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Relativistic Quantum Response of a Strongly Magnetised Plasma. I. Mildly Relativistic Electron Gas

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Abstract

Approximate analytic expressions are derived for the linear response 4-tensor of a strongly magnestised, mildly relativistic electron plasma. The results are obtained within the framework of quantum plasma dynamics, thus the response contains relativistic and quantum effects that are essential in a super-strong magnetic field. The response is obtained in terms of relativistic plasma dispersion functions known as Shkarofsky functions. These functions allow the wave properties of the plasma to be studied without resorting to complicated numerical schemes. The response derived is valid for radiation with frequency up to about the cyclotron frequency and is of use in the theory of spectra formation in X-ray pulsars. In addition, a simple graphical technique is introduced that allows one to visually locate the roots of the resonant denominator occurring in the response, as well as determine the conditions under which both roots are valid and contribute to absorption.

1. Introduction

The discovery of neutron stars possessing enormous magnetic fields, with typically $B \gtrsim 10^8$ T, has stimulated great interest in understanding physical processes in super-strong magnetic fields and the effects they have on spectrum formulation, especially as it was realised that such fields can drastically alter physical processes. This can manifest itself in two ways: (i) Processes which already exist in the absence of a magnetic field and which are modified by the presence of a field. For example, the Compton scattering cross section exhibits strong anisotropy and polarisation dependence as well as having resonances near the cyclotron harmonics. (ii) Processes which are kinetically forbidden in the absence of a magnetic field become allowed in such fields, to the extent that they have measurable effects. One example of this is the quantum-electrodynamical process of single photon pair production, which is considered paramount for generating the pulses in radio pulsars (Toll 1952; Erber 1966; Ruderman and Sutherland 1975; Daugherty and Harding 1983).

Super-strong magnetic fields, aside from introducing purely quantum processes such as single photon pair production and photon splitting, also produce quantum modifications to familiar astrophysical processes. The reason that quantum effects are of importance is due to the quantisation (into 'Landau levels') of the electron motion perpendicular to the field. If the energy spacing of the Landau levels is greater than the kinetic energy of the electron, the quantisation of the electron

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orbit is important. Most electrons occupy the lowest few Landau levels, and the plasma becomes one-dimensional. It is expected that the plasma in X-ray pulsar accretion columns and radio pulsars is one-dimensional.

The main objective of the work reported here is the derivation of analytic results for the linear response tensor of a strongly magnetised electron plasma. The response is evaluated for a mildly relativistic electron gas. Application to an ultrarelativistic electron–positron pair plasma is discussed in an accompanying paper (Padden 1992; present issue p. 165). Knowledge of the response tensor of the plasma is essential to the study of the dispersion and absorption of waves in a plasma. Given the response tensor, one can calculate all the properties of the natural wave modes; in particular the dispersion relation (e.g. the refractive indices), the absorption coefficient and the polarisation vectors. This work is intended to provide the basis for the investigation of the propagation and absorption of X-ray radiation in super-strong magnetic fields. An important aim of the work is to obtain expressions which permit straightforward computation of the wave propertise of the plasma, particularly in the vicinity of the cyclotron frequency. It is hoped that the results obtained will allow a clearer picture of the physics to emerge without being obscured in numerical details. The primary motivation for this work comes from application to problems associated with the formulation of spectra in the accretion columns of X-ray pulsars. There is an extensive literature on the theory and observations of X-ray pulsars, e.g. see the reviews of Börner (1980), Meszaros (1984) and Hayakawa (1985).

Quantum effects on the dispersion of waves close to the cyclotron frequency in strongly magnetised (i.e. $B \geq 10^8$ T) plasmas has been investigated by Canuto and Ventura (1977), Kirk (1980) and Pavlov *et al.* (1980). These authors employed a nonrelativistic approximation to the resonant denominators appearing in the response. The use of a nonrelativistic approach is generally justified by noting that, for magnetic field strengths of interest, the cyclotron energy is typically one order of magnitude smaller than the electron rest mass energy, i.e. $\Omega_e/m \simeq 0{\cdot}1$. However, the use of nonrelativistic results close to the cyclotron frequency can lead to incorrect results. This was first shown in the case of a classical, magnetised plasma by Wu and Lee (1979). This situation also occurs with the resonance condition that arises in a quantum plasma. Herold *et al.* (1981) investigated photon propagation in a strongly magnetised plasma within the framework of relativistic quantum theory, using parameters appropriate to X-ray pulsars. By keeping relativistic effects in the resonant denominators of the response, significant differences were found from the nonrelativistic treatment of Kirk (1980), particularly for quasi-perpendicular propagation. The work of Herold *et al.* (1981) was extended by Kirk and Cramer (1985) who numerically integrated the susceptibility, retaining the exact resonant denominators.

