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# Effect of Small-scale Inhomogeneities on the Dispersive Properties of a Plasma

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*Abstract*

A theory is developed to describe the effect of small-scale inhomogeneities or 'ripples' in an unmagnetised plasma on the properties of transverse waves with wavelength long compared with the scale length of the ripples. The procedure used is based on a covariant version of nonlinear plasma kinetic theory. A simpler theory, which is more physically transparent, is explored in an appendix, and is found to reproduce the results of the more rigorous theory only in the simplest limiting cases. It is found (a) that an anisotropic distribution of density fluctuations causes an otherwise isotropic plasma to become birefringent, (b) that one of the two modes has two resonances associated with the frequencies where three-wave matching conditions are satisfied, together with stopbands and cutoff frequencies, and (c) that there is a crossover frequency. The smearing out of these resonances due to various effects is discussed, and it is concluded that effects of the resonances may be observable under idealised conditions.

## 1. Introduction

Small-scale inhomogeneities in a medium affect the propagation of waves in the medium. It is convenient to identify three limiting cases. (i) One limit is that where high frequency waves are scattered by inhomogeneities with scale size much larger than the wavelength of the wave. Specifically, the scale length of the inhomogeneity, $L$ say, is much larger than the inverse of the wavenumber $|\boldsymbol{k}|$ of the scattered wave: $|\boldsymbol{k}|L \gg 1$. Such scattering is attributed to local variations in the refractive index associated with the inhomogeneities, and is often described in terms of the propagation of waves in random media (Chandrasekhar 1943; Tatarski 1961; Chernov 1960). The important simplifying approximation in this case is that the propagation of the waves may be described using geometric optics, with small local deviations in the ray path. (ii) A second limit corresponds to the inhomogeneities being associated with a spectrum of low frequency waves, so that one has $L \approx 1/|\boldsymbol{k}_L|$, where $|\boldsymbol{k}_L|$ is a characteristic wave number of the low frequency waves. Then three-wave interactions between one of these low frequency waves and a test wave with wave vector $\boldsymbol{k}$ can produce a scattered or 'daughter' wave with wave vector $|\boldsymbol{k}_D| = |\boldsymbol{k} \pm \boldsymbol{k}_L|$. This three-wave matching condition (Manley-Rowe condition) requires that at least one of the high frequency waves has a wave number comparable with that of the low frequency wave. Scattering of waves due to

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three-wave interactions is particularly important in plasmas (e.g., Tsytovich 1970; Davidson 1972). (iii) The third limit corresponds to inhomogeneities with scale lengths shorter than the wavelengths of waves, that is to $|\boldsymbol{k}|L \lesssim 1$. Specific examples include rippled media and composite media in which local variations in density and composition are present with scale lengths shorter than the wavelength of waves of interest in the medium. There is an extensive literature on the transport properties of composite media, e.g., McPhedran *et al.* (1983).

The discussion in the present paper is concerned with plasmas with local inhomogeneities in the density or other plasma parameters. We are concerned with the case where the local inhomogeneities have a spatially periodic structure, which corresponds to a 'rippled' plasma. Our purposes are threefold: (i) to develop a systematic procedure for treating this case based on weak-turbulence theory; (ii) to explore the validity of a simpler and more intuitive theory, based loosely on a standard method for treating Rayleigh scattering; and (iii) to discuss some simple examples of the effects of ripples.

The motivation for our investigation of the limit $|\boldsymbol{k}|L \lesssim 1$ is connected with two possible effects of small-scale inhomogeneities on the dispersion of waves in plasmas. One effect is relevant to an otherwise isotropic medium with an anistropic distribution of inhomogeneities. On a scale long compared with the scale length of the ripples, the medium has an anisotropy due to the distribution of ripples. The medium is then birefringent, and this implies a breaking of the degeneracy between the two states of transverse polarisation. Such birefringency has possibly important physical conseqences on the polarisation of radiation passing through a rippled medium. The second effect of small-scale inhomogeneities on the dispersion of waves concerns the cutoff frequencies for high frequency waves in plasmas. In the absence of a magnetostatic field transverse waves have two states of polarisation; the (doubly degenerate) transverse mode has a cutoff at the plasma frequency $\omega_p$. At a cutoff the wavelength of the waves becomes infinite, and hence for frequencies just above any cutoff the wavelength is necessarily greater than the scale length of any inhomogeneities. This leads one to expect the presence of ripples to modify the cutoff frequency. Even a small change in a cutoff frequency may affect the coupling of waves across the stopband below the cutoff, with possibly important consequences in allowing waves below the cutoff (which could not otherwise escape) to escape by tunnelling across the stopband.

The type of inhomogeneity that we have in mind involves an idealised spectrum of 'ripples', which are purely spatial, undamped, periodic variations, as in a grating. Such ripples may be described in terms of standing waves. One example is standing lower hybrid waves, corresponding to ripples perpendicular to the ambient magnetic field in a thermal plasma. Another example is the standing wave pattern in four-wave mixing, in connection with phase-conjugate reflection in a nonlinear medium.

The method used here to treat the effects of the small-scale inhomogeneities or ripples is based on nonlinear plasma theory. The inhomogeneities are described by their electromagnetic field, and this field is included in the appropriate nonlinear response of the medium. One then takes an average over the inhomogeneities to derive a nonlinear correction to the linear response tensor, e.g., to the dielectric tensor. Thus the inhomogeneities or ripples are described in terms of the correlation function for the electromagnetic field associated with them. This procedure is rigorous in the sense that it is founded on a systematic method, namely weak-turbulence theory.

An alternative method, that is often simpler but is less rigorous, is essentially that used in a standard treatment of Rayleigh scattering, e.g., Landau and Lifshitz (1960, p. 387). This procedure involves assuming that the linear response tensor is proportional to the density of scatterers, and including the inhomogeneities as a perturbation in the linear response tensor, through this functional dependence. This method is outlined in the Appendix, where it is shown that it reproduces the results of the more rigorous theory in the simplest case, but seemingly only in the simplest case.

The formal procedure based on the weak-turbulence expansion is developed in Section 2. The particular forms for the nonlinear responses used is the covariant form developed in earlier papers (Melrose 1981, 1982, 1983, 1986*a*). The method leads to a cumbersome formula that is impracticable to apply. Appropriate approximations are developed in Section 2. The autocorrelation function for the inhomogeneities or ripples is discussed in Section 3. Possible applications are outlined in Section 4. The procedure adopted and the results obtained are discussed in Section 5, and the conclusions are summarised in Section 6.

