Aust. J. Phys., 1989, **42**, 171–86

Hadronisation of QCD

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Abstract

Functional integral calculus (FIC) methods are used to transform the meson-diquark bosonisation of quantum chromodynamics into a meson-baryon effective action description of the low energy states of QCD—the *hadronisation* of QCD.

1. Introduction

The hadron family of particles includes the well known mesons $\pi, \rho, \omega, \ldots$ and baryons $p, n, \ldots$ and together they play key roles in the quark sector of matter, for the hadrons are bound states of quarks. The quarks, and the gluons through which they interact, correspond to the quantum fields of quantum chromodynamics (QCD), but are themselves not directly observable. The main problem of QCD is to extract its mass spectrum, and other observables, and to establish that these correspond to the hadronic phenomenon. One way to define this problem, and also a means for proceeding to a solution, is to consider the functional integral calculus formulation of the mass spectrum problem. We have in, Euclidean metric,

$$\sum_n \exp(-E_n T) = \int D\bar{q}DqDA^a_\mu \Delta_f[A^a_\mu] \exp(-S[A^a_\mu, \bar{q}, q]), \tag{1}$$

where T is a finite Euclidean time variable and $\{E_n\}$ is the energy spectrum of QCD. Lattice gauge theory (LGT) is a fairly direct numerical modelling of (1), but an accurate lattice study of the low energy mass spectrum is computationally formidable, more so in the chiral limit as the low current masses of the u and d quarks are approached and the pions become very low mass and finally (massless) Nambu-Goldstone (NG) bosons.

A more powerful and practical way to proceed is to apply the new techniques of functional integral calculus (FIC) to (1). This integral and differential calculus is the ideal mathematical language for quantum field theories, and its development appears to mirror that of ordinary calculus as the language of Newtonian physics. As individual quarks and gluons are not directly observable they do not appear as states represented in the QCD spectrum, and hence, while they are the defining fields of QCD, they are inappropriate variables of integration on which to base approximations to

0004-9506/89/020171$03.00

(1), in contrast to the conventional approach to quantum electrodynamics, for example. The idea of the FIC approach to QCD is to change variables in (1) in such a way that the new variables correspond to quantum fields whose mass spectrum is that of the low energy states of QCD—presumably the hadronic states already mentioned. The choice of the fields introduced through successive changes of FIC variables is actually controlled by the dynamics of the quantum field theory under consideration, that is, it is not an arbitrary choice if a meaningful analysis is to proceed. For example, in Cahill and Roberts (1985) and Roberts and Cahill (1987), a change of variables to colour $\mathbf{1}_c$ and $\mathbf{8}_c$ bilocal $\overline{q}q$ variables was made which, while useful in extracting meson observables, was also limited by the fact that the $\mathbf{8}_c$ bilocal fields do not appear to play any meaningful role in QCD (related to the fact that gluon exchange is repulsive for these $\overline{q}q$ states); hence this choice of FIC variables halts any further analysis, and in particular the introduction of baryonic FIC variables. Kleinert (1976) and Schrauner (1977) have studied the use of bilocal fields, but without consideration of the colour algebra.

However in Cahill *et al.* (1989*b*, present issue p. 161) $\mathbf{1}_c$ meson and $\overline{\mathbf{3}}_c$ and $\mathbf{3}_c$ diquark FIC variables are introduced in place of those in (1). This is very significant as all of these variables correspond to bound states of QCD, though in the diquark case they occur as constituents of $\mathbf{1}_c$ baryons. Thus (1) may be written

$$\sum_n \exp(-E_n T) = \int D\pi D\rho ... Dd .. Dd^\star \exp(-S[\pi, ..., d, .. d^\star]), \tag{2}$$

where the d and $d^\star$ variables represent all the local diquark fields (successive effective actions are labelled by their arguments). Equation (2) is a meson-diquark bosonisation of QCD, that is, the FIC variables are all bose fields. While the diquarks are not expected to be strictly represented in the QCD spectrum, as their colour charge is expected to lead to their confinement, there is growing experimental and phenomenological evidence that they correspond to qq correlations in baryons, which are qqq bound states. It also appears that they may be assigned effective masses. However one of the fundamental difficulties in FIC analysis up to now has been the complete lack of any convincing way of introducing a change of FIC variables so that they now include Grassmannian baryonic variables. Clearly (2) represented progress in this direction, but how are Grassmannian fields to arise from the above bose fields? The answer to this riddle actually lies in the fact that the effective action $S[\pi, ..., d, .. d^\star]$ in (2) is an infinite series in the diquark fields, and that the diquark integrations in (2) produce functionals which are naturally representable in terms of baryonic FIC variables. Hence the purpose of this work is to show that (1), via (2), may be *hadronised*, that is, written as

$$\sum_n \exp(-E_n T) = \int D\pi D\rho .. D\overline{N} .. DN .. \exp(-S[\pi, ..., \overline{N}, N, ..]), \tag{3}$$

where N and $\overline{N}$ represent the complete set of $\mathbf{1}_c$ baryon variables (each corresponding to a particular baryon). As we will note, the FIC change of variables fully specifies the meson-baryon effective action in (3), that is, its form together with the values of the various parameters such as bare masses and coupling constants, all of which are ultimately determined by the underlying quark-gluon dynamics. Of course the FIC technique provides a systematic procedure for calculating the effective action, but one which must be truncated to be practical. The truncation criterion is to be based on the long wavelength approximation. There are two aspects of (3) not resolved in this paper; first whether, in addition to the usual hadronic variables in (3), there must also be present variables describing exotic states such as diquark–anti-diquark states etc., and second, how the local gauge invariance of QCD manifests itself in (3). However FIC analysis is still in its early days and rapid development can be foreseen. One of the important features of this analysis already established is that it is ideally suited to the chiral limit of QCD, unlike LGT, in that the degenerate vacuum structure of QCD is most easily studied and the consequences of the hidden chiral symmetry carried through to the effective action in (3). As well we have Lorentz covariance and the dynamical consequences of the colour algebra, which is so significant to the formation of the baryon states. Finally we draw attention to the fact that because the FIC analysis uses non-local field variables, which are then reduced to an infinite set of local field variables, all integral equations and functional expressions are finite—no regularisation is needed. This is to be compared with FIC analyses which proceed entirely through local field variables, leading to divergent expressions for observables. Our experience with the non-local FIC analysis of QCD points to the conclusion that divergences are not necessarily a part of quantum field theories in four dimensional space-time. They are more likely a consequence of bad analysis or the wrong theory.