In this paper, the full 4-tensor expression for the linear response is derived. The previous analytical treatments of a magnetised quantum plasma by Canuto and Ventura (1977), Kirk (1980) and Pavlov *et al.* (1980) have only obtained results for the 3-tensor. It should be noted that the use of a covariant formulation is not necessary for the inclusion of relativistic effects in the response. However, using a covariant theory allows one to make a Lorentz transformation to a new frame. This greatly facilitates the inclusion of a streaming motion for any particle species in the plasma.

The layout of the paper is as follows. In Section 2, general expressions for the components of the linear response 4-tensor of a magnetised plasma are presented. These results are based on the formalism of quantum plasmadynamics (QPD): a synthesis of the theory of quantum electrodynamics and classical plasmadynamics. Quantum plasmadynamics reduces to the theory of a collisionless plasma in the classical limit and to conventional quantum electrodynamics *in vacuo*. It is a fully covariant and gauge invariant theory which has been developed in a series of papers by Melrose (1974, 1983) and Melrose and Parle (1983*a*, 1983*b*). In Section 3, a nonrelativistic Maxwellian distribution function is introduced for the electrons. The resonant denominator is treated in Section 4 using a semirelativistic approximation that preserves the essential relativistic effects required near the cyclotron frequency. In Section 4*a*, a simple graphical technique is introduced for locating the roots of the resonance condition. The relativistic plasma dispersion functions of Shkarofsky (1966; see also Robinson 1986) are introduced in Section 5 which allow the evaluation of the momentum integrals appearing in the expressions for the linear response tensor. In Section 6, specific results are given for the components of the linear response 4-tensor that are valid in the case of quasi-perpendicular propagation (i.e. for angles not too far from 90°). The behaviour of the antihermitian part of the response is examined for frequencies close to the cutoff frequency for cyclotron absorption in Section 6*a*. In Section 6*b*, results for exact perpendicular propagation are written down as a special case, as they are considerably simpler than the more general results. Finally in Appendix B, a 4-tensor generalisation of the results given in Section 3 of Pavlov *et al.* (1980) for the polarisability of a magnetised electron gas is presented. It is believed that the work reported here in this paper is the first time that the complete 4-tensor has been written down for a strongly magnetised quantum plasma.

The notation used in this paper follows that of Berestetskii *et al.* (1982). The 3-vectors are written in bold script, while 4-vectors are in normal typeface. Unless otherwise stated, natural units $\hbar = 1$, $c = 1$ are used in all formulas, which are based on SI units. In natural units the fine structure constant is given by $\alpha = e^2/4\pi\epsilon_0$. The metric tensor adopted here is given by $g^{\mu\nu} = \mathrm{diag}(1,\ -1,\ -1,\ -1)$.

2. Linear Response Tensor for a Strongly Magnetised Electron Gas

In this section expressions are presented for the components of the linear response 4-tensor of a magnetised plasma. As a starting point, the general expression for the linear response tensor of a magnetised medium as given by equation (34) of Melrose and Parle (1983*b*) is employed, viz.

$$\alpha^{\mu\nu}(k) = -\frac{e^3 B}{2\pi} \sum_{Q,Q'} \int \frac{dp_{||}}{2\pi} \, \frac{\{\frac{1}{2}(\epsilon'-\epsilon) + \epsilon N_q^{\epsilon} - \epsilon' N_{q'}^{\epsilon'}\}}{\omega - \epsilon\epsilon_q + \epsilon'\epsilon_{q'} + i0} \, [\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{K})]^{\mu} [\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{K})]^{*\nu}, \quad (1)$$

where Q denotes the quantum numbers ϵ, n, σ with ϵ labelling the sign of the energy, $\epsilon = 1$ for electrons, $\epsilon = -1$ for positrons, n labels the Landau levels and σ is the spin quantum number ($\sigma = \pm 1$), also q denotes $n, p_{||}, \sigma$. The energy eigenvalues $\epsilon_q, \epsilon_{q'}$ are given by

$$\epsilon(n, p_{||}) = [m^2 + p_{||}^2 + 2n\Omega m]^{\frac{1}{2}} . \quad (2)$$

