aligned}\tfrac{2}{3}\{1-P_2(\cos\theta)\}\,P_l(\cos\theta) &= -(l-1)\,l\,P_{l-2}(\cos\theta) + \frac{2l^2+2l-1}{(2l-1)(2l+3)}\,P_l(\cos\theta) \\ &\qquad - \frac{(l+1)(l+2)}{(2l+1)(2l+3)}\,P_{l+2}(\cos\theta)\end{aligned} \tag{A19}$$

in the following equation:

$$\begin{aligned}\frac{\mathrm{d}N_l^{\mathrm{o}}(k)}{\mathrm{d}t} &= \frac{2l+1}{2}\int_{-1}^{1} \mathrm{d}(\cos\theta)\; P_l(\cos\theta)\,\frac{\mathrm{d}N^{\mathrm{o}}(k,\theta)}{\mathrm{d}t} \\ &= \frac{8\pi}{3}\, r_0^2\, n_{\mathrm{e}}\, c\,\frac{2l+1}{2}\int_{-1}^{1} \mathrm{d}(\cos\theta)\; P_l(\cos\theta)\tfrac{2}{3}\{1-P_2(\cos\theta)\} \\ &\qquad \times \left(N_0^{\mathrm{o}}(k) - \tfrac{1}{5}N_2^{\mathrm{o}}(k) - \sum_{l'=0}^{\infty} N_{l'}^{\mathrm{o}}(k)\, P_{l'}(\cos\theta)\right). \end{aligned} \tag{A20}$$

Hence, we get

$$\begin{aligned}\frac{\mathrm{d}N_0^{\mathrm{o}}(k)}{\mathrm{d}t} &= \sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c\tfrac{1}{2}\int_{-1}^{1} \mathrm{d}(\cos\theta)\; \tfrac{2}{3}\{1-P_2(\cos\theta)\} \\ &\qquad \times \left(N_0^{\mathrm{o}}(k) - \tfrac{1}{5}N_2^{\mathrm{o}}(k) - \sum_{l'=0}^{\infty} N_{l'}^{\mathrm{o}}(k)\, P_{l'}(\cos\theta)\right) = 0\,, \end{aligned} \tag{A21}$$

$$\begin{aligned}\frac{\mathrm{d}N_2^{\mathrm{o}}(k)}{\mathrm{d}t} &= \sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c\tfrac{5}{2}\int_{-1}^{1} \mathrm{d}(\cos\theta)\left(-\frac{2}{3\times 5} + \frac{11}{3\times 7}\,P_2(\cos\theta) - \frac{3\times 4}{5\times 7}\,P_4(\cos\theta)\right) \\ &\qquad \times \left(N_0^{\mathrm{o}}(k) - \tfrac{1}{5}N_2^{\mathrm{o}}(k) - \sum_{l'=0}^{\infty} N_{l'}^{\mathrm{o}}(k)\, P_{l'}(\cos\theta)\right) \\ &= \frac{\sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c}{105}\{41N_2^{\mathrm{o}}(k)-20N_4^{\mathrm{o}}(k)\}\,. \end{aligned} \tag{A22}$$

More generally we have

$$\frac{\mathrm{d}N_l^{\mathrm{o}}(k)}{\mathrm{d}t} = \frac{\sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c}{2}(2l+1)\int_{-1}^{1}\mathrm{d}(\cos\theta)\Bigg(-\frac{(l-1)l}{(2l+1)(2l-1)}\,P_{l-2}(\cos\theta)$$

$$+\frac{2l^2+2l-1}{(2l-1)(2l+3)}\,P_l(\cos\theta)-\frac{(l+1)(l+2)}{(2l+1)(2l+3)}\,P_{l+2}(\cos\theta)\Bigg)$$

$$\times\left(N_0^{\mathrm{o}}(k)-\tfrac{1}{5}N_2^{\mathrm{o}}(k)-\sum_{l'=0}^{\infty}N_{l'}^{\mathrm{o}}(k)\,P_{l'}(\cos\theta)\right)$$

$$=\frac{\sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c}{2}(2l+1)\Bigg(\frac{(l-1)l}{(2l-1)(2l+1)}\,\frac{2N_{l-2}^{\mathrm{o}}(k)}{2(l-2)+1}-\frac{2l^2+2l-1}{(2l-1)(2l+3)}\,\frac{2N_l^{\mathrm{o}}(k)}{2l+1}$$

$$+\frac{(l+1)(l+2)}{(2l+1)(2l+3)}\,\frac{2N_{l+2}^{\mathrm{o}}(k)}{2(l+2)+1}\Bigg),$$

or finally

$$\frac{\mathrm{d}N_l^{\mathrm{o}}(k)}{\mathrm{d}t} = (2l+1)\sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c\Bigg(\frac{(l-1)l\,N_{l-2}^{\mathrm{o}}(k)}{(2l-3)(2l-1)(2l+1)}-\frac{(2l^2+2l-1)N_l^{\mathrm{o}}(k)}{(2l-1)(2l+1)(2l+3)}$$

$$+\frac{(l+1)(l+2)N_{l+2}^{\mathrm{o}}(k)}{(2l+1)(2l+3)(2l+5)}\Bigg), \tag{A23}$$

for l odd or $l \geqslant 4$ and even.