In Section 2 we recall and extend the FIC analysis that led to (2). Section 3 presents the analysis which extracts baryon states from the diquark integrations and gives the hadronisation of QCD.

2. Meson-Diquark Bosonisation

Here we outline the FIC analysis which takes us from (1) to (2) (see Cahill *et al.* 1989*b*), but in doing so we provide a new general technique which implements the transformation from bilocal to local FIC boson variables. It is convenient to take $T \to \infty$ as we shall be able to infer the mass spectrum from the resulting effective actions. This avoids the need to deal with boundary conditions. We need only keep T finite if we are interested in genuine finite temperature and density effects. The defining QCD action is

$$S[A^a_\mu, \bar{q}, q] = \int d^4x \left(\tfrac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 + \bar{q}\{\gamma_\mu(\partial_\mu - ig\frac{\lambda^a}{2}A^a_\mu)\}q \right),$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \qquad a = 1, 2 \ldots 8,$$

and we add source terms to S, viz.

$$-\int d^4x (\bar{\eta} q + \bar{q}\eta + J^a_\mu A^a_\mu),$$

and in the presence of sources we call the RHS of (1) $Z[J,\overline{\eta},\eta]$. Here we consider massless quarks and as well as the local colour symmetry the action then has global chiral symmetry $G = U_L(N_f) \otimes U_R(N_f)$. As shown in Cahill *et al.* (1989*b*) (1) may be written, using various FIC techniques,

$$Z = \int D\mathcal{B}D\mathcal{D}D\mathcal{D}^{\star} \exp\Big(\mathrm{TrLn}(G[\mathcal{B}]^{-1}) + \tfrac{1}{2}\mathrm{TrLn}(\mathbf{1} + \overline{\mathcal{D}}G[\mathcal{B}]^T\mathcal{D}G[\mathcal{B}])$$

$$-\int\int \frac{\mathcal{B}^\theta\mathcal{B}^\theta}{2D} - \int\int \frac{\mathcal{D}^\phi\mathcal{D}^{\phi\dagger}}{2D} - R[\mathcal{B},\mathcal{D},\overline{\mathcal{D}}]\Big), \tag{4}$$

in which we finally put $J = 0, \eta = 0$ and $\overline{\eta} = 0$. Here $\mathcal{B}$ is a matrix valued colour-singlet bilocal meson field, while $\mathcal{D}$ and $\overline{\mathcal{D}}$ are colour 3 and $\overline{3}$ bilocal diquark fields, respectively. To complete the meson-diquark bosonisation we must now write (4) explicitly in terms of the (in principle) infinite set of meson and diquark states, as in (2). To illustrate the general technique consider the meson sector of (4) only (i.e. with $\mathcal{D}, \mathcal{D}^\star = 0$);

$$Z = \int D\mathcal{B}\exp(-S[\mathcal{B}]). \tag{5}$$

First the 'vacuum' configurations must be determined as the solutions of the Euler-Lagrange (EL) equations $\delta S/\delta\mathcal{B} = 0$, which become

$$\mathcal{B}^\theta(x,y) = D(x-y)\left[\mathrm{tr}\left(G(x,y,[\mathcal{B}])\frac{M_m^\theta}{2}\right) - \frac{\delta R[\mathcal{B},0,0]}{\delta\mathcal{B}^\theta(y,x)}\right],$$

which is the Schwinger-Dyson equation. This a non-linear equation for the $\{\mathcal{B}^\theta\}$, and only translation invariant solutions, depending only on $x-y$, are known. This equation has degenerate solutions and an analysis, similar to that in Roberts and Cahill (1987), shows that in the vacuum G has the form

$$G(q)^{-1} = iA(q)q.\gamma + VB(q),$$

where $V = \exp(i\sqrt{2}\gamma_5\pi^a F^a)$ and $\{\pi^a\}$ are arbitrary real constants $|\pi| \in [0, 2\pi]$. Thus in the chiral limit the vacuum is degenerate and is the manifold G/H where G is the chiral group and $H = U_V \subset G$. Thus the chiral symmetry is represented as a hidden symmetry. Let us now change variables in (5) so that $\mathcal{B} = 0$ is now the vacuum. It is convenient here to give the quarks small current masses to avoid dealing with the degenerate vacuum. Expanding S about its minimum gives

$$S[\mathcal{B}] = \sum_{n=0,2,3..} S_n[\mathcal{B}],$$

where S_n is of order n in $\mathcal{B}$ and we write $S_2 = \frac{1}{2}\int \mathcal{B}^\theta(\Delta_m^{-1})^{\theta\psi}\mathcal{B}^\psi$. Introducing bilocal source terms in (5) we have, with $S' = S - S_2$,

$$Z[J] = \int D\mathcal{B} \exp\left(-S'[\mathcal{B}] - S_2[\mathcal{B}] + \int \mathcal{B}^\theta J^\theta\right)$$

$$= \exp(-S'[\frac{\delta}{\delta J}]) \int D\mathcal{B} \exp\left(-\int \tfrac{1}{2}\mathcal{B}^\theta (\Delta_m^{-1})^{\theta\psi} \mathcal{B}^\psi + \int \mathcal{B}^\theta J^\theta\right)$$

$$= \exp(-S'[\frac{\delta}{\delta J}]) \exp\left(-\tfrac{1}{2}\mathrm{TrLn}(\Delta_m^{-1}) + \tfrac{1}{2}\int J\Delta_m J\right). \tag{6}$$

