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FLRW Cosmological Models in Lyra's Manifold with Time Dependent Displacement Field

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Abstract

Friedmann–Lemaitre–Robertson–Walker (FLRW) cosmological models are derived in Lyra's manifold assuming a time dependent displacement field. The models obtained solve the singularity, entropy and horizon problems which exist in the standard models of cosmology based on Riemannian geometry. The asymptotic behaviour of the models is also examined.

1. Introduction

In 1918, shortly after Einstein's theory of general relativity, Weyl (1918) introduced a generalisation of Riemannian geometry in an attempt to unify gravitation and electromagnetism. Weyl's theory was not taken seriously because the electromagnetic field implies a nonintegrable length. This in turn means that different atoms should not be emitting radiation at the same frequency since they have different histories. This is in contradiction with the well-known observational result that sharp spectral lines are observed even in the presence of an electromagnetic field.

Lyra (1951) (see also Scheibe 1952) proposed a modification of Riemannian geometry which bears a close resemblance to Weyl's geometry. However, in contrast to Weyl's geometry, in Lyra's geometry the connection is metric preserving as in Riemannian geometry, and length transfers are integrable. Lyra introduced a gauge function into the structureless manifold, as a result of which a displacement field arises naturally.

Historically, Einstein introduced the cosmological constant in an *ad hoc* fashion into his field equations in order to obtain a static model of the universe since, as is well known, his equations without the cosmological term admit only nonstatic cosmological models for a nonzero energy density. After the discovery of the redshift of the galaxies and its interpretation as being due to the expansion of the universe, Einstein regretted the introduction of the constant. In Lyra's geometry the displacement field arises naturally and several authors (Sen 1957; Bhamra 1974; Kalyanshetti and Waghmode 1982; Reddy and Innaiah 1985; Beesham 1986*a*, 1986*b*, 1986*c*; Reddy and Venkateswarlu 1987) have studied cosmology in Lyra's geometry with a constant displacement field which plays the same role as the cosmological constant in the normal treatment. The fact that the field arises naturally from the geometry was regarded as an advantage (Sen 1957; Halford 1970). Sen (1957) constructed a static model similar to the static Einstein universe, but a significant difference was that the model exhibited redshift and had finite density.

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In general relativity Einstein succeeded in geometrising gravitation by identifying the metric tensor with the gravitational potentials. In the scalar tensor theory of Brans–Dicke on the other hand, the scalar field remains alien to the geometry. Lyra's geometry is more in keeping with the spirit of Einstein's principle of geometrisation since both the scalar and tensor fields have more or less intrinsic geometrical significance. Furthermore, the present theory predicts the same effects, within observational limits, as far as the classical Solar System tests are concerned, as well as tests based on the linearised form of the field equations (Halford 1972). For further physical motivation, we refer to some excellent references (Sen 1960; Sen and Dunn 1971; Manoukian 1972; Sen and Vanstone 1972; Hudgin 1973).

As we have already mentioned, several authors have studied cosmology in Lyra's geometry with a constant displacement field. However, this restriction of the displacement field to be constant is merely one of convenience and there is no *a priori* reason for it. Halford (1970) suggested that this assumption could be relaxed but the suggestion does not seem to have been followed up. As a first step in this direction, we allow the displacement field to be time dependent and derive the Friedmann–Lemaitre–Robertson–Walker (FLRW) models. The models have the $k = -1$ geometry, are free of the big-bang singularity and solve the entropy and horizon problems which beset the standard models based on Riemannian geometry.

In Section 2 we outline the main features of Lyra's geometry; Section 3 deals with the vacuum FLRW models. The nonempty FLRW models are derived and considered in Section 4, while Section 5 concerns the asymptotic behaviour. Our conclusions are discussed in Section 6.

2. Lyra's Geometry

In this section we give a very brief outline of Lyra's geometry, which is a modification of Riemannian geometry, and which bears a close similarity to Weyl's geometry. For further details we refer to the literature (Lyra 1951; Scheibe 1952; Sen 1957, 1960; Halford 1970; Sen and Dunn 1971; Manoukian 1972; Sen and Vanstone 1972; Hudgin 1973).