## 2. Nonlinear Correction to the Linear Response Tensor

The effect of inhomogeneities in a medium on the linear response tensor is included here as follows. The linear response is described in terms of the induced current; the relevant current is that which is linear in a *test field.* The inhomogeneities are assumed to be described by the electromagnetic field associated with them; this field is called the *fluctuating field.* Nonlinear plasma theory is used to find the nonlinear correction to the linear response tensor by expanding in the total (test plus fluctuating) field. Consider the nonlinear current that is cubic in the total field. It contains a term that is quadratic in the fluctuating field and linear in the test field, and this is the nonlinear current of relevance here. One finds the nonlinear correction to the linear response function by performing an ensemble average over the product of the fluctuating fields in this term in the nonlinear current.

### *Weak Turbulence Expansion and the Wave Equation*

The basic equations used here are those of weak-turbulence theory in 4-tensor notation. These equations have been written down and discussed previously (e.g., Melrose 1982, 1983; Melrose and Kuijpers 1984). They are as follows. The weak-turbulence expansion of the Fourier transform of the induced 4-current $J^{\mu}_{\rm ind}(k)$ in powers of the 4-potential $A^{\mu}(k)$ is

$$J^{\mu}_{\rm ind}(k) = \alpha^{\mu\nu}(k) + \int d\lambda^{(2)}\alpha^{(2)\mu\nu\rho}(k,k_1,k_2)A_\nu(k_1)A_\rho(k_2)$$
$$+ \int d\lambda^{(3)}\alpha^{(3)\mu\nu\pi\sigma}(k,k_1,k_2,k_3)A_\nu(k_1)A_\rho(k_2)A_\sigma(k_3) + \cdots, \tag{1}$$

where the $n$th order convolution integral is defined by

$$d\lambda^{(n)} = \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}(2\pi)^4\delta^4(k-k_1-k_2-\cdots-k_n). \tag{2}$$

The expansion (1) defines the linear response tensor $\alpha^{\mu\nu}(k)$ and a hierarchy of nonlinear response tensors, including the quadratic response tensor $\alpha^{(2)\mu\nu\rho}(k,k_1,k_2)$ and the cubic response tensor $\alpha^{(3)\mu\nu\rho\sigma}(k,k_1,k_2,k_3)$. The nonresonant parts of the nonlinear response tensors are the only parts of relevance here, and these satisfy the symmetry properties (Melrose and Kuijpers 1984)

$$\alpha^{(2)\mu\nu\rho}(k,k_1,k_2) = \alpha^{(2)\mu\rho\nu}(k,k_2,k_1) = \alpha^{(2)\nu\mu\rho}(-k_1,-k,k_2),$$
$$\alpha^{(3)\mu\nu\rho\sigma}(k,k_1,k_2,k_3) = \alpha^{(3)\mu\nu\sigma\rho}(k,k_1,k_3,k_2) = \alpha^{(3)\mu\rho\nu\sigma}(k,k_2,k_1,k_3)$$
$$= \alpha^{(3)\nu\mu\rho\sigma}(-k_1,-k,k_2,k_3). \tag{3}$$

In the following the formal solution of the wave equation is required. The Fourier transformed form of Maxwell's equations may be reduced to the wave equation

$$\Lambda^{\mu\nu}(k)A_\nu(k) = -\mu_0 J^{\mu}_{\rm ext}(k), \tag{4}$$

with

$$\Lambda^{\mu\nu}(k) = k^2 g^{\mu\nu} - k^\mu k^\nu + \mu_0\,\alpha^{\mu\nu}(k). \tag{5}$$

The current is separated into the linear response, which is included on the left hand side, and all remaining terms are included in $J^{\mu}_{\rm ext}(k)$, which is regarded as an arbitrary extraneous current which acts as a source term. The inhomogeneous wave equation (4) may be solved by introducing the Green's function or photon propagator $D^{\mu\nu}(k)$ (Melrose 1983). The solution is then

$$A^{\mu}(k) = D^{\mu}{}_{\nu}(k)J^{\nu}_{\rm ext}(k). \tag{6}$$

*Nonlinear Correction to the Linear Response Tensor*

The relevant source term for a four-wave interaction involving the coalescence of waves in three modes labelled $M_1$, $M_2$, $M_3$, into one wave is given by the cubic term in the weak turbulence expansion (1), in which one writes $A \to A_{M_1} + A_{M_2} + A_{M_3}$ and keeps only the cross terms that are proportional to $A_{M_1}A_{M_2}A_{M_3}$. There are six such terms which contribute equally in view of the symmetry property (3). Additional source terms arise from the quadratic response operating twice. For example, the quadratic response due to $A_{M_1}$ and $A_{M_2}$ gives a beat field that can combine with $A_{M_3}$, again due to the quadratic response, to give an additional cubic response term. There are three such combinations, and the sum of all three and the cubic response itself defines an *effective cubic response tensor*, denoted by $\tilde{\alpha}^{(3)\mu\nu\rho\sigma}(k,k_1,k_2,k_3)$. The effective cubic response tensor was written down by Melrose (1986*a*) using the symmetrised forms of the nonlinear response tensors. The relevant symmetry, which is included in the properties (3), follows from the cubic term in (1) by permuting the arguments $k_1$, $k_2$, $k_3$.

In the present context, in which different wave vectors correspond to distinguishable fields, it is convenient to use unsymmetrised forms. The relation written down by Melrose (1986*a*) is then replaced by

$$\begin{aligned}6\tilde{\alpha}^{(3)\mu\nu\rho\sigma}(k,k_1,k_2,k_3) &= \alpha^{(3)\mu\nu\rho\sigma}_{\rm unsym}(k,k_1,k_2,k_3)\\ &+\alpha^{(2)\mu\nu\theta}_{\rm unsym}(k,k_1,k_2+k_3)D_{\theta\eta}(k_2+k_3)\alpha^{(2)\eta\rho\sigma}_{\rm unsym}(k_2+k_3,k_2,k_3)\\ &+\alpha^{(2)\mu\rho\theta}_{\rm unsym}(k,k_2,k_1+k_3)D_{\theta\eta}(k_1+k_3)\alpha^{(2)\eta\nu\sigma}_{\rm unsym}(k_1+k_3,k_1,k_3)\\ &+\alpha^{(2)\mu\sigma\theta}_{\rm unsym}(k,k_3,k_1+k_2)D_{\theta\eta}(k_1+k_2)\alpha^{(2)\eta\nu\rho}_{\rm unsym}(k_1+k_2,k_1,k_2)\,,\end{aligned}\tag{7}$$

where 'unsym' refers to unsymmetrised forms. The factor 6 on the left hand side of (7) is included for notational consistency; this factor was not included in the corresponding definition in Melrose (1986*a*), where the appropriate factor was included elsewhere in the analysis.