Now

$$\mathrm{TrLn}(\Delta_m^{-1}) = \sum_k \int d^4x \int \frac{d^4P}{(2\pi)^4} \ln(\lambda_k(P^2))$$

$$= \sum_k \mathrm{TrLn}(\lambda_k(\Box)\delta^4(x-y))$$

(see Appendix), and hence we may construct the local-boson-field FIC representation

$$\exp(-\tfrac{1}{2}\mathrm{TrLn}(\Delta_m^{-1})) = \int \prod Dm_k \exp\left(-\sum_k \tfrac{1}{2}\int m_k(x)\lambda_k(\Box)\delta^4(x-y)m_k(y)\right), \tag{7}$$

where $\Box = -\partial^2$ and where $\lambda_k(P^2)$ are the eigenvalues of Δ_m^{-1} (in momentum space);

$$\int \frac{d^4q}{(2\pi)^4} (\Delta_m^{-1})^{\theta\psi}(p,q;P)\Gamma_k^\psi(q;P) = \lambda_k(P^2)\Gamma_k^\theta(p;P). \tag{8}$$

We have the orthonormality relation

$$\int \frac{d^4q}{(2\pi)^4} \Gamma_k(q;P)\Gamma_l(q;P) = N\delta_{kl},$$

and the corresponding completeness relation

$$\sum_k \Gamma_k(q;P)\Gamma_k(p;P) = N\delta^4(q-p).$$

Note that in (7) the LHS functional Tr involves a double space-time trace, viz. $\int d^4x d^4y$, appropriate to bilocal fields, whereas the RHS has only one space-time trace. Equation (7) is a fundamental identity as it implements the reduction of the bilocal FIC meson (and diquark) formulation to local FIC variables, and exposes the physical content of (4). In fact we can write (6) as

$$Z[J] = \exp(-S'[\frac{\delta}{\delta J}]) \int \prod Dm_k \exp\Big(- \sum_k \tfrac{1}{2} \int \; m_k(x)\lambda_k(\Box)m_k(x) +$$

$$+ \int \; J^\theta \Gamma_k^\theta m_k \Big), \tag{9}$$

where $\{m_k(x)\}$ is an infinite set of local meson fields, each corresponding to one physical meson state, and we have used the spectral expansion

$$\Delta_m^{\theta\psi}(p,q;P) = \frac{1}{N} \sum_k \Gamma_k^\theta(p;P)\lambda_k(P^2)^{-1}\Gamma_k^\psi(q;P).$$

Applying the functional operator $\exp(-S'[\delta/\delta J])$ and then, with $J \to 0$, we obtain

$$Z = \int \prod Dm_k \exp\Big(- \sum_k \tfrac{1}{2} \int \; m_k(x)\lambda(\Box)m_k(x) - S'[\Gamma_k^\theta m_k]\Big) .$$

By explicit evaluation of $S'[\Gamma_k^\theta m_k]$, and identifying the mesons by their quantum numbers, we obtain the local FIC representation of the meson sector of (4)

$$Z = \int \; D\pi D\rho D\omega ... \exp(-S[\pi,\rho,\omega,..]), \tag{10}$$

where

$$S[\pi,\rho,\omega,...] = \int \; d^4x \Big[\frac{f_\pi^2}{2}[(\partial_\mu\pi)^2 + m_\pi^2\pi^2] + \frac{f_\rho^2}{2}[-\rho_\mu\Box\rho_\mu +$$

$$+(\partial_\mu\rho_\mu)^2 + m_\rho^2\rho_\mu^2] + \frac{f_\omega^2}{2}[\rho \to \omega] - f_\rho f_\pi^2 g_{\rho\pi\pi}\rho_\mu.\pi \times \partial_\mu\pi +$$

$$-if_\omega f_\pi^3 \epsilon_{\mu\nu\sigma\tau}\omega_\mu\partial_\nu\pi.\partial_\sigma\pi \times \partial_\tau\pi - if_\omega f_\rho f_\pi G_{\omega\rho\pi}\epsilon_{\mu\nu\sigma\tau}\omega_\mu\partial_\nu\rho_\sigma.\partial_\tau\pi +$$

$$+\frac{\lambda i}{80\pi^2}\epsilon_{\mu\nu\sigma\tau}\mathrm{tr}(\pi.\tau\partial_\mu\pi.\tau\partial_\nu\pi.\tau\partial_\sigma\pi.\tau\partial_\tau\pi.\tau) + ...\Big] + S[0,...], \tag{11}$$

where we have written $\lambda_j(P^2) = (P^2 + m_j(P^2)^2)N^{-\frac{1}{2}}f_j^2$ where $m_j(P^2)$ are the running meson masses, and $N^{-\frac{1}{2}}f_j^2$ is the slope of $\lambda_j(P^2)$ at its zero $P_j^2 = -m_j^2$, and we have rescaled the fields $m(x) \to N^{\frac{1}{4}}m(x)$. We choose N such that the above f_π^2 agrees with the value in Cahill *et al.* (1987).