(1) The displacement vector between two neighbouring points x^a and $x^a + dx^a$ is defined by its components $\psi\, dx^a$, where $\psi = \psi(x^a)$ is a gauge function. The coordinate system x^a and the gauge function ψ together form a reference system (ψ, x^a). Transformation to a new reference system $(\psi', x^{a'})$ is given by

$$\psi' = \psi'(\psi, x^a), \qquad x^{a'} = x^{a'}(x^a), \tag{1}$$

where

$$\partial\psi'/\partial\psi \neq 0, \qquad \det|\partial x^{a'}/\partial x^a| \neq 0 .$$

(2) The connections ${}^*\Gamma^a_{bc}$ are given by

$${}^*\Gamma^a_{bc} = \psi^{-1}\Gamma^a_{bc} - \tfrac{1}{2}(\delta^a_b\phi_c + \delta^a_c\phi_b - g_{cb}\phi^a), \tag{2}$$

where the Γ^a_{bc} are defined in terms of the metric tensor g_{ab} as in Riemannian geometry and ϕ_a is a displacement vector field. Lyra (1951) and Sen (1957) have shown that in any general reference system the vector field quantities ϕ_a arise as a natural consequence of the introduction of the gauge function ψ into the structureless manifold. The $^*\Gamma^a_{bc}$ are symmetric in their lower two indices.

(3) The metric is given by

$$\mathrm{d}s^2 = \psi^2 g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b, \tag{3}$$

and is invariant under both coordinate and gauge transformations.

(4) Parallel transport of a vector ξ^a is given by

$$\mathrm{d}\xi^a = -\tilde{\Gamma}^a_{bc}\xi^b\psi\,\mathrm{d}x^c, \tag{4}$$

where

$$\tilde{\Gamma}^a_{bc} = {}^*\Gamma^a_{bc} - \tfrac{1}{2}\delta^a_b\phi_c. \tag{5}$$

The $\tilde{\Gamma}^a_{bc}$ are not symmetric in b and c. The length of a vector does not change under parallel transport unlike in Weyl's geometry.

A curvature tensor is defined by means of parallel transport of a vector along a closed curve:

$${}^*R^a_{bcd} = \psi^{-2}\{-(\psi\tilde{\Gamma}^a_{bc})_{,d} + (\psi\tilde{\Gamma}^a_{bd})_{,c} - \psi^2(\tilde{\Gamma}^e_{bc}\tilde{\Gamma}^a_{ed} - \tilde{\Gamma}^e_{bd}\tilde{\Gamma}^a_{ce})\}, \tag{6}$$

where the $\tilde{\Gamma}^a_{bc}$ are as given in equation (5). Contraction of (6) leads to the curvature scalar

$${}^*R = \psi^{-2}R + 3\psi^{-1}\phi^a_{;a} + \tfrac{3}{2}\phi^a\phi_a + 2\psi^{-1}(\log\psi^2)_{,a}\psi^a, \tag{7}$$

where R is the Riemann curvature scalar and the semicolon denotes covariant differentiation with respect to the Christoffel symbols of the second kind in the Riemannian sense.

(5) The invariant volume integral is given by

$$I = \int L(-g)^{\frac{1}{2}}\psi^4\,\mathrm{d}^4x, \tag{8}$$

where d^4x is the volume element and L is a scalar.

(6) The normal gauge used is (Sen 1957) $\psi = 1$, and we put (Halford 1970) $L = {}^*R$ in equation (8). Putting $\psi = 1$ in (7) leads to

$${}^*R = R + 3\phi^a_{;a} + \tfrac{3}{2}\phi^a\phi_a. \tag{9}$$

The field equations are obtained from the variational principle

$$\delta(I+J) = 0, \tag{10}$$

where I is given by (8) and J is related to the Lagrangian density $\mathscr{L}$ of matter by the usual equation

$$J = \int \mathscr{L}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x. \tag{11}$$

The field equations are thus

$$R_{ab}-\tfrac{1}{2}Rg_{ab}+\tfrac{3}{2}\phi_a\phi_b-\tfrac{3}{4}g_{ab}\phi^c\phi_c = T_{ab}, \tag{12}$$

where we are using units in which $c = 8\pi G = 1$.