Equation (1) corrects a minor error in the expression given by Melrose and Parle (1983*b*), who have interchanged the order of quantum numbers in the second of the gauge-independent vertex factors $[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^{*\nu}$. The infinitesimal imaginary part $i0$ in the resonant denominator arises from imposing the casual condition on the response tensor. The terms proportional to $\frac{1}{2}(\epsilon'-\epsilon)$ in equation (1) give the vacuum polarisation tensor. A large literature exists on vacuum polarisation in a magnetic field and, in particular, it has been extensively studied by Melrose and Stoneham (1976, 1977) and Stoneham (1978) amongst others. As such, only the contribution from the plasma to the linear response is examined in this work. This contribution arises from the terms proportional to $\epsilon N^{\epsilon}_{q} - \epsilon' N^{\epsilon'}_{q'}$, where N^{ϵ}_{q} represents the particle distribution function.

The gauge-independent vertex factors in equation (1) depend explicitly on the choice of spin eigenfunction. Spin-summed results for the linear response are presented here, so that the results are independent of the spin eigenfunctions. Before the sum over the spins in equation (1) is performed, the sum over ϵ and ϵ' is first evaluated. One obtains

$$\begin{aligned}\alpha^{\mu\nu}(k) = -\frac{e^3B}{2\pi}\sum_{n,n'}\int\frac{dp_{||}}{2\pi}\Bigg\{&\frac{N^{+}_{q}-N^{+}_{q'}}{\omega-\epsilon_q+\epsilon_{q'}+i0}\sum_{\sigma,\sigma'}[\Gamma^{++}_{q'q}(\mathbf{k})]^{\mu}[\Gamma^{++}_{q'q}(\mathbf{k})]^{*\nu}\\ &+\frac{N^{+}_{q}+N^{-}_{q'}}{\omega-\epsilon_q-\epsilon_{q'}+i0}\sum_{\sigma,\sigma'}[\Gamma^{-+}_{q'q}(\mathbf{k})]^{\mu}[\Gamma^{-+}_{q'q}(\mathbf{k})]^{*\nu}\\ &-\frac{N^{-}_{q}+N^{+}_{q'}}{\omega+\epsilon_q+\epsilon_{q'}+i0}\sum_{\sigma,\sigma'}[\Gamma^{+-}_{q'q+}(\mathbf{k})]^{\mu}[\Gamma^{+-}_{q'q}(\mathbf{k})]^{*\nu}\\ &-\frac{N^{-}_{q}+N^{+}_{q'}}{\omega+\epsilon_q-\epsilon_{q'}+i0}\sum_{\sigma,\sigma'}[\Gamma^{--}_{q'q+}(\mathbf{k})]^{\mu}[\Gamma^{--}_{q'q}(\mathbf{k})]^{*\nu}\Bigg\}\end{aligned} \tag{3}$$

In writing down equation (3) it is assumed that the distribution functions are independent of the spin of the particles. The sums over the spin quantum number are performed using equations (18) and (50) of Melrose and Parle (1983*a*) and Johnson and Lippmann (1949) eigenfunctions, as these are the simplest spin eigenfunctions. Also the magnetic field is chosen to lie along the z-axis as well as a gauge in which the photon wavevector lies in the xz plane, i.e.

$$\mathbf{k} = (k_x,\ 0,\ k_z) = (k_\perp,\ 0,\ k_{||}).$$

Although there are 16 terms in the linear response tensor, these are not all independent, in the sense that one may employ the Onsager relations to show that only 10 terms need be evaluated; the 4 diagonal terms α^{00}, α^{11}, α^{22}, α^{33} and 6 off-diagonal terms α^{01}, α^{02}, α^{03}, α^{13}, α^{23}. The other 6 terms may be calculated using the Onsager relations, which imply

$$\alpha^{\mu\nu}(\omega,\ \mathbf{k};\ \mathbf{B}) = \alpha^{\nu\mu}(\omega,\ -\mathbf{k};-\mathbf{B});$$

these relations are a result of time-reversal invariance.