The total field $A^\mu(k)$ is the sum of the test field and the fluctuating field, denoted by $A^\mu_{\rm F}(k)$. On making the replacement $A\rightarrow A+A_{\rm F}$ in the effective cubic response term in (1), only the terms quadratic in $A_{\rm F}$ are retained. There are three such terms and they contribute equally (e.g., Melrose 1986*a*). An ensemble average over the fluctuations is performed, assuming that the fluctuations are statistically stationary and uniform. The ensemble average, which is denoted by angle brackets, reduces to

$$\langle A^\mu_{\rm F}(k)A^\nu_{\rm F}(k')\rangle=(2\pi)^4\,\delta^4(k+k')\langle A_{\rm F}A_{\rm F}\rangle^{\mu\nu}(k)\,,\tag{8}$$

where $\langle A_{\rm F}A_{\rm F}\rangle^{\mu\nu}(k)$ denotes the Fourier transform of the autocorrelation function of the fluctuations. This gives

$$\alpha^{\mu\nu}_{\rm NL}(k)=3\int\frac{d^4k'}{(2\pi)^4}\,\tilde{\alpha}^{(3)\mu\nu\rho\sigma}(k,k,k',-k')\langle A_{\rm F}A_{\rm F}\rangle_{\rho\sigma}(k')\,,\tag{9}$$

where the factor 3 arises from the three equal terms.

### *Approximations for the Effective Cubic Response Tensor*

The general form of the effective cubic response tensor (7) is too cumbersome to be of direct use in most applications. Appropriate approximations are required. An important distinction that may be made is between 'fast' and 'slow' disturbances, which are defined depending on whether the phase speed is, respectively, much greater than or much less than the thermal speed of electrons $V_e$. Melrose (1986*a*) derived approximations for the nonlinear response tensors for several cases involving fast and slow disturbances.

In the present context, the fluctuations or ripples are assumed to be slow ($\omega'/|\boldsymbol{k}'| \ll V_e$), with low frequencies ($\omega' \ll \omega$) and large wavenumbers ($|\boldsymbol{k}'| \gtrsim |\boldsymbol{k}|$). The waves, described by the test field, are fast ($\omega/|\boldsymbol{k}| \gg V_e$), and the beats at $\omega \pm \omega', \boldsymbol{k} \pm \boldsymbol{k}'$ are also assumed to be fast. On setting $k = k_1$ and $k_2 = -k_3 = k'$ in (7), it follows that approximations are required for the quadratic response tensor with two fast ($k$, $k_1$) and one slow ($k_2$) disturbance, and for the cubic response tensor with two fast ($k$, $k_1$) and two slow ($k_2$, $k_3$) disturbances and where the beats between $k_1$, $k_2$, $k_3$ are fast. In the rest frame of a nonrelativistic, thermal, unmagnetised plasma with $\bar{u}^\mu = [1, \boldsymbol{0}]$, these approximations are, respectively,

$$\alpha^{(2)\mu\nu\rho}_{\rm unsym}(k, k_1, k_2) \approx \frac{e^3 n_e}{m_e^2 V_e^2}\, a^{\mu\nu}(k, k_1, \bar{u})\, \frac{k_2\bar{u}\, k_{2\alpha} G^{\alpha\rho}(k_2, \bar{u})}{k_2^2 - (k_2\bar{u})^2}, \tag{10}$$

$$\alpha^{(3)\mu\nu\rho\sigma}_{\rm unsym}(k, k_1, k_2, k_3) \approx -\frac{e^4 n_e}{m_e^3 V_e^4}\, a^{\mu\nu}(k, k_1, \bar{u})$$
$$\times \frac{k_2\bar{u}\, k_{2\alpha} G^{\alpha\rho}(k_2, \bar{u})}{k_2^2 - (k_2\bar{u})^2}\, \frac{k_3\bar{u}\, k_{3\beta} G^{\beta\sigma}(k_3, \bar{u})}{k_3^2 - (k_3\bar{u})^2}, \tag{11}$$

where we use the notation

$$G^{\mu\nu}(k, \bar{u}) = g^{\mu\nu} - \frac{k^\mu \bar{u}^\nu}{k\bar{u}}, \tag{12}$$

$$a^{\mu\nu}(k, k_1, \bar{u}) = g^{\mu\nu} - \frac{k_1^\mu \bar{u}^\nu}{k_1\bar{u}} - \frac{k^\nu \bar{u}^\mu}{k\bar{u}} + \frac{kk_1\, \bar{u}^\mu \bar{u}^\nu}{k\bar{u}\, k_1\bar{u}}, \tag{13}$$

and retain only contributions from electrons with charge $-e$ mass $m_e$. Substituting (10) and (11) into (7), and using the symmetry properties (3), one obtains

$$6\bar{\alpha}^{(3)\mu\nu\rho\sigma}(k, k_1, k_2, k_3) \approx -\frac{e^4 n_e}{m_e^3 V_e^4}\, \frac{k_2\bar{u}\, k_{2\alpha} G^{\alpha\rho}(k_2, \bar{u})}{k_2^2 - (k_2\bar{u})^2}\, \frac{k_3\bar{u}\, k_{3\beta} G^{\beta\sigma}(k_3, \bar{u})}{k_3^2 - (k_3\bar{u})^2} \Big[ a^{\mu\nu}(k, k_1, \bar{u})$$
$$- \frac{e^2 n_e}{m_e} \sum_{a=2,3} a^{\mu\theta}(k, k_1 + k_a, \bar{u}) D_{\theta\eta}(k_1 + k_a) a^{\eta\nu}(k_1 + k_a, k, \bar{u}) \Big]. \tag{14}$$

The contribution to the effective cubic response of the term involving the beat at $k_2 + k_3$ in (7) is omitted because it is purely static when we set $k_2 = k'$, $k_3 = -k'$ below; such a static term is inconsistent with the approximation that the beats are fast.