3. Vacuum FLRW Models

We let ϕ_a be the time-like vector

$$\phi_a = (\beta, 0, 0, 0), \tag{13}$$

where β is a function of time alone. For the Robertson–Walker metric,

$$ds^2 = -dt^2 + R^2(t)\{dr^2/(1-kr^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\}, \tag{14}$$

the field equations (12) become, in the vacuum case,

$$3(\dot{R}/R)^2 + 3k/R^2 - \tfrac{3}{4}\beta^2(t) = 0, \tag{15}$$

$$2\ddot{R}/R + (\dot{R}/R)^2 + k/R^2 + \tfrac{3}{4}\beta^2(t) = 0. \tag{16}$$

From these we derive an equation that is independent of β:

$$2\ddot{R}/R + 4\dot{R}^2/R^2 + 4k/R^2 = 0. \tag{17}$$

This equation may be written as

$$d\dot{R}^2/dR + 4\dot{R}^2/R = -4k/R, \tag{18}$$

which may readily be integrated to yield

$$\dot{R}^2 = (A - kR^4)/R^4, \tag{19}$$

where A is a constant of integration.

We now give the solutions to (19) for the various values of k. For $k = 0$, we obtain

$$R = A^{\frac{1}{6}}(3t)^{\frac{1}{3}}, \qquad \beta = 2/3t. \tag{20, 21}$$

For $k = \pm 1$, it is convenient to make the substitution

$$d\tau = R\,dt, \tag{22}$$

so that (19) becomes

$$d\tau = R^3(A - kR^4)^{-\frac{1}{2}}\,dR. \tag{23}$$

For $k = -1$, the solution is

$$R^4 = 4\tau^2 - A, \qquad \beta^2 = 4A/(4\tau^2 - A)^{\frac{3}{2}}, \tag{24, 25}$$

whereas for $k = +1$, we obtain

$$R^4 = A - 4\tau^2, \qquad \beta^2 = 4A/(A - 4\tau^2)^{\frac{3}{2}}. \tag{26, 27}$$

In each of the solutions (20), (24) and (26), we have chosen a constant of integration to be zero.

4. Nonempty FLRW Models

We assume a perfect fluid form for the energy–momentum tensor:

$$T_{ab} = (\mu + p) u_a u_b + p g_{ab}. \tag{28}$$

The field equations (12) become, for the Robertson–Walker metric (14), the perfect fluid form (28), and field (13),

$$3(\dot{R}/R)^2 + 3k/R^2 - \tfrac{3}{4}\beta^2(t) = \mu, \tag{29}$$

$$2\ddot{R}/R + (\dot{R}/R)^2 + k/R^2 + \tfrac{3}{4}\beta^2(t) = -p. \tag{30}$$

We first demonstrate how a big bang singularity may be averted. From (29) and (30), we derive the following Raychaudhuri-type equation:

$$3\ddot{R}/R = -\tfrac{1}{2}(\mu + 3p) - \tfrac{3}{2}\beta^2. \tag{31}$$

Now in the case of Riemannian geometry without a cosmological constant, we have $\ddot{R}/R < 0$ always as $\mu + 3p$ is generally assumed to be positive (Hawking and Ellis 1973). Thus, for a presently expanding universe, a big bang singularity is unavoidable in the FLRW models (Ellis 1971, 1973; Hawking and Ellis 1973). However, in Lyra's manifold, it is possible to have a bounce at a minimum (at which $\dot{R}/R = 0$) provided that $\ddot{R}/R > 0$, or equivalently, from equation (31), provided that

$$\beta^2 < -3(\mu + 3p). \tag{32}$$

Pure imaginary values for β^2 have already been considered by Sen (1957) and by Kalyanshetti and Waghmode (1982). Although the physical significance of such an imaginary quantity is not very clear, we remark that manifolds with imaginary connection have been considered (Moffat 1956). As we shall see later, the metric always turns out to be real in the models we derive. Perhaps a more convenient way of looking at this difficulty is to let $-\beta^2 = \alpha$ (say), which is real, and which can then be thought of as a variable cosmological-type term. In the standard models, the cosmological term is regarded as the vacuum energy density of the quantum field and the problem is that of very fine tuning, i.e. why is the term so small now. As we shall see later in our models, β^2 was large during the early stages of the universe and has decayed to its low value at present. Thus the problem of fine tuning has a ready explanation.