On performing the spin sums, it is found that the linear response tensor may be expressed as follows:

$$\alpha^{\mu\nu}(k) = -\frac{e^3B}{2\pi}\sum_{n=0,n'=0}^{\infty}\int\frac{dp_{||}}{2\pi}\left\{\frac{N_q^+ - N_{q'}^+}{\omega-\epsilon_q+\epsilon_{q'}+i0}Q_+^{\mu\nu}(n',n)\right.$$
$$+\frac{N_q^+ + N_{q'}^-}{\omega-\epsilon_q-\epsilon_{q'}+i0}Q_-^{\mu\nu}(n',n) - \frac{N_{q'}^+ + N_q}{\omega+\epsilon_q+\epsilon_{q'}+i0}Q_-^{\mu\nu}(n',n)$$
$$\left.-\frac{N_q^- - N_{q'}^-}{\omega+\epsilon_q-\epsilon_{q'}+i0}\eta(\mu,\nu)Q_+^{\mu\nu}(n',n)\right\}. \tag{4}$$

In (4), $\eta(\mu,\nu)$ is defined by

$$\eta(\mu,\nu) = \begin{cases} +1 & \mu\nu = 00,\ 11,\ 22,\ 33,\ 03,\ 12, \\ -1 & \mu\nu = 01,\ 02,\ 13,\ 23, \end{cases} \tag{5}$$

while the $Q_\pm^{\mu\nu}(n',n)$ are given by

$$Q_\pm^{00}(n',n) = \tfrac{1}{2}\left(1 \pm \frac{m^2 \pm p_{||}'p_{||}}{\epsilon_{q'}\epsilon_{q'}}\right)[(J_{n'-n}^{n-1})^2 + (J_{n'-n}^{n})^2]$$
$$\pm\frac{p_{n'}p_n}{\epsilon_{q'}\epsilon_{q'}}J_{n'-n}^{n-1}J_{n'-n}^{n}\,, \tag{6}$$

$$Q_\pm^{11}(n',n) = \tfrac{1}{2}\left(1 \mp \frac{m^2 \pm p_{||}'p_{||}}{\epsilon q'}\epsilon q'\right)[(J_{n'-n-1}^{n})^2 + (J_{n'-n+1}^{n-1})^2]$$
$$\pm\frac{p_{n'}p_n}{\epsilon_{q'}\epsilon_{q'}}J_{n'-n-1}^{n}J_{n'-n+1}^{n-1}\,, \tag{7}$$

$$Q_\pm^{22}(n',n) = Q_\pm^{11}(n',n) \mp 2\frac{p_{n'}p_n}{\epsilon_{q'}\epsilon_{q'}}J_{n'-n-1}^{n}J_{n'-n+1}^{n-1}\,, \tag{8}$$

$$Q_\pm^{33}(n',n) = \tfrac{1}{2}\left(1 \mp \frac{m^2 \mp p_{||}'p_{||}}{\epsilon_{q'}\epsilon_{q'}}\right)[(J_{n'-n}^{n-1})^2 + (J_{n'-n}^{n})^2]$$
$$\mp\frac{p_{n'}p_n}{\epsilon_{q'}\epsilon_{q'}}J_{n'-n}^{n-1}J_{n'-n}^{n}\,, \tag{9}$$

$$Q_\pm^{01}(n',n) = -\frac{p_n}{2\epsilon_q}[(J_{n'-n}^{n-1}J_{n'-n-1}^{n}) + (J_{n'-n}^{n}J_{n'-n+1}^{n-1})]$$
$$\mp\frac{p_{n'}}{2\epsilon_{q'}}[(J_{n'-n}^{n-1}J_{n'-n+1}^{n-1}) + (J_{n'-n}^{n}J_{n'-n-1}^{n})]\,, \tag{10}$$

$$Q_\pm^{02}(n',n) = i\frac{p_n}{2\epsilon_q}[(J_{n'-n}^{n}J_{n'-n+1}^{n-1}) - (J_{n'-n}^{n-1}J_{n'-n-1}^{n})]$$
$$\pm i\frac{p_{n'}}{2\epsilon_{q'}}[(J_{n'-n}^{n-1}J_{n'-n+1}^{n-1}) - (J_{n'-n}^{n}J_{n'-n-1}^{n})]\,, \tag{11}$$