The nonlinear correction to the linear response tensor is obtained by setting $k_1 = k$, $k_2 = k'$, $k_3 = -k'$ in (14) and inserting the result into (9). This gives

$$\alpha^{\mu\nu}_{\rm NL}(k) \approx -\frac{e^4 n_e}{2m_e^3 V_e^4}\int \frac{d^4k'}{(2\pi)^4}\left(\frac{k'\bar{u}}{k'^2}\right)^2 [k'^2-(k'\bar{u})^2]\Big[a^{\mu\nu}(k,k,\bar{u}) - \frac{e^2 n_e}{m_e}\frac{k'^4}{[k'^2-(k'\bar{u})^2]^2}\sum_{\pm} a^{\mu\theta}(k,k\pm k',\bar{u})D_{\theta\eta}(k\pm k')a^{\eta\nu}(k\pm k',k,\bar{u})\Big] \times L^{\rho\sigma}(k',\bar{u})\langle A_{\rm F}A_{\rm F}\rangle_{\rho\sigma}(k'), \tag{15}$$

with

$$L^{\rho\sigma}(k,\bar{u}) := \frac{k_\alpha G^{\alpha\rho}(k,\bar{u})k_\beta G^{\beta\sigma}(k,\bar{u})}{k^2-(k\bar{u})^2}. \tag{16}$$

In applications one of the beats at $k \pm k'$ is often relatively close to a pole in the propagator $D_{\theta\eta}(k\pm k')$. For example, suppose that $k\pm k'$ is close to the pole at $k\pm k' = k_M$, which corresponds to waves in the mode $M$; then the dominant term in (11) is that involving the appropriate propagator $D_{\theta\eta}(k\pm k')$.

*Longitudinal Approximation for the Propagator*

One further simplifying assumption is that the propagator in (15) may be approximated by its longitudinal part. In an isotropic plasma one may separate the propagator into a longitudinal $D^L(k)$ part and a transverse part $D^T(k)$ by writing (Melrose 1982)

$$D^{\mu\nu}(k) = D^L(k)\,L^{\mu\nu}(k,\bar{u}) + D^T(k)\,T^{\mu\nu}(k,\bar{u}), \tag{17}$$

with $L^{\mu\nu}(k,\bar{u})$ given by (16) and with

$$T^{\mu\nu}(k,\bar{u}) = a^{\mu\nu}(k,k,\bar{u}) - L^{\mu\nu}(k,\bar{u}), \tag{18}$$

with $a^{\mu\nu}(k,k,\bar{u})$ implied by (13). On inserting (17) into (15), a lengthy calculation gives

$$\begin{aligned}\alpha^{\mu\nu}_{\rm NL}(k) \approx &-\frac{e^4 n_e}{2m_e^3 V_e^4}\int \frac{d^4k'}{(2\pi)^4}\left(\frac{k'\bar{u}}{k'^2}\right)^2 [k'^2-(k'\bar{u})^2] \\ &\times\Big\{a^{\mu\nu}(k,k,\bar{u})\Big[1-\frac{e^2n_e}{m_e}\sum_{\pm} D^T(k\pm k')\frac{k'^4}{[k'^2-(k'\bar{u})^2]^2}\Big] \\ &-\frac{e^2n_e}{m_e}\frac{k'^4}{[k'^2-(k'\bar{u})^2]^2}\sum_{\pm}\frac{(k\pm k')_\alpha G^{\alpha\mu}(k,\bar{u})\,(k\pm k')_\beta G^{\beta\nu}(k,\bar{u})}{(k\pm k')^2-[(k\pm k')\bar{u}]^2} \\ &\times\Big[\frac{(k\pm k')^4}{[(k\pm k')\bar{u}]^4}D^L(k\pm k') - D^T(k\pm k')\Big]\Big\}L^{\rho\sigma}(k'.\bar{u})\langle A_{\rm F}A_{\rm F}\rangle_{\rho\sigma}(k').\end{aligned} \tag{19}$$

On making the longitudinal approximation to the propagator, only the term involving $D^L(k\pm k')$ is retained in the curly brackets in (19). Our justification for this assumption is that the effect of interest is strongest when $k\pm k'$ is close to a pole of $D^L(k \pm k')$, when only the term involving $D^L(k \pm k')$ in (19) is important. With this assumption we have

$$D^L(k) = -\frac{(k\bar{u})^4}{k^4} \frac{\mu_0}{\Lambda^L(k)}, \tag{20}$$

with $\Lambda^L(k) = (k\bar{u})^2 + \mu_0 \alpha^L(k)$, where $\Lambda^L(k)$ and $\alpha^L(k)$ are the longitudinal parts of the tensors $\Lambda^{\mu\nu}(k)$ and $\alpha^{\mu\nu}(k)$, respectively, cf. (5). On making the longitudinal approximation, (19) reduces to

$$\alpha^{\mu\nu}_{\mathrm{NL}}(k) \approx -\frac{\mu_0 e^6 n_e^2}{2m_e^4 V_e^4} \int \frac{d^4k'}{(2\pi)^4} \frac{(k'\bar{u})^2}{k'^2 - (k'\bar{u})^2}$$
$$\times \sum_{\pm} \frac{(k \pm k')_\alpha G^{\alpha\mu}(k,\bar{u})\,(k \pm k')_\beta G^{\beta\nu}(k,\bar{u})}{(k \pm k')^2 - [(k \pm k')\bar{u}]^2} \frac{L^{\rho\sigma}(k',\bar{u})\langle A_F A_F\rangle_{\rho\sigma}(k')}{[(k \pm k')\bar{u}]^2 + \mu_0 \alpha^L(k \pm k')}. \tag{21}$$

Further simplification is made in Section 4 after choosing the rest frame of the plasma.

## 3. Correlation Function for Density Fluctuations

The correlation function for the fluctuations or ripples appears in the expression (8) for the nonlinear correction to the linear response tensor. This correlation function is evaluated here for density fluctuations.