The general result shows that odd and even l are uncoupled; for example, (A23) implies that for the two lowest order odd terms

$$\frac{\mathrm{d}N_1^{\mathrm{o}}(k)}{\mathrm{d}t} = \sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c\{-\tfrac{3}{5}N_1^{\mathrm{o}}(k)+\tfrac{6}{35}N_3^{\mathrm{o}}(k)\}, \tag{A24}$$

$$\frac{\mathrm{d}N_3^{\mathrm{o}}(k)}{\mathrm{d}t} = \sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c\{\tfrac{2}{5}N_1^{\mathrm{o}}(k)-\tfrac{23}{45}N_3^{\mathrm{o}}(k)+\tfrac{20}{99}N_5^{\mathrm{o}}(k)\}, \tag{A25}$$

while for $l = 4$ we obtain

$$\frac{\mathrm{d}N_4^{\mathrm{o}}(k)}{\mathrm{d}t} = \sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c\{\tfrac{12}{35}N_2^{\mathrm{o}}(k)-\tfrac{39}{77}N_4^{\mathrm{o}}(k)+\tfrac{30}{143}N_6^{\mathrm{o}}(k)\}. \tag{A26}$$

We see that the scattered o-mode photons approach isotropy on a timescale of order $\approx 2/\sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c$. The scattering of z-mode photons is isotropic (60) and the distribution of z-mode photons becomes isotropic on a timescale of order Y^2 larger than do the o-mode photons. The o–z and z–o scatterings occur on the same timescale as that for z–z and hence complete isotropy is approached only on the longer timescale of $\approx 2Y^2/\sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c$.

Appendix 3: Extension of the Kompaneets Equation to Weak Anisotropies

Here we present the results of calculations to generalise the Kompaneets equation for unmagnetised scattering media to include a weak anisotropy due to spatial gradients. These gradients could, for example, arise from a bulk motion of the scattering media. Such results could be of relevance in problems of accretion onto black holes, where the magnetic fields are thought to be relatively weak, or to any problem involving a streaming motion of the scattering media. We also reproduce a result by Blandford and Payne (1981) who took a somewhat different approach.

The approach outlined here is similar to that in Appendix 2 with the difference that we now include all terms in the kinetic equation. The relevant kinetic equation with the effects of quantum recoil included is for unmagnetised media

$$\frac{\mathrm{d}N_\sigma(\boldsymbol{k})}{\mathrm{d}t} = \int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \int \mathrm{d}^3 p\; w_{\sigma\sigma'}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{k}')\Big(\{N_{\sigma'}(\boldsymbol{k}') - N_\sigma(\boldsymbol{k})\} f(\boldsymbol{p})$$

$$+ N_{\sigma'}(\boldsymbol{k}')\, N_\sigma(\boldsymbol{k})\, \hbar(\boldsymbol{k} - \boldsymbol{k}') \,.\, \frac{\partial f(\boldsymbol{p})}{\partial \boldsymbol{p}} + \tfrac{1}{2}\{N_{\sigma'}(\boldsymbol{k}') + N_\sigma(\boldsymbol{k})\}\hbar(\boldsymbol{k} - \boldsymbol{k}') \,.\, \frac{\partial f(\boldsymbol{p})}{\partial \boldsymbol{p}}\Big), \qquad \text{(A27)}$$

where $w_{\sigma\sigma'}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{k}')$ is the scattering probability and for the case of Thomson scattering of transverse waves may be expressed as

$$w_{\sigma\sigma'}(\boldsymbol{p}, \boldsymbol{k}, \boldsymbol{k}') = \frac{(2\pi)^3}{m_\mathrm{e}^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{|a_{\sigma\sigma'}(\boldsymbol{k}, \boldsymbol{k}'; \boldsymbol{v})|^2}{\omega\omega'} \delta\{\omega(1 - \boldsymbol{\kappa} \,.\, \boldsymbol{\beta}) - \omega'(1 - \boldsymbol{\kappa}' \,.\, \boldsymbol{\beta})\}, \qquad \text{(A28)}$$