$$Q^{03}_{\pm}(n',n) = \left(\frac{p'_{||}}{2\epsilon_{q'}} + \frac{p_{||}}{2\epsilon_q}\right)[(J^{n-1}_{n'-n})^2 + (J^{n}_{n'-n})^2], \tag{12}$$

$$Q^{12}_{\pm}(n',n) = i\tfrac{1}{2}\left(1 \mp \frac{m^2 \pm p'_{||}p_{||}}{\epsilon_{q'}\epsilon_{q'}}\right)[(J^{n}_{n'-n-1})^2] \tag{13}$$

$$\begin{aligned} Q^{13}_{\pm}(n',n) &= -\frac{p'_{||}p_n}{2\epsilon_{q'}\epsilon_q}[(J^{n}_{n'-n-1}J^{n-1}_{n'-n}) + (J^{n-1}_{n'-n+1}J_{n'-n})] \\ &\quad \mp \frac{p_{||}p_{n'}}{2\epsilon_{q'}\epsilon_q}[(J^{n}_{n'-n-1}J^{n}_{n'-n}) + (J^{n-1}_{n'-n+1}J^{n-1}_{n'-n})], \end{aligned} \tag{14}$$

$$\begin{aligned} Q^{23}_{\pm}(n',n) &= i\frac{p'_{||}p_n}{2\epsilon_{q'}\epsilon_q}[(J^{n}_{n'-n-1}J^{n-1}_{n'-n}) - (J^{n-1}_{n'-n+1}J^{n}_{n'-n})] \\ &\quad \pm \frac{p_{||}p_{n'}}{2\epsilon_{q'}\epsilon_q}[(J^{n}_{n'-n-1}J^{n}_{n'-n}) - (J^{n-1}_{n'-n+1}J^{n-1}_{n'-n})], \end{aligned} \tag{15}$$

where $p_n = (2neB)^{\frac{1}{2}}$. The argument of the $J^n_{n'}(k_x^2/2eB)$ functions is omitted in the above expressions. These functions are related to the generalised Laguerre polynomials $L^{n'}_n(u)$ defined in Abramowitz and Stegun (1965), via

$$J^n_{n'}(u) = \left[\frac{n!}{(n+n')!}\right]^{\frac{1}{2}} \exp(-\tfrac{1}{2}u)u^{n'/2}L^{n'}_n(u). \tag{16}$$

In the literature one may come across functions denoted by either $I_{n',n}(u)$ or $F_{n,n'}(u)$; these are related to the $J^n_{n'}(u)$ by

$$I_{n,n'}(u) = J^{n'}_{n-n'}(u).$$

Some of the more important properties of the $J^n_{n'}(u)$ functions are given in Melrose and Parle (1983*a*).

Terms proportional to N^+_q in equation (4) are due to the electron plasma component, while those proportional to N^-_q are due to the positron plasma component. In this paper only a pure electron plasma is considered, so that the N^-_q is set to zero. As such, equation (4) is replaced by

$$\begin{aligned} \alpha^{\mu\nu}(k) = -\frac{e^2 m\Omega_e}{4\pi^2}\sum_{n=0,n'=0}^{\infty}\int dp_{||}\Bigg\{&\frac{f(\epsilon_q)-f(\epsilon_{q'})}{\omega-\epsilon_q+\epsilon_{q'}+i0}Q^{\mu\nu}_{+}(n',n) \\ &+\frac{f(\epsilon_q)Q^{\mu\nu}_{-}(n',n)}{\omega-\epsilon_q-\epsilon_{q'}+i0} - \frac{f(\epsilon_{q'})Q^{\mu\nu}_{-}(n',n)}{\omega+\epsilon_q+\epsilon_{q'}+i0}\Bigg\}, \end{aligned} \tag{17}$$

where Ω_e denotes the electron cyclotron frequency and the electron distribution function is now denoted by $f(\epsilon_q)$. The normalisation of the distribution function is given by

$$\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} dp_{||}g_n f(\epsilon_q) = \frac{4\pi^2 N_e}{\epsilon_0 m\Omega_e}, \tag{18}$$

where $g_n = 2 - \delta_{n0}$ is the degeneracy factor for the Landau levels and N_e is the number density of electrons.