*Density Fluctuations*

Fluctuations in the charge density are related to the electric field by Poisson's equation. We assume that the charge density is due entirely to fluctuations $\delta n_e$ in the electron number density. The Fourier transform of the charge density is then identified as $\rho(k) = -e\delta n_e(k)$. The charge density is also identified as the $\mu = 0$ component of the induced current $J^\mu_{\mathrm{ind}}(k)$ in (1), and is related to the the electrostatic potential $\Phi(k) = A^0(k)$ in the Coulomb gauge (when the other components of $A^\nu(k)$ for a longitudinal field are zero) by $J^\mu_{\mathrm{ind}}(k) = \alpha^{\mu\nu}(k) A_\nu(k)$. Thus we have

$$-e\delta n_e(k) = \alpha^{00}(k)\,\Phi(k). \tag{22}$$

Furthermore, the $\mu = 0$, $\nu = 0$ component of $\alpha^{\mu\nu}(k)$ is related to the longitudinal part of the tensor $\alpha^{\mu\nu}(k)$ by (Williams and Melrose 1989)

$$\alpha^{00}(k) = -\frac{|\boldsymbol{k}|^2}{\omega^2} \alpha^{\mathrm{L}}(k). \tag{23}$$

In (22) and (23) $k$ refers to a slow ($\omega/|\boldsymbol{k}| \ll V_e$) longitudinal field, and then the relevant approximation to $\alpha^{\mathrm{L}}(k)$ in a thermal plasma is $\approx -\varepsilon_0 \omega^2/|\boldsymbol{k}|^2 \lambda_D^2$. This implies

$$\alpha^{00}(k) \approx \frac{\varepsilon_0}{\lambda_D^2} = \frac{\varepsilon_0 \omega_p^2}{V_e^2}. \tag{24}$$

On taking the correlation function of (22) with itself, dividing by $n_e^2$, and using (24), one obtains

$$\frac{\langle(\delta n_e)^2\rangle(k)}{n_e^2} = \left(\frac{\varepsilon_0}{e\lambda_D^2 n_e}\right)^2 \langle(\Phi)^2\rangle(k) = \left(\frac{e}{m_e V_e^2}\right)^2 \langle A_F A_F\rangle_{00}(k). \tag{25}$$

*Gaussian Model for Density Fluctuations*

One specific model for the density fluctuations is for the case where their spectrum is gaussian. A general form is

$$\frac{\langle(\delta n_e)^2\rangle(k)}{n_e^2} = \frac{\eta^2(\Delta k)^3}{(2\pi)^{3/2}} 2\pi\delta(\omega-\omega_0) \exp\left[-\frac{(\boldsymbol{k}-\boldsymbol{k}_0)^2}{2(\Delta k)^2}\right]. \tag{26}$$

The parameters $\omega_0$, $\boldsymbol{k}_0$ and $\Delta k$ are interpreted as the frequency, wave vector and spread in wave number of the density fluctuations, respectively. The parameter

$$\eta^2 = \langle(\delta n_e)^2\rangle/n_e^2 = \int \frac{d^4\boldsymbol{k}}{(2\pi)^4} \frac{\langle(\delta n_e)^2\rangle(k)}{n_e^2} \tag{27}$$

is interpreted as the relative level of the density fluctuations.

## 4. Applications

In applying the results derived in Section 2 to specific situations, we first write the correction to the linear response tensor in terms of 3-tensor notation as a correction to the dielectric tensor. The applications discussed here are restricted to cases that may be described in terms of the gaussian spectrum (26) of density fluctuations.

*Nonlinear Correction to the Dielectric Tensor*

The translation of (21) into 3-tensor notation involves the following steps. First, space components $\mu = i$, $\nu = j$ of $\alpha^{\mu\nu}(k)$ are rewritten as minus the corresponding components of the 3-tensor $\alpha_{ij}(k)$. (The minus sign arises because of the equality between the mixed tensor components $\mu = i$, $\nu = j$ of $\alpha^{\mu}{}_{\nu}(k)$ and the components of $\alpha_{ij}(k)$.) Then one uses the relation $\alpha_{ij} = \varepsilon_0\omega^2(K_{ij}-\delta_{ij})$ to introduce the equivalent dielectric tensor. Thus writing (21) in terms of a correction $\delta K_{ij}(k)$ to the dielectric tensor $K_{ij}(k)$ gives

$$\delta K_{ij}(k) = -\frac{\omega_p^4}{2\omega^2} \int \frac{d^4k'}{(2\pi)^4} \frac{\langle(\delta n_e)^2\rangle(k')}{n_e^2} \sum_{\pm} \frac{(\boldsymbol{k}\pm\boldsymbol{k}')_i(\boldsymbol{k}\pm\boldsymbol{k}')_j}{|\boldsymbol{k}\pm\boldsymbol{k}'|^2(\omega\pm\omega')^2 K^{\mathrm{L}}(k\pm k')}, \tag{28}$$

where $K^{\mathrm{L}}(k)$ is the longitudinal part of the dielectric tensor of the background system. The background system is assumed to be an isotropic, unmagnetised, thermal electron gas at a temperature $T_e = m_e V_e^2$ (in units with Boltzmann's constant equal to unity). The appropriate approximate form for the real part of $K^L(k)$ for a fast disturbance is

$$\mathrm{Re}K^L(k) \approx 1 - \frac{\omega_L^2(\boldsymbol{k})}{\omega^2}, \tag{29}$$

where $\omega_L(\boldsymbol{k}) = [\omega_p^2 + 3|\boldsymbol{k}|^2 V_e^2]^{1/2}$ is the frequency of Langmuir waves. The thermal corrections are unimportant when treating transverse waves, and then the dielectric tensor for the background system may be taken to be $K_{ij}(\omega) = (1 - \omega_p^2/\omega^2)\delta_{ij}$.