The imaginary terms in this meson action are the chiral anomalies of QCD. In (11) we give the long wavelength limit of the effective action—in general the coupling constants, such as $g_{\rho\pi\pi}$ are non-local (i.e. momentum dependent). The leading terms in the expressions for the various meson parameters in (11) are given in Praschifka *et al.* (1987*a*, 1987*b*), where it is the good agreement with the experimental values of these parameters which implies that we have indeed identified a viable truncation procedure. All these parameters are given by convergent integrals, and no regularisation is required. This follows from the presence of the Γ_k, which are the meson form factors, and indeed it is seen that (8), for $\lambda(P^2) = 0$, is the Bethe-Salpeter equation for the on mass-shell form factors. We have transformed (8) (Cahill *et al.* 1987) into a mass functional variational problem, but it may also be solved directly. By keeping the momentum dependence of $g_{\rho\pi\pi}$ Roberts *et al.* (1989) have shown that the $\pi - \pi$ loop contribution to the ρ propagator, which arises from the π integration in (10), generates a (finite) mass splitting between the ω and ρ, comparable with the observed splitting.

When the quarks are massless special techniques are needed to deal with the degenerate vacuum manifold, for in this case the NG bosons form homogeneous Riemann coordinates on the vacuum manifold, and we must use the matrix field $U(x) = \exp(i\sqrt{2}\pi^a(x)F^a)$ where $V(x) = P_L U(x)^\dagger + P_R U(x) = \exp(i\sqrt{2}\gamma_5\pi^a(x)F^a)$. In the chiral limit it is important to note that $B(q)$ of the quark propagator is also the NG boson form factor, as may be found from (8). This is discussed more explicitly in Cahill *et al.* (1987). For this reason the ground state pseudoscalars, in the chiral limit, play a dual role: they are at the same time both the NG bosons associated with the hidden chiral symmetry and also $\overline{q}q$ bound states. This is important also in the non-local coupling of these pseudoscalars to the baryons. To maintain the hidden chiral symmetry necessitates a derivative expansion in $\partial_\mu U(x)$, and we obtain (Roberts *et al.* 1988), for the NG boson sector only (here $\mathcal{M}$ is the quark mass matrix)

$$\int d^4x \frac{f_\pi^2}{2}[(\partial_\mu\pi)^2 + m_\pi^2\pi^2] \to \int d^4x \Big(\frac{f_\pi^2}{4}\mathrm{tr}(\partial_\mu U \partial_\mu U^\dagger) + \kappa_1 \mathrm{tr}(\partial^2 U \partial^2 U^\dagger) +$$

$$+\kappa_2 \mathrm{tr}([\partial_\mu U \partial_\mu U^\dagger]^2) + \kappa_3 \mathrm{tr}(\partial_\mu U \partial_\nu U^\dagger \partial_\mu U \partial_\nu U^\dagger) +$$

$$+\frac{\rho}{4}\mathrm{tr}([2\mathbf{1} - U - U^\dagger]\mathcal{M}) + ... \Big). \tag{12}$$

As derived in Roberts and Cahill (1987) under a chiral transformation $G = U_L \otimes U_R$, $U(x) \to U_L U(x) U_R^\dagger$, and (12) is of course invariant under this mapping (except for the mass term). From this action we easily obtain the Gell-Mann–Okubo mass formula for the pseudoscalar octet $4m_\kappa^2 = m_\pi^2 + 3m_\eta^2$ and expressions for the $\pi - \pi$ scattering lengths (a_0^0 and a_0^2) which reproduce the Weinberg result $a_0^0/a_0^2 = -7/2$. The action in (12) is that of the non-linear σ-model (NLSM), with all parameters $f_\pi, \kappa_1, ...\rho, ..$ being finite and fully specified by the underlying quark-gluon dynamics. However (12) is only the long wavelength limit of this derived NLSM. If one wishes to study the quantum mechanics of this system, by carrying out functional integrations over the matrix field $U(x)$, it will be necessary to keep the full non-locality of the above parameters, or equivalently, in momentum space, their momentum dependence. In this way we would see that there are no divergences associated with the quantised NLSM, even in four dimensional space-time. Again this is due to the extended nature of the NG bosons.

A similar transformation to local FIC variables is also possible for the diquarks, but this is a part of the baryonisation of QCD and is considered in the next section.

3. Baryons

Here we show that the diquark sector of the meson-diquark bosonisation (4) generates the colour singlet baryon states of QCD. The meson dressing of these (bare) baryons also follows from (4). As we wish to be explicit we shall not discuss those extra (and more complicated) processes arising from $\mathcal{R}[\mathcal{B},\mathcal{D},\overline{\mathcal{D}}]$. Hence we will consider

$$Z[J^\star,J] = \int D\mathcal{D}D\mathcal{D}^\star \exp\Big(\tfrac{1}{2}\mathrm{TrLn}(\mathbf{1}+\overline{\mathcal{D}}G^T\mathcal{D}G) - \int \frac{\mathcal{D}\mathcal{D}^\dagger}{2D} + \int (J^\star\mathcal{D}+\mathcal{D}^\star J)\Big), \tag{13}$$

where the bilocal diquark source terms facilitate the analysis, and in which $G = G[\mathcal{B}_v]$, where $\mathcal{B}_v$ are the vacuum configurations for the meson sector. Meson couplings arise from retention of the full $\mathcal{B}$. We will not discuss here the possibility that there may be non-zero diquark field configurations in the vacuum, i.e. diquark condensates. The existence of these can be studied using the full EL equations for the total action in (4).