with

$$|a_{\sigma\sigma'}(\boldsymbol{k}, \boldsymbol{k}'; \boldsymbol{v})|^2 = \tfrac{1}{2}(1 - \beta^2)\left\{1 + \left(1 - \frac{(1-\beta^2)(1 - \boldsymbol{\kappa} \,.\, \boldsymbol{\kappa}')}{(1 - \boldsymbol{\kappa} \,.\, \boldsymbol{\beta})(1 - \boldsymbol{\kappa}' \,.\, \boldsymbol{\beta})}\right)^2\right\}, \qquad \text{(A29)}$$

$$\boldsymbol{\kappa} = \boldsymbol{k}/|\boldsymbol{k}|, \qquad \boldsymbol{\kappa}' = \boldsymbol{k}'/|\boldsymbol{k}'|, \qquad \boldsymbol{\beta} = \boldsymbol{v}/c.$$

If we assume that $N(\boldsymbol{k})$ is only weakly dependent on angle θ we can write, as in Appendix 2,

$$N(k, \theta) = \sum_{l=0}^{\infty} N_l(k)\, P_l(\cos\theta). \qquad \text{(A30)}$$

The usual form of the Kompaneets equation (1) applies for $N_l(k) = 0$ $(l \geqslant 1)$, and we write down equations which include terms to $N_3(k)$. We should note that $N_1(k) \neq 0$ corresponds to a net streaming speed u of the photons with

$$u = c \int_{-1}^{1} \mathrm{d}(\cos\theta)\, \cos\theta\; N(k, \theta) \Big/ \int_{-1}^{1} \mathrm{d}(\cos\theta)\; N(k, \theta) = \frac{c}{3}\,\frac{N_1(k)}{N_0(k)}. \qquad \text{(A31)}$$

If we again assume a nonrelativistic electron distribution and expand $|a_{\sigma\sigma'}(\boldsymbol{k}, \boldsymbol{k}'; \boldsymbol{v})|^2$ in powers of β up to second order, we obtain after a simple but lengthy calculation the result

$$\frac{\mathrm{d}N(\omega,\theta)}{\mathrm{d}t} = \tfrac{3}{4}\sigma_{\mathrm{T}}\, n_{\mathrm{e}}\, c\Bigg[\tfrac{4}{3}\{N_0(\omega)-N(\omega,\theta)\}+\tfrac{2}{15}N_2(\omega)\,P_2(\cos\theta)$$

$$-\frac{T_{\mathrm{e}}}{m_{\mathrm{e}}\,c^2}[6\{N_0(\omega)-N(\omega,\theta)\}-\tfrac{6}{5}N_1(\omega)\,P_1(\cos\theta)-\tfrac{6}{5}N_2(\omega)\,P_2(\cos\theta)$$

$$+\tfrac{8}{35}N_3(\omega)\,P_3(\cos\theta)]+\frac{4T_{\mathrm{e}}}{m_{\mathrm{e}}\,c^2}\Bigg\{\omega\frac{\partial}{\partial\omega}+\tfrac{1}{4}\omega^2\frac{\partial^2}{\partial\omega^2}$$

$$+\frac{\hbar\omega}{T_{\mathrm{e}}}\{N(\omega,\theta)+\tfrac{1}{2}\}\left(1+\tfrac{1}{2}\omega\frac{\partial}{\partial\omega}\right)\Bigg\}\{\tfrac{4}{3}N_0(\omega)-\tfrac{8}{15}N_1(\omega)\,P_1(\cos\theta)$$

$$+\tfrac{2}{15}N_2(\omega)\,P_2(\cos\theta)-\tfrac{2}{35}N_3(\omega)\,P_3(\cos\theta)\}+\tfrac{8}{3}\frac{\hbar\omega}{m_{\mathrm{e}}\,c^2}N(\omega,\theta)\Bigg]. \tag{A32}$$