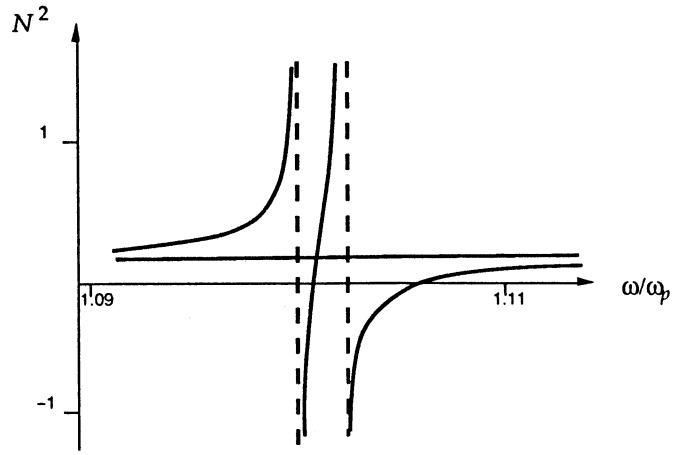


**Fig. 1.** Plot of the square of the refractive index for the two transverse wave modes in a rippled medium (with collimated low frequency density fluctuations). The 'extraordinary' mode has polarisation orthogonal to the direction of propagation of the ripples, and it has two resonances and associated cutoffs and stop bands. The 'ordinary' mode, described by the nearly horizontal curve, is unaffected by the ripples. The parameters chosen in the correction (31) to the dielectric tensor are $\omega_L(\boldsymbol{k}_0)/\omega_p = 1 \cdot 1$, $\omega_0/\omega_p = 0 \cdot 005$, $(\delta n_e/n_e)^2 \sin^2\theta = 0 \cdot 001$, where $\theta$ is the angle between the direction $\boldsymbol{k}$ of wave propagation and the direction $\boldsymbol{k}_0$ of the ripples.

We make the further simplifying approximation that the wavelength of the waves of interest is long compared with the characteristic wavelength associated with the density fluctuations. This corresponds to assuming $|\boldsymbol{k}| \ll |\boldsymbol{k}'|$ in (28). When the gaussian spectrum (26) is inserted into (28), and the long-wavelength approximation and the approximation (29) are made, the integrals in (28) may be performed explicitly in terms of the plasma dispersion function $\phi(z)$:

$$\phi(z) = -\frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dt\, e^{-t^2}}{t - (z + i0)} = 2ze^{-z^2} \int_0^z dt\, e^{t^2} - i\sqrt{\pi} z e^{-z^2}. \tag{30}$$

The resulting general expression is not written down here; some of its properties are used in the following discussion.

### *Anisotropic Density Fluctuations $|\boldsymbol{k}_0| \gg \Delta k$*

The most interesting case is when the spread in wavenumbers is small and the mean wavenumber of the fluctuations is small compared with the Debye wavenumber, that is, for $\Delta k \ll |\boldsymbol{k}_0| \ll \lambda_D^{-1}$. In this case, the general result (28) simplifies to

$$\delta K_{ij}(k) = -\eta^2 \frac{\omega_p^4}{2\omega^2} \sum_{\pm} \frac{(\boldsymbol{k}_0)_i(\boldsymbol{k}_0)_j}{|\boldsymbol{k}_0|^2\left[(\omega \pm \omega_0)^2 - \omega_L^2(\boldsymbol{k}_0)\right]}. \tag{31}$$

On adding (31) to the dielectric tensor $K_{ij}(\omega) = (1-\omega_p^2/\omega^2)\delta_{ij}$ for the background plasma, the additional term causes the plasma to become birefringent. There is an *ordinary* mode which is unaffected by the fluctuations and has refractive index squared $N^2 = (1-\omega_p^2/\omega^2)$. The *extraordinary* mode has resonances at $\omega = \omega_L(\boldsymbol{k}_0) \pm \omega_0$ with associated cutoffs and stopbands. The form of the dispersion relations in the frequency range near $\omega = \omega_L(\boldsymbol{k}_0)$ is illustrated in Fig. 1.

There are several interesting features of the dispersion relations plotted in Fig. 1. One is the appearance of two resonances, which occur at $\omega = \omega_L \pm \omega_0$. The frequencies of these resonance correspond to the conditions for the three-wave matching conditions to be satisfied. That is, if one regards the ripples as a distribution of waves with frequency $\omega_0$ and wave vector $\boldsymbol{k}_0$, then the resonances occur where one or other of the conditions $\omega \pm \omega_0 = \omega_L(\boldsymbol{k} \pm \boldsymbol{k}_0)$ is satsified, so that the beat is on the dispersion curve for Langmuir waves. Another interesting feature in Fig. 1 is that there is a crossover between the two modes, that is, a point at which the refractive indices are equal. Strong coupling between the modes is expected to occur in the vicinity of the crossover.

### *Smearing out of Resonances*

A resonance corresponds to $|\boldsymbol{k}| \to \infty$, and this violates the assumption $|\boldsymbol{k}| \ll |\boldsymbol{k}_0|$ made in deriving (31). The resonance and associated cutoffs can be smeared out by at least three effects:

(i) the spread $\Delta k$ in wave numbers for the density fluctuations,
(ii) damping of the virtual Langmuir waves at $\omega = \omega_L(\boldsymbol{k} \pm \boldsymbol{k}_0)$, and
(iii) damping of the density fluctuations.

The first of these effects is included by evaluating the integral in (28) with (26) in terms of the plasma dispersion function and then expanding for small $\Delta k$. The second of the effects is included by retaining the imaginary part of $K^L(k)$ on the left hand side of (29); this corresponds to retaining the damping of the (virtual) Langmuir waves on the right hand side. Damping of the density fluctuations may be included by regarding $\omega_0$ as the frequency of ion sound waves and including the appropriate damping as the imaginary part of this frequency. Landau damping of the density fluctuations is a relatively strong effect in a thermal plasma, and this damping can place severe constraints on the conditions under which the resonances and cutoffs illustrated in Fig. 1 might be observable.

On including any of the effects listed above, provided they are small, the resonance is replaced by a maximum in the dispersion curve, followed by a region of inverse dispersion leading to a minimum, as is familiar for the response of any damped oscillator. The smeared-out resonance is important only if the peak value of $N^2$ is greater than the unperturbed value in the absence of the fluctuations. It then follows from (i), (ii) and (iii) that for the resonance to be important one requires

$$\frac{(\delta n_e)^2}{n_e^2}\,\frac{\omega_p^2}{3k_0^2V_e^2} \gg \max\left[\frac{3(\Delta k)^2V_e^2}{\omega_p^2},\frac{\gamma_L(\boldsymbol{k}_0)}{\omega_p},\frac{\gamma_S(\boldsymbol{k}_0)}{\omega_p}\right], \tag{32}$$

where $\gamma_L(\boldsymbol{k}_0)$ and $\gamma_S(\boldsymbol{k}_0)$ are the absorption coefficients for the virtual Langmuir waves and the density fluctuations, respectively, assuming that the density fluctuations are associated with ion sound waves.