We now proceed to derive some singularity-free models. Equations (29) and (30) are two equations in the four unknowns R, β, p and μ. Assuming the usual equation of state of the form

$$p = (\gamma - 1)\mu; \qquad 1 < \mu < 2, \tag{33}$$

we reduce the number of unknowns to three. In the absence of another independent equation for β, we are forced to impose some condition. Following Ozer and Taha (1986, 1987), we assume that the energy density of the universe is always equal to its 'critical' value,

$$\mu = 3(\dot{R}/R)^2, \tag{34}$$

for which there seems to be some observational evidence (Ozer and Taha 1986, 1987). This ansatz yields viable cosmological models that are free of some of the main problems of the standard models.

Using the relation (34) in (29) we obtain

$$\beta^2 = 4k/R^2. \tag{35}$$

Now in order to solve the entropy problem of the standard model, it is necessary to have $\mathrm{d}S > 0$ for at least a part of the evolution of the universe. In Riemannian geometry without a cosmological constant, we have

$$T\,\mathrm{d}S \equiv \mathrm{d}(\mu R^3) + p\,\mathrm{d}R^3 = 0, \tag{36}$$

for all time. In Lyra's geometry with $\beta \equiv \beta(t)$, we derive from (29) and (30) the following conservation-type equation:

$$\mathrm{d}(\mu R^3) + p\,\mathrm{d}R^3 + \tfrac{3}{4}R^3\,\mathrm{d}\beta^2 + \tfrac{9}{2}\beta^2 R^2\,\mathrm{d}R = 0. \tag{37}$$

Thus for $\mathrm{d}S > 0$, we must have

$$\tfrac{3}{4}R^3(\beta^2)\dot{R} + \tfrac{9}{2}\beta^2 R^2 \dot{R} < 0. \tag{38}$$

The inequality (38) is satisfied for $\beta^2 < 0$. From (35) it then follows that $k = -1$. Equation (35) then becomes

$$\beta^2 = -4R^2. \tag{39}$$

From (29) and (30) we obtain

$$6\ddot{R}/R + \mu + 3\beta^2 = -3p. \tag{40}$$

During the radiation dominated era, $p = \mu/3$ and, with the aid of (34) and (39), we obtain

$$2\ddot{R}/R + 2(\dot{R}/R)^2 = 4/R^2. \tag{41}$$

This equation can be written as

$$d\dot{R}^2/dR + 2\dot{R}^2/R = 4/R, \tag{42}$$

which can readily be integrated to yield

$$\dot{R}^2 = A/R^2 + 2, \tag{43}$$

where A is a constant of integration. A further integration then yields

$$R^2 = 2t^2 + R_0^2, \tag{44}$$

where we have chosen $R = R_0$ at time $t = 0$ and the second constant of integration to be zero. From (34) and (44) we obtain, for the energy density,

$$\mu = (6/R^2)(1 - R_0^2/R^2). \tag{45}$$

This model is singularity free with $\mu = T = S = 0$ and $R = R_0$ at $t = 0$. The quantities μ, T, S and R are nonzero and finite for all finite nonzero values of time. As we have already demonstrated, the model solves the entropy problem since $dS/dt > 0$. Since the integral

$$\int_{t_1}^{t} \frac{dt'}{R(t')}$$

diverges as $t_1 \to -\infty$ (Weinberg 1972; as the model is singularity free, we consider $t_1 \to -\infty$), it follows that our model does not possess particle horizons (Rindler 1956).