First consider the physical meaning of the $\frac{1}{2}$TrLn in (13), which on expansion

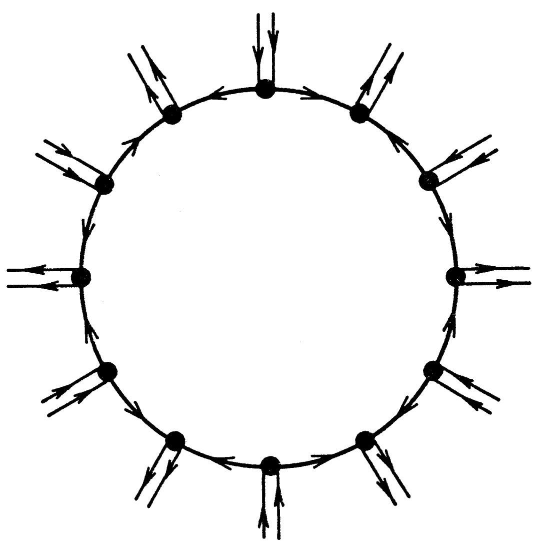


Fig. 1. Single loop processes corresponding to the terms in (14). Note that the quark lines alternate in direction in accord with their coupling to the diquark fields. These loops also represent the terms in S' of (19), in which case there are functional derivatives at each vertex.

gives

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{(n+1)}}{n}\mathrm{Tr}(\overline{\mathcal{D}}G^{T}\mathcal{D}G)^{n}, \tag{14}$$

and which are single loop processes (see Fig. 1), but with the quark lines alternating in direction, in accord with their coupling to the diquark and anti-diquark fields. These loop structures are the key to noticing that $Z[0,0]$ contains contributions representable in terms of baryonic FIC variables. The derived effective action for these fields is that of all the known baryon states.

Using (14) the action in (13) has the expansion

$$S[\mathcal{D}^{\star},\mathcal{D}]=\sum_{n}S_{n}[\mathcal{D}^{\star},\mathcal{D}],$$

and we write $S_1=\int \mathcal{D}^{\phi\star}(\Delta_d^{-1})^{\phi\psi}\mathcal{D}^{\psi}$. Defining $S'=S-S_1$, (13) can be written

$$Z[J^{\star},J]=\exp(S'[\frac{\delta}{\delta J^{\star}},\frac{\delta}{\delta J}])\int D\mathcal{D}D\mathcal{D}^{\star}\exp\left(-\int \mathcal{D}^{\star}\Delta_d^{-1}\mathcal{D}+\int (J^{\star}\mathcal{D}+\mathcal{D}^{\star}J)\right),$$

$$=\exp(S'[\frac{\delta}{\delta J^{\star}},\frac{\delta}{\delta J}])\exp\left(-\mathrm{TrLn}(\Delta_d^{-1})+\int J^{\star}\Delta_d J\right), \tag{15}$$

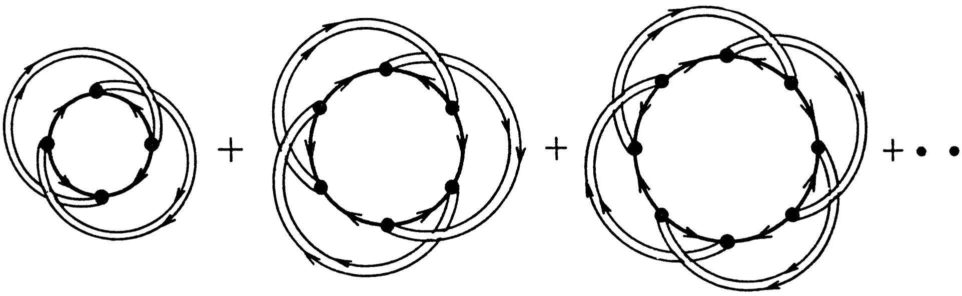
where Δ_d^{-1} has eigenvalues and eigenvectors (which are the diquark form factors) from

$$\int\frac{d^4q}{(2\pi)^4}(\Delta_d^{-1})^{\phi\psi}(p,q;P)\Gamma_k^{\psi}(q;P)=\lambda_k(P^2)\Gamma_k^{\phi}(p;P), \tag{16}$$

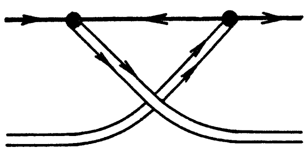
and whence the spectral expansion,

$$\Delta_d^{\phi\psi}(p,q;P)=\frac{1}{N}\sum\Gamma_k^{\phi}(p;P)\lambda_k^{-1}(P^2)\Gamma_k^{\psi}(q;P)^{\star}. \tag{17}$$

Praschifka *et al.* (1988*a*, 1988*b*) have extensively studied (16) in its mass functional form (Cahill *et al.* 1987) and effective masses and form factors for the $0^+, 0^-, 1^+$ and 1^- ground state diquarks have been obtained. One expects that (17) may be truncated after only one or two terms as the higher mass diquark states will not contribute to the lower mass baryon states. It is also important to note that the sum in (17) is purely discrete, that is, there is no continuum contribution. This is because the quark propagators $G[\mathcal{B}_v]$ are confining, that is, the propagators have no poles on the $q^2<0$ axis. Hence there is no 'ionisation' threshold leading to a continuum. To simplify the notation we will now only consider one term in the sum (17) and, with the $J^P=\frac{1}{2}^+$ baryon octet $(p,n,\Lambda,\Sigma,..)$ in mind, we keep the first $J^P=0^+$ scalar diquark (which has an effective mass of ≈ 0.6 GeV). The above comments imply that the qq correlations in baryons are influenced by only a few diquark states, and this should have strong implications for the baryon structure functions. In fact (Praschifka *et al.* 1988*a*, 1988*b*) the calculated diquark form factors show strong peaking, which causes a quark constituent mass effect in diquarks.