Next if we calculate the quantity

$$\frac{\mathrm{d}N_l(\omega)}{\mathrm{d}t}=\frac{2l+1}{2}\int_{-1}^{1}\mathrm{d}(\cos\theta)\,\frac{\mathrm{d}N(\omega,\theta)}{\mathrm{d}t}\,P_l(\cos\theta),$$

we obtain

$$\frac{\mathrm{d}N_0(\omega)}{\mathrm{d}t}=\frac{\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,c\,T_{\mathrm{e}}}{m_{\mathrm{e}}\,c^2}\omega\left(4+\omega\frac{\partial}{\partial\omega}\right)\Bigg(\frac{\partial N_0(\omega)}{\partial\omega}+\frac{\hbar}{T_{\mathrm{e}}}[N_0(\omega)\{1+N_0(\omega)\}$$

$$-\tfrac{2}{15}N_1^2(\omega)+\tfrac{1}{50}N_2^2(\omega)-\tfrac{3}{490}N_3^2(\omega)]\Bigg), \tag{A33}$$

$$\frac{\mathrm{d}N_1(\omega)}{\mathrm{d}t}=\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,c\,N_1(\omega)\left(-1+\frac{4T_{\mathrm{e}}}{m_{\mathrm{e}}\,c^2}+\frac{2\hbar\omega}{m_{\mathrm{e}}\,c^2}\right)+\frac{\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,c\,T_{\mathrm{e}}}{m_{\mathrm{e}}\,c^2}$$

$$\times\Bigg\{\left(\omega\frac{\partial}{\partial\omega}+\tfrac{1}{4}\omega^2\frac{\partial^2}{\partial\omega^2}+\frac{\hbar}{T_{\mathrm{e}}}\{N_0(\omega)+\tfrac{1}{2}\}\hat{D}\right)\{-\tfrac{8}{5}N_1(\omega)\}$$

$$+\frac{\hbar\omega}{T_{\mathrm{e}}}\{N_1(\omega)\,\hat{D}\,4N_0(\omega)-N_1(\omega)\hat{D}\,\tfrac{2}{5}\alpha_{112}\,N_2(\omega)-N_2(\omega)\hat{D}\tfrac{6}{35}\alpha_{123}\,N_3(\omega)$$

$$-N_2(\omega)\hat{D}\,\tfrac{8}{5}\alpha_{112}\,N_1(\omega)+N_3(\omega)\hat{D}\,\tfrac{2}{5}\alpha_{123}\,N_2(\omega)\}\Bigg\}, \tag{A34}$$

where

$$\hat{D}=1+\tfrac{1}{2}\omega\,\partial/\partial\omega,$$

$$\alpha_{1lm}=\tfrac{3}{2}\int_{-1}^{1}\mathrm{d}(\cos\theta)\,P_1(\cos\theta)\,P_l(\cos\theta)\,P_m(\cos\theta). \tag{A35}$$

We can see for the case of very weak anisotropies $N_l \ll N_0$ ($l>1$) that (A33) reduces to the standard Kompaneets equation, while (A34) implies that any streaming motion dies away on a timescale of order $\sim\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,c$.

Next we derive an equation describing how the isotropic part of the photon distribution $N_0(\omega)$ changes when we include the effects of non-uniform fluid flow of characteristic speed u. Since we are only considering weak anisotropies, then basically this is a problem of geometric optics in which we are considering the propagation of radiation through a slowly varying medium. The problem may be formulated by applying mechanical concepts to a system of photons. If we regard the dispersion relation $\omega = \omega_\sigma$ as a function of position and time so that $\omega = \omega_\sigma(\boldsymbol{k}; \boldsymbol{x}, t)$, we can take $\omega_\sigma(\boldsymbol{k}; \boldsymbol{x}, t)$ as the Hamiltonian of the system of photons. Also, regarding the photon distribution $N_\sigma(\boldsymbol{k})$ as a function of position and time and making the identification

$$\frac{\mathrm{d}}{\mathrm{d}t} \to \frac{\partial}{\partial t} + \dot{\boldsymbol{x}}.\frac{\partial}{\partial \boldsymbol{x}} + \dot{\boldsymbol{k}}.\frac{\partial}{\partial \boldsymbol{k}}, \tag{A36}$$

we obtain as our equation for the photon distribution

$$\frac{\partial N(\omega,\theta)}{\partial t} + \dot{\boldsymbol{x}}.\frac{\partial N(\omega,\theta)}{\partial \boldsymbol{x}} + \dot{\boldsymbol{k}}.\frac{\partial N(\omega,\theta)}{\partial \boldsymbol{k}} = \left[\frac{\mathrm{d}N(\omega,\theta)}{\mathrm{d}t}\right]_{\mathrm{R}}, \tag{A37}$$

where

$$\dot{\boldsymbol{x}} = \partial\omega(\boldsymbol{k}; \boldsymbol{x}, t)/\partial \boldsymbol{k}, \qquad \dot{\boldsymbol{k}} = \partial\omega(\boldsymbol{k}; \boldsymbol{x}, t)/\partial \boldsymbol{x}. \tag{A38}$$

and $[\mathrm{d}N/\mathrm{d}t]_{\mathrm{R}}$ indicates the right-hand side of (A32).