We now turn to the matter dominated era during which $p = 0$. Equation (40) becomes

$$6\ddot{R}/R + \mu + 3\beta^2 = 0. \tag{46}$$

Substituting (34) and (39) into the above equation we obtain, after some simplification and rearranging,

$$2\ddot{R} + \dot{R}^2/R = 4/R. \tag{47}$$

Integrating, we get

$$\dot{R}^2 = A/R + 4, \tag{48}$$

where A is an integration constant. In order to integrate (48), we make the substitution

$$d\tau = R^{\frac{1}{2}}\, dt, \tag{49}$$

and obtain the solution

$$R = \tau^2 + R_0, \tag{50}$$

where we have chosen $R = R_0$ at $\tau = 0$ and the second constant of integration to be zero. The energy density is given by

$$\mu = (12/R^2)(1 - R_0/R). \tag{51}$$

Finally we turn to the stiff matter solution, but digress for a moment to point out the importance of stiff matter in general. The possibility $p = \mu$ during the early stages of the evolution of the universe seems to have been considered first by Zeldovich (1972). Solutions have since then been discussed by a number of authors in cosmology both in general relativity (Barrow 1977, and references therein; Maartens and Nel 1978; Wainwright *et al.* 1979), and in other theories as well, notably the Brans–Dicke theory (Ram and Singh 1984; Lorenz-Petzold 1984, 1985). Other important applications are the relativistic degenerate Fermi gas (Zeldovich and Novikov 1971), and probably superdense degenerate baryon matter at low temperaturaes (Zeldovich 1962; Bludman and Ruderman 1970; Walecka 1974). Substituting $p = \mu$ and (34) and (39) into equation (40), we derive the following equation for R:

$$2\ddot{R} + 4\dot{R}^2/R = 4/R. \tag{52}$$

Integrating we find that

$$\dot{R}^2 = A/R^4 + 1. \tag{53}$$

Introducing a new time variable τ given by the transformation (49), we can integrate (53):

$$R^4 = 4\tau^2 + R_0^4, \tag{54}$$

where, once again, we have chosen $R = R_0$ at $\tau = 0$ and the second constant of integration to be zero. The energy density of the model is given by

$$\mu = (3/R^2)(1 - R_0^4/R^4). \tag{55}$$

We remark that it is also possible to derive solutions for $k = +1$, but these solutions are not as interesting physically as the $k = -1$ solutions.

5. Asymptotic Behaviour of the Models

We now discuss the asymptotic behaviour of the models that we have derived, starting with the vacuum models. Equation (19) may be written as

$$\dot{R}^2 = A/R^4 - k/R^2. \tag{56}$$

For small R, the term A/R^4 dominates the dynamics of the model and the asymptotic equation is

$$\dot{R}^2 \approx A/R^4. \tag{57}$$

Since the term in 'k' becomes negligible, the asymptotic solution is the same as the

$k = 0$ vacuum solution (20), namely

$$R \propto t^{\frac{1}{3}}. \tag{58}$$

For large R, we may write (56) as

$$R^2 \dot{R}^2 = A/R^2 - k. \tag{59}$$

The asymptotic equation for large R is thus

$$R^2 \dot{R}^2 \approx -k. \tag{60}$$

For $k = -1$, the asymptotic solution is

$$R \propto t^{\frac{1}{2}}. \tag{61}$$

The nonempty models exhibit the same asymptotic behaviour for small R as the FLRW models in Riemannian geometry (Ellis 1971, 1973). For large R, the asymptotic equation, which follows from equations (43), (48) and (53), is

$$\dot{R}^2 \approx \text{const.} \tag{62}$$

The asymptotic solution is thus the Milne universe $R \sim t$.

6. Conclusions

In conclusion, we note that the models that we have derived with a time dependent displacement field solve the singularity, entropy and horizon problems. Further, the models possess the open topology since $k = -1$. This is in contrast to the model of Ozer and Taha (1986, 1987) in which $k = +1$. The observational evidence seems to favour open models (Gott *et al.* 1974). We remark that the inflationary universe scenario, which also claims to solve some of the problems of the standard model, is beset with a number of difficulties (Ellis and Rothmam 1985; Olivo-Melchiorri and Melchiorri 1985). Further it does not avoid the initial singularity, does not solve completely the horizon problem and the difficulty of fine tuning still exists (Olivo-Melchiorri and Melchiorri 1985).

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