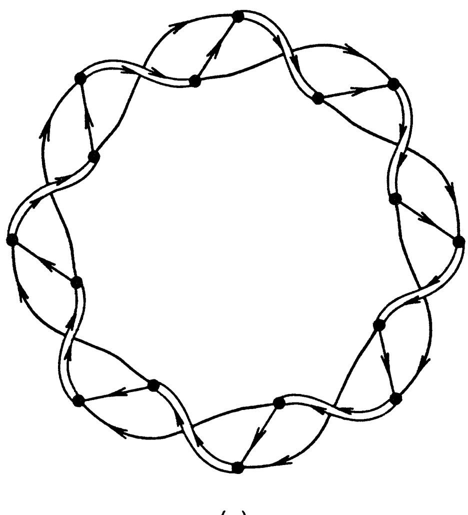


(a)

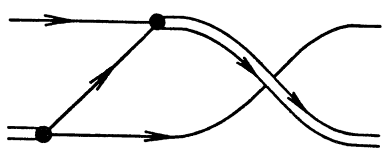


(b)

Fig. 2. (*a*) Three-quark loops arising from Fig.1 when diquark propagators are attached between vertices. Shown are orders $n = 2, 3, 4$ with clockwise diquark propagators. The anti-clockwise diagrams contribute equally, giving a factor of 2 (except for $n = 3$ for which there is only one diagram). While the even order amplitudes may be drawn as planar graphs, odd orders are necessarily non-planar. The cross-over pattern for the diquarks has no significance other than to see how Fig. 3*a* is obtained. (*b*) The kernel of the above loops.



(a)



(b)

Fig. 3. (*a*) A closed *double helix* of QCD—a baryon loop. To reveal the three-quark or baryon loop form of the diagrams in Fig. 2*a*, and in a form valid for all orders, they have been deformed into the above (an 8th order diagram is shown). (*b*) The kernel of the above loops, which is topologically identical to that in Fig. 2*b*.

With the above truncation we can obtain from (15) a simplified version of (13), using the general methods of Section 2,

$$Z[j^\star,j] = \int DdDd^\star \exp\Big(\tfrac{1}{2}\mathrm{TrLn}(\mathbf{1} + \tfrac{1}{4}d^\star \Gamma M_0 C^T G^T C^T M_0 \Gamma d G) +$$

$$- \int d^\star d \int \frac{\Gamma\Gamma}{2D} + \int (j^\star d + d^\star j)\Big), \tag{18}$$

where j is a local source for the one scalar diquark, and $M_0^{\rho f} = -\sqrt{\frac{1}{3}} i\gamma_5 \epsilon^\rho \epsilon^f$ where $\rho = 1,2,3$ and $f = 1,2,3$ are the colour and flavour indicies of the $\overline{\mathbf{3}}_c$ and $\overline{\mathbf{3}}_f$ scalar diquarks, and Γ is the form factor. The second order term in (18) has the form $-\int d^\star \lambda_0 d$. Then

$$Z[j^\star,j] = \exp\Big(-S'[\frac{\delta}{\delta j^\star}, \frac{\delta}{\delta j}]\Big) \exp\Big(\int j^\star \lambda_0^{-1} j\Big), \tag{19}$$

in which S' is a sum of loops, as in Fig.1, but with the functional derivative operators $\delta/\delta j^\star$ and $\delta/\delta j$ at alternating vertices, and with M_0... at each vertex.

Evaluating (19), and with $j^\star, j \to 0$, we find that $Z[0,0]$ has the form $\exp(W)$ where W is the sum of connected loop diagrams, with the vertices now joined by the diquark propagators. Of particular significance is that infinite subset of diagrams which will be seen to have the form of three-quark (i.e. baryon) loops, as in Fig. 2a. These come with a combinatoric factor of 2 (except for the order $n = 3$ diagram) which cancels the $\frac{1}{2}$ coefficient of the TrLn in (18). These 3-loops are planar for even order, but non-planar, with one twist, for odd order. To exhibit the three-quark loop structure in Fig. 3a we show a typical diagram from Fig. 2b after deformation, revealing a closed double helix: a diagram of order n is drawn on a Mobius strip of $n-1$ twists. This infinite series may be summed as the diagrams are generated by the kernel K, defined by the one-twist diagram, shown in Figs 2b and 3b. The weightings are such that (except for the third order diagram) all the double helix diagrams may be summed to $\mathrm{TrLn}(\mathbf{1}+K) - \mathrm{Tr}K = W_B - \mathrm{Tr}K$. Thus

$$Z[0,0] = \exp(W_B + W_R),$$

in which W_R is the sum of the remaining diagrams (see discussion later). To determine the content of W_B we consider the eigenvalue problem for $\mathbf{1}+K$;

$$(\mathbf{1}+K)\Psi_k = \lambda_k \Psi_k, \tag{20}$$

which in mometum space has the detailed form (see Fig. 4)

$$\int \frac{d^4 q}{(2\pi)^4} K(p,q;P)^{\beta j,\rho h}_{\alpha f,\gamma l} \Psi^{\gamma l}_{\rho h}(q;P) = (\lambda_k(P^2) - 1)\Psi^{\beta j}_{\alpha f}(p;P), \tag{21}$$

$$K(p,q;P)^{\beta j,\rho h}_{\alpha f,\gamma l} = \sum_{i\delta} \tfrac{1}{12} \Gamma \epsilon_{\gamma\alpha\delta}\epsilon_{lfi} i\gamma_5 C^T G^T C^T i\gamma_5 \epsilon_{\beta\delta\rho}\epsilon_{jih} \Gamma G \lambda_0^{-1},$$

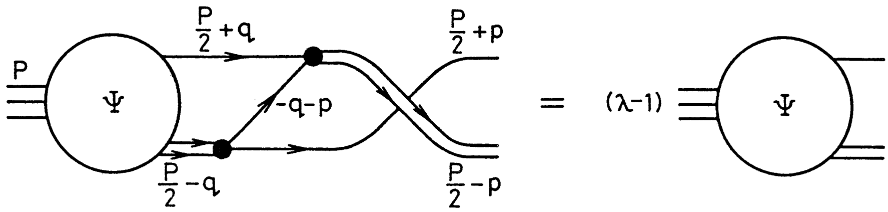


Fig. 4. The eigenvalue equation which determines the spectral content of the infinite sum of double helix amplitudes of the form in Fig. 3*a*. The eigenvectors Ψ are the baryon-quark-diquark form factors. The baryon masses are found from $\lambda(-M^2)=0$.