Let us suppose the plasma has velocity $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ and that the photons have a frequency ω in the frame moving with the plasma and that they have wavevector $\boldsymbol{k}$. Since we are interested in the spectrum of radiation emerging from the plasma, the frequency of radiation as seen in a frame which is at rest with respect to the plasma flow is simply given by the Doppler formula

$$\omega' = \gamma(\omega + \boldsymbol{k}.\boldsymbol{u}), \tag{A39}$$

where $\gamma = (1 - u^2/c^2)^{-\frac{1}{2}}$.

For simplicity we assume that the plasma is moving at nonrelativistic speeds and take $\gamma \approx 1$, so that $\omega' \approx \omega + \boldsymbol{k}.\boldsymbol{u}$. Using this result in (A38) we find that

$$\dot{\boldsymbol{x}} = \boldsymbol{v}_{\mathrm{g}} + \boldsymbol{u}, \qquad \dot{\boldsymbol{k}} = -\boldsymbol{k}.\nabla \boldsymbol{u} - \boldsymbol{k} \times (\nabla \times \boldsymbol{u}), \tag{A40}$$

where $\boldsymbol{v}_{\mathrm{g}}$ is the group velocity. Next, assuming that the photon distribution is axisymmetric and weakly anisotropic, so that

$$N(\omega,\theta) \approx N_0(\omega) + N_1(\omega)\cos\theta, \tag{A41}$$

then (A40) and (A41) in (A37) gives

$$\frac{\partial N(\omega,\theta)}{\partial t} + (\boldsymbol{v}_{\mathrm{g}} + \boldsymbol{u}).\frac{\partial N(\omega,\theta)}{\partial \boldsymbol{x}} - (\boldsymbol{k}.\nabla \boldsymbol{u} + \boldsymbol{k} \times \nabla \times \boldsymbol{u}).\frac{\partial N(\omega,\theta)}{\partial \boldsymbol{k}} = \left[\frac{\mathrm{d}N}{\mathrm{d}t}\right]_{\mathrm{R}}. \tag{A42}$$

Now, if we average (A42) over all angles the right-hand side simply becomes the right-hand side of the Kompaneets equation. In addition if we note that $N_0(\omega)$ is isotropic and assuming that ω is only a function of $|\boldsymbol{k}|$, we find on performing the averages over angles that (A42) yields

$$\frac{\partial N_0(\omega)}{\partial t}+\boldsymbol{u}.\nabla N_0(\omega)+\tfrac{1}{3}\frac{\partial\omega}{\partial k}\frac{\partial N_1(\omega)}{\partial z}=\tfrac{1}{3}(\nabla.\boldsymbol{u})k\frac{\partial N_0(\omega)}{\partial k}=\left[\frac{\mathrm{d}N_0}{\mathrm{d}t}\right]_{\mathrm{KOM}}. \tag{A43}$$

Next multiplying (A42) by $\cos\theta$ and averaging over all directions of $\boldsymbol{k}$, we obtain

$$\frac{\partial N_1(\omega)}{\partial t}+\boldsymbol{u}.\nabla N_1(\omega)+\frac{\partial\omega}{\partial k}\frac{\partial N_0(\omega)}{\partial z}=-\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,cN_1(\omega). \tag{A44}$$

We neglect the convective derivative of $N_1(\omega)$ and substitute (A44) in (A43). Also our analysis has been for an axisymmetric distribution of photons; for a general distribution we make the replacement $\partial^2/\partial z^2\rightarrow\nabla^2$. Hence our final result for the scattering of the isotropic component of the photon distribution in a non-uniform plasma flow is

$$\frac{\mathrm{D}N_0(\omega)}{\mathrm{D}t}=\frac{1}{3\sigma_{\mathrm{T}}\,n_{\mathrm{e}}\,c}\frac{\partial\omega}{\partial k}\nabla^2 N_0(\omega)-\tfrac{1}{3}(\nabla.\boldsymbol{u})\omega\frac{\partial\omega}{\partial k}\frac{\partial N_0(\omega)}{\partial\omega}+\left[\frac{\mathrm{d}N_0}{\mathrm{d}t}\right]_{\mathrm{KOM}}, \tag{A45}$$

with $\mathrm{D}/\mathrm{D}t$ the convective derivative. Equation (A45) is basically the same as that derived by Blandford and Payne (1981) in their equation (18), except that we have allowed for the possibility that the refractive index of the plasma may not be unity, whereas they have a frequency-dependent cross section $\sigma(\omega)$ instead of the classical Thomson cross section σ_{T}.

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