## 5. Discussion

We discuss two aspects of the foregoing results in further detail: the formal derivation of the correction to the response tensor, and the significance of the possible appearance of additional resonances and cutoffs in a rippled plasma.

The procedure adopted here for including the effect of local spatial variations in the plasma parameters has the advantage of being of wide validity. However, it has the disadvantage of being relatively cumbersome and difficult to interpret intuitively. A much simpler and more intuitive method is developed in the Appendix. This method involves including the fluctuations through local variations in the plasma parameters on which the linear response tensor is functionally dependent. In the case of a cold, unmagnetised, electronic plasma, the only such parameter is the electron number density. It is of interest to compare the results obtained using these two methods. Specifically, we compare (28) and (A8). In making the comparison the static limit ($\omega_0 = 0$) is made in (28), and in (A8) the function $\langle(\delta n_e)^2\rangle(\boldsymbol{k})/n_e^2$ is related to the function $\langle(\delta n_e)^2\rangle(k)/n_e^2$ in (28) by integrating the latter over $d\omega/2\pi$. In this limit the two results agree. We have attempted to generalise the procedure developed in the Appendix in various ways. It is not difficult to include the frequency of the fluctuations, corresponding to $\omega_0$ in (28). Otherwise however, when one attempts to include fluctuations with arbitrary $k'$, as in (9), the method developed in the Appendix becomes ill-defined and ambiguous. Except in the simplest case just discussed, this alternative method does not appear to reproduce the results of the more general theory developed in Section 2.

We have shown that for density fluctuations with a well-defined $\boldsymbol{k}_0$ the plasma becomes birefringent, and the 'extraordinary' mode has two resonances and two cutoffs, which are separated by two stopbands, near the frequency $\omega_L(\boldsymbol{k}_0)$ of virtual Langmuir waves with this wave vector. The appearance of these resonance may be interpreted in terms of three-wave interactions. In (28) the resonances occur at $K^{\mathrm{L}}(k \pm k') = 0$: these conditions correspond to the Manley-Rowe matching conditions being satisfied for the three-wave interaction between a low frequency wave (ion sound wave) with wave 4-vector $k'^{\mu}$ to beat with a Langmuir wave with wave 4-vector $k^{\mu} \pm k'^{\mu}$ to produce a test wave with wave 4-vector $k^{\mu}$. These three-wave interactions and the associated resonances in the refractive index may be attributed to the resistive and reactive parts of the same current. Put another way, a three-wave interaction corresponds to a resonance (expressing the Manley-Rowe matching condition), and any resonance contributes both a real and an imaginary part to the dispersion curve.

If present these resonances and cutoffs would have a major effect on the escape of radiation generated near the plasma frequency. Specifically, radiation generated below the resonance due to fundamental plasma emission (e.g., Melrose 1980, 1985, 1986*b*, 1987) in solar radio bursts encounters a resonance if it is in the 'extraordinary mode'. Although it would require a detailed calculation (including the effect of an ambient magnetic field) to study the propagation in detail, one prediction can be made from qualitative arguments: one would expect any escaping radiation to be predominantly in the 'ordinary' mode, which is linearly polarised in the approximations made here. This would lead to a reduction in the degree of circular polarisation, leading to a possible resolution of a long-standing difficulty in the interpretation of the polarisation of fundamental plasma emission, e.g., Melrose (1987, 1989).

However, these resonances are well defined only under quite restrictive conditions: (i) the damping of the density fluctuations and of the virtual Langmuir waves must be sufficiently weak, and (ii) the spread $\Delta k$ in the wave vector of the density fluctuations must be sufficiently small. The actual limits imposed by these requirements are summarised by condition (32). The most restrictive of these requirements is that the damping of the density fluctuations be sufficiently small. The density fluctuations are subject to Landau damping, which cannot be avoided in an unmagnetised plasma. Two cases where this effect is weak might be mentioned. One favourable case is for density fluctuations associated with lower hybrid waves in a magnetised plasma. Lower hybrid waves propagate nearly perpendicular to the magnetic field, where Landau damping is relatively weak. Another case where the damping of the density fluctuations may be unimportant is when they are driven, e.g., in a four-wave mixing, or in any other driven wave-wave interactions. In a steady state, Landau damping of the density fluctuations associated with low frequency waves is balanced by their growth due to the wave-wave interaction. Even if the damping of the density fluctuations is unimportant, the other conditions for the resonances to be observable remain severe: relatively large amplitude, highly structured (well-defined $\boldsymbol{k}_0$) ripples are required.

## 6. Conclusions

The effect of density fluctuations on the scattering of high frequency waves and in causing wave-wave interactions in plasmas is well known, but the associated effect on the dispersive properties of waves has been ignored in the past. Here we have developed a systematic theory for treating the effects of any fluctuations in the background plasma of the properties of high frequency waves. The procedure adopted is based on nonlinear plasma theory, and although there is an alternative simpler procedure, discussed in the Appendix, it appears to be of very restricted validity.

The most notable feature of the applications discussed here is that ripples in the form of anisotropic density fluctuations cause an otherwise isotropic plasma to become birefringent, and can lead to the appearance of resonances, cutoffs and stopbands near the frequency of a virtual Langmuir wave with wave vector equal to that of the ripples. These effects may have profound consequences on the propagation of radiation near the plasma frequency. However, the conditions under which the resonances and cutoffs are observable are quite restrictive. Two favourable cases for observing these effects is for density fluctuations (associated with lower hybrid waves) aligned perpendicular to an ambient magnetic field, and for ripples driven by wave-wave interactions as in degenerate four-wave mixing.

The investigation reported here is part of an ongoing study of nonlinear effects in plasmas which is currently directed towards including magnetic effects in the Zakharov equations and in the theory of phase conjugation by four-wave mixing in collisionless plasmas.