in which the (momentum) arguments may be inferred from Fig. 4 and (23), and λ_0 is the scalar diquark eigenvalue, (16). Equation (21) is a bound state equation for a three-quark state in which the paired quarks form a scalar diquark. It is a simple matter to generalise (21) to include further diquark states from the spectral expansion (17), but in doing so we note that the generalised (coupled) equations still have only one $\int d^4q$ because (17) has only a discrete spectrum, due to the confining nature of the quark propagator. In (21) Ψ is a Dirac spinor and as well a second rank tensor in both colour and flavour. To isolate the baryon octet we extract

$$\Psi_f^j = \Psi_{\alpha f}^{\alpha j} - \tfrac{1}{3}\Psi_{\alpha i}^{\alpha i}\delta_{fj},$$

which transforms as a colour singlet and flavour octet. Equation (21) then becomes, after simplifying the Dirac algebra,

$$\int \frac{d^4q}{(2\pi)^4} K(p,q;P)\Psi(q;P) = (\lambda(P^2)-1)\Psi(p;P), \tag{22}$$

$$K(p,q;P) = -\tfrac{1}{6}\Gamma\left(\frac{P}{4}+q+\frac{p}{2};\frac{P}{2}-p\right)\Gamma\left(\frac{P}{4}+p+\frac{q}{2};\frac{P}{2}-q\right)\lambda_0^{-1}\left(\left(\frac{P}{2}-q\right)^2\right)$$
$$\times G(-q-p)G\left(\frac{P}{2}+q\right). \tag{23}$$

Equation (22) is the basic equation for the $J^P=\frac{1}{2}^+$ baryon octet where Ψ is the nucleon-quark-diquark form factor, and has been proposed by Cahill *et al.* (1989*a*, present issue p. 129) using non-FIC methods. Initial numerical studies of (22) by Burden *et al.* (1989, present issue p. 147) show that it has a solution, that is, the quark-diquark rearrangement mechanism is attractive, and has a lowest mass state, determined by $\lambda(-M^2)=0$ of ≈ 1 GeV. In (23) λ_0^{-1} is the diquark propagator, and we find, using the mass functional formulation of (16) that $\lambda_0^{-1}(p^2)\approx 18 f_0^{-2}(p^2+m_0^2)^{-1}$, where $f_0[\Gamma]$ is the functional defined in Cahill *et al.* (1987). One can also show (Cahill *et al.* 1989*a*) that the baryon eigenvalues $\lambda(P^2)$ of (22) are chiral invariants, and hence so are the baryon masses. The baryon equation (20) is thus Lorentz and chiral covariant, incorporates the colour algebra (without this algebra the exchange of spin-1 bosons between 2 fermions is repulsive), and has bound state solutions.

Let us now construct an appropriate FIC representation for exp(TrLn($\mathbf{1}+$K)). To this end note that analysis of (22) shows that an eigenvalue for positive energy solutions, with degeneracy 2 (for spin ↑ and ↓), has the form

$$\lambda_+^{\uparrow\downarrow}(P^2) = (M(P^2) + i\sqrt{P^2}a(P^2))F$$

(define F so that $a = 1$ when $\lambda = 0$, then $M_k(P^2)$ will be seen to be baryon running masses), while for negative energy solutions (anti-baryons) $\lambda_-^{\uparrow\downarrow} = (\lambda_+^{\uparrow\downarrow})^*$. Thus, from the spectral representation for $\mathbf{1}+K$,

$$\exp(\mathrm{TrLn}(\mathbf{1}+K)) = \exp\Big(\sum_k n\int d^4x\int \frac{d^4P}{(2\pi)^4}[\ln(\lambda_+^{\uparrow\downarrow}(P^2)^2) + \ln(\lambda_-^{\uparrow\downarrow}(P^2)^2)]\Big),$$

where k sums the ground state and excited baryons states of (22), and the squares in the ln's arise from the spin degeneracy, and $n = 8$ from the flavour degeneracy (put $a = 1$ for simplicity), giving

$$= \exp\Big(\sum_k n\int d^4x\int \frac{d^4P}{(2\pi)^4}\ln[(P^2 + M_k(P^2)^2)^2F_k^4]\Big)$$

$$= \exp\Big(\sum_k n\mathrm{TrLn}[(\gamma.\partial + M_k(\Box))F_k^2\delta^4(x-y)]\Big)$$

$$= \int \prod D\overline{N}_k DN_k \exp\Big(-\sum_k\int d^4x\overline{N}_k(x)(\gamma.\partial + M_k(\Box))F_k^2N_k(x)\Big), \qquad (24)$$

in terms of $\overline{N}_k$ and N_k each of which is a flavour octet of local baryonic spin-$\frac{1}{2}$ FIC variables. Hence the exponentiated sum of the closed double helix diagrams is representable as a (free) baryon field theory. The F_k may be absorbed with a re-definition of the nucleon fields. Other more complicated (including baryon multi-loops) diagrams are present and constitute a wealth of dressings and interactions between these (bare) baryons. Of some interest would be the terms describing nucleon-nucleon interactions via quark and/or diquark exchange. These are not meson mediated interactions and would affect the short range nucleon-nucleon force. As for the meson-baryon couplings, these may be systematically extracted by keeping the full $\mathcal{B}$ dependence in $G[\mathcal{B}]$ in (13). Such couplings are non-local, which makes all quantum fluctuation calculations, such as pion dressing of the nucleon, finite. For small quark current masses we have already shown (Cahill *et al.* 1989*a*) that the baryon equation (21) leads to multiplet splitting in accord with the well known Gell-Mann–Okubo and Coleman-Glashow formulae.