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## Appendix: Ensemble Average of the Wave Equation

In an alternative method for including the effect of the fluctuations one assumes that there are fluctuations in the linear response tensor due to fluctuations in those plasma parameters on which it has a functional dependence. This is a standard method for treating classical versions of Rayleigh and

Raman scattering, e.g., Landau and Lifshitz (1960, p. 387), Herzberg (1945, p. 239). The method is well defined only if the Fourier components of the test field and of the fluctuations are well separated in both $\omega$ and $\boldsymbol{k}$; to a first approximation, the frequency of the fluctuations needs to be negligible in comparison with that of the test field, and the wave number of the test field needs to be negligible in comparison with that of the fluctuations. In view of these different approximations to $\omega$ and $\boldsymbol{k}$, a 3-tensor description is more convenient than the 4-tensor description used in the rest of this paper. The treatment outlined here is a minor extension of a method due originally to Rax (1986, personal communication).

In the absence of fluctuations, the medium is assumed not to be spatially dispersive so that its response 3-tensor $\alpha_{ij}(\omega)$ does not depend on $\boldsymbol{k}$. The wave equation, after Fourier transforming in time but not in space, is

$$\left\{\delta_{ij}\partial^2 - \partial_i\partial_j + (\omega^2/c^2)\delta_{ij} + \mu_0\alpha_{ij}(\omega)\right\}A_j(\omega,\boldsymbol{x}) = 0. \tag{A1}$$

Fluctuations $\delta Q_A(\boldsymbol{x})$, $\delta Q_B(\boldsymbol{x})$, ... in plasma parameters $Q_A$, $Q_B$, ... on which $\alpha_{ij}(\omega)$ has a functional dependence are introduced by making the replacement

$$\alpha_{ij}(\omega) \to \left[1 + \sum_A \delta Q_A(\boldsymbol{x})\frac{\partial}{\partial Q_A} + \frac{1}{2}\sum_{AB}\delta Q_A(\boldsymbol{x})\delta Q_B(\boldsymbol{x})\frac{\partial^2}{\partial Q_A\partial Q_B} + \cdots\right]\alpha_{ij}(\omega). \tag{A2}$$

The test field $\boldsymbol{A}(\omega,\boldsymbol{x})$ is replaced according to

$$\boldsymbol{A}(\omega,\boldsymbol{x}) \to \boldsymbol{A}^{(0)}(\omega) + \boldsymbol{A}^{(1)}(\omega,\boldsymbol{x}) + \boldsymbol{A}^{(2)}(\omega,\boldsymbol{x}) + \cdots, \tag{A3}$$

where the superscript denotes the order in a perturbation expansion in the fluctuations. The spatial dependence of $\boldsymbol{A}^{(0)}(\omega)$ in (A3) is ignored; this corresponds to making the dipole approximation in the classical treatments cited above for Rayleigh and Raman scatterings. After making the replacements (A2) and (A3) in (A1), an ensemble average (denoted by angle brackets) in performed. The ensemble average of the first order quantities is assumed to be zero, and the difference between a second order term and its ensemble average is assumed to be of higher order. On subtracting the ensemble average from the unaveraged equation, to first order one obtains

$$\left\{\delta_{ij}\partial^2 - \partial_i\partial_j + (\omega^2/c^2)\delta_{ij} + \mu_0\alpha_{ij}(\omega)\right\}A_j^{(1)}(\omega,\boldsymbol{x}) + \mu_0\sum_A \delta Q_A(\boldsymbol{x})\frac{\partial\alpha_{ij}(\omega)}{\partial Q_A}A_j^{(0)}(\omega) = 0. \tag{A4}$$

After Fourier transforming, (A4) may be solved for

$$A_i^{(1)}(\omega,\boldsymbol{x}) = -\frac{\mu_0 c^2}{\omega^2}\int\frac{d^3\boldsymbol{x}'d^3\boldsymbol{k}}{(2\pi)^3}\sum_A \delta Q_A(\boldsymbol{x}')\frac{\partial\alpha_{ir}(\omega)}{\partial Q_A}$$

$$\left[\frac{\kappa_r\kappa_s}{K^L(\omega,|\boldsymbol{k}|)} + \frac{\delta_{rs}-\kappa_r\kappa_s}{K^T(\omega,|\boldsymbol{k}|) - |\boldsymbol{k}|^2c^2/\omega^2}\right]A_s^{(0)}(\omega), \tag{A5}$$

where the $\boldsymbol{\kappa}$ is a unit vector along $\boldsymbol{k}$. The averaged equation includes a term that involves the product of the first order terms from (A2) and (A3), and (A5) is used to evaluate this product in terms of the correlation function for the fluctuations:

$$W_{AB}(\boldsymbol{x}-\boldsymbol{x}') := \langle \delta Q_A(\boldsymbol{x}) \delta Q_B(\boldsymbol{x}') \rangle . \tag{A6}$$

As a result of these calculations, the correction to the response tensor is found to be

$$\delta\alpha_{ij}(\omega) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \sum_{AB} W_{AB}(\boldsymbol{k}) \left[ \frac{1}{2} \frac{\partial^2 \alpha_{ij}(\omega)}{\partial Q_A \partial Q_B} \right.$$

$$\left. - \frac{\mu_0 c^2}{\omega^2} \frac{\partial \alpha_{ir}(\omega)}{\partial Q_A} \left[ \frac{\kappa_r \kappa_s}{K^L(\omega, |\boldsymbol{k}|)} + \frac{\delta_{rs} - \kappa_r \kappa_s}{K^T(\omega, |\boldsymbol{k}|) - |\boldsymbol{k}|^2 c^2/\omega^2} \right] \frac{\partial \alpha_{sj}(\omega)}{\partial Q_B} \right] . \tag{A7}$$

The simplest case is where (i) only density fluctuations are considered, (ii) the cold plasma approximation is made in the form $\alpha_{ij}(\omega) = (e^2 n_e/m_e)\delta_{ij}$, and (iii) the term involving $K^T$ in (A7) is neglected. Then (A7) reduces to

$$\delta\alpha_{ij}(\omega) = -\frac{\mu_0 c^2 e^4 n_e^2}{m_e^2 \omega^2} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{\langle (\delta n_e)^2 \rangle (\boldsymbol{k})}{n_e^2} \frac{\kappa_i \kappa_j}{K^L(\omega, |\boldsymbol{k}|)} . \tag{A8}$$

The correction to the dielectric tensor is given by $\delta K_{ij}(k) = (\mu_0/\omega^2)\delta\alpha_{ij}(\omega)$.

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