The long wavelength limit of the NG-boson–baryon coupling may be inferred from the chiral invariance of (23). Now

$$\mathrm{TrLn}[(\gamma.\partial + M(\Box))\delta^4(x-y)] = \mathrm{TrLn}[(\gamma.\partial + \mathcal{V}M(\Box))\delta^4(x-y)]$$

reflects that invariance in (29), where $\mathcal{V} = \exp(i\sqrt{2}\gamma_5\pi^aT^a)$, with $\{T^a\}$ the generators of $SU(3_f)$ **8** representation. Then combining the results above and from Section 2, we finally obtain, on expanding $\mathcal{V}$ to first order in π^a,

$$Z = \int D\pi ... D\overline{N} DN .. \exp\Big(-\int [\frac{f_\pi^2}{2}((\partial_\mu \pi)^2 + m_\pi^2 \pi^2) + ..$$

$$+ \mathrm{tr}\{\overline{N}(\gamma.\partial + m_N + \Delta m_N + m_N \sqrt{2} i\gamma_5 \pi^a \mathcal{T}^a + ..)N\} + ..]\Big), \quad (25)$$

in which the baryon octet is finally written as a 2nd rank tensor, $N = N^a \mathcal{T}^a$, where the $\{\mathcal{T}^a\}$ are generators of the $SU(3_f)$ **3** representation (Cahill *et al.* 1989*a*). We have shown m_π and Δm_N which are mass terms from the chiral symmetry breaking quark current masses, while m_N is the 'chiral mass' of the baryons, determined by $\lambda(-m_N^2) = 0$. Keeping in (25) the momentum dependence of $M(P^2)$ leads to the form factor $F_{\pi NN}$.

The extraction of the full meson-baryon and baryon-baryon interactions requires more detailed techniques, but the derivation here of (25) from (1) does demonstrate the hadronisation of QCD.

The detailed form of (20) is easily determined for the other baryon multiplets, particularly the well known $\frac{3}{2}^+$. An important application of this formalism is to the magnetic moments and structure functions of the baryons.

4. Conclusions

We have presented here the FIC techniques that reveal the low energy states of quantum chromodynamics—the colour singlet $\overline{q}q$ mesons and qqq baryons. The necessary intermediate stage of a $\mathbf{1}_c - \overline{\mathbf{3}}_c - \mathbf{3}_c$ meson-diquark bosonisation leads directly to qqq states—the baryons. The resulting baryon equations are simple to solve and easily extended to include numbers of diquark states. They are Lorentz and chiral covariant and include the colour algebra, which is the key to the existence of qqq bound states. The resulting hadronisation is succinctly and elegantly stated as a dynamically determined change of integration variables;

$$\int D\overline{q} Dq DA \exp(-S[A, \overline{q}, q]) = \int D\pi D\rho .. D\overline{N} DN .. \exp(-S[\pi, ., \overline{N}, N, .]).$$

The complete meson-baryon effective action has all parameters determined by the underlying quark-gluon dynamics and is ideally suited to the $u - d$ mesons and baryons, as this analytic approach allows a complete description of the hidden chiral symmetry, and a simple perturbative account of the consequences of the small chiral symmetry breaking quark current masses. The natural non-locality of all hadron couplings means that as a matter of principle there are no divergent expressions in the quark physics. Hence QCD has been an ideal problem in which to develop a mathematically and physically sensible analysis of how quantum field theories behave: *It is not necessary for QFTs in four-dimensional space-time to be associated with infinities.*

The above integral identity is perhaps also archetypal to our understanding of matter. We might expect to the left of this equation another identity relating quarks and gluons to some more fundamental variables, while to the right further changes of integration variables leading to the field variables of nuclear physics. The mappings of the effective actions $.. \rightarrow S[A,\overline{q},q] \rightarrow S[\pi,..,\overline{N},N,..] \rightarrow ...$ correspond to an increasing scale of the collective phenomenon: at the appropriate wavelengths we see meson-baryon processes while at shorter wavelengths we see quark-gluon processes. It is in the hadronisation of QCD where we see most clearly the development of such mappings and the FIC techniques which bring them about.

Acknowledgment

The author thanks the Australian Research Grant Scheme for support.

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Appendix

Consider, for example, a $2 \rightarrow 2$ particle translation invariant kernel, $K(x,y;u,v)$. Then the momentum space eigenvalue equation is

$$\int \frac{d^4q}{(2\pi)^4} K(p,q;P)\Psi_k(q;P) = \lambda_k(P^2)\Psi_k(p;P).$$

The completeness relation is then

$$K(p,q;P) = \sum_k \Psi_k(p;P)\lambda_k(P^2)\Psi_k(q;P)^\dagger,$$

where the orthonormality relation is $(\Psi_k,\Psi_l) = \delta_{kl}$. Then

$$\mathrm{TrLn}(K) = \mathrm{TrLn}(\mathbf{1} + (K - \mathbf{1}))$$

$$= \mathrm{Tr} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(K - \mathbf{1})^n$$

$$
= \sum_n \frac{(-1)^{n+1}}{n} \int d^4x d^4y (K - \mathbf{1})^n (x, y; x, y)
$$

$$
= \sum_n \sum_k \frac{(-1)^{n+1}}{n} \int d^4z \int \frac{d^4P}{(2\pi)^4} (\lambda_k(P^2) - 1)^n
$$

$$
= \sum_k \int d^4z \int \frac{d^4P}{(2\pi)^4} \ln(\lambda_k(P^2))
$$

$$
= \sum_k \mathrm{TrLn}(\lambda_k(\Box)\delta^4(x-y)),
$$

where $\Box = -\partial_\mu \partial_\mu$, and any degeneracies are included in the k-sum. The $\int d^4z$ factor is the space-time volume.

Manuscript received 22 July, accepted 7 December 1988