As equation (17) stands at the moment, one is to interpret the primed quantities as representing final states with momentum $p'_{||} = p_{||} - k_{||}$. This is the correct interpretation if one is interested in cyclotron emission. If, on the other hand, one is interested in cyclotron absorption, then primed quantities should be reinterpreted as initial states with momentum denoted by $p_{||}$ and unprimed quantities as final states with momentum $p'_{||} = p_{||} + k_{||}$. Thus on interchanging primed and umprimed quantities equation (17) is now written

$$\alpha^{\mu\nu}(k) = -\frac{e^2 m\Omega_e}{4\pi^2} \sum_{n=0,n'=0}^{\infty} \int dp_{||} \left\{ \frac{f(\epsilon_{q'}) - f(\epsilon_q)}{\omega - \epsilon_{q'} + \epsilon_q + i0} Q_+^{\mu\nu}(n', n) \right.$$
$$\left. + \frac{f(\epsilon_{q'}) Q_-^{\mu\nu}(n', n)}{\omega - \epsilon_{q'} - \epsilon_q + i0} - \frac{f(\epsilon_q) Q_-^{\mu\nu}(n', n)}{\omega + \epsilon_{q'} + \epsilon_q + i0} \right\}, \tag{19}$$

with $p'_{||} = p_{||} + k_{||}$ now implicit.

In order to see that equation (19) reproduces the results of Pavlov *et al.* (1980) and Kirk and Cramer (1985), note that the 3-tensor component of a 4-tensor $\beta^{\mu\nu}(k)$ is defined by

$$\beta_{ij}(k) = [\beta^{\mu}_{\nu}(k)]_{\mu=i,\nu=j} = -[\beta^{\mu\nu}(k)]_{\mu=i,\nu=j} \,. \tag{20}$$

Thus if one examines the $\mu = i, \nu = j$ components of equation (19) and used equation (20), the results of Pavlov *et al.* (1980) and Kirk and Cramer (1985) are reproduced (note that the susceptibility is related to the linear response by $\chi_{ij} = \alpha_{ij}/\omega^2$).

So far, the only assumption that has been made in arriving at equation (19) is that the distribution function does not depend on the spin of the electron. At this point a major simplification is introduced into the linear response tensor. It is assumed that the initial distribution of electrons is restricted to the ground state Landau level $n = 0$, there being no initial distribution of particles in excited states. This is usually justified as follows: The quantum theory of cyclotron emission shows that the decay rate or inverse lifetime of the first excited state given by

$$\Gamma_1 \simeq \tfrac{4}{3}\alpha\Omega_e \frac{B}{B_{cr}} \simeq 4\times 10^{15} B_8^2 \ \mathrm{s}^{-1} \,, \tag{21}$$

where B_8 is the field in units of 10^8 T. Thus for the fields thought typical of X-ray pulsars, $B \simeq (2-6)\times 10^8$ T, it is seen that the electrons radiate away any transverse momentum they acquire, on a timescale of order 10^{-17} s. For mildly relativistic temperatures, this timescale is significantly shorter than the average time between collisions in a plasma even as dense as 10^{32} m^{-3}, which is much higher than the plasma density of X-ray pulsar accretion columns. Thus one expects most electrons to be in the ground state. It must be pointed out, however, that this argument fails to take account of the possibility that if the mean cyclotron photon occupation number is comparable with unity, then a significant population of electrons in the first excited state can exist. A rough estimate of the importance of this effect can be obtained by estimating the brightness temperature at the cyclotron frequency and comparing it with $\hbar\Omega_e$. This possibility is ignored here as the aim of this paper to derive the simplest results of general applicability rather than examine a specific X-ray pulsar. If there is a significant number of electrons in the first excited state, it is a relatively straightforward task to include this effect in the results. Qualitatively it would have the effect of reducing the magnitude of the response tensor (see equation 19) due to the reduced phase space for particles in the ground state if they make a transition to the first excited state.

As a further simplification, only absorption of radiation from the ground state to the first excited state $n' = 1$ is considered. Under these new assumptions one has

$$Q^{00}_{\pm}(1,0) = \tfrac{1}{2}\left(1 \pm \frac{m^2 \pm p'_{||}p_{||}}{\epsilon_{q'}\epsilon_{q'}}\right)(J^0_1)^2, \tag{22}$$

$$Q^{11}_{\pm}(1,0) = \tfrac{1}{2}\left(1 \mp \frac{m^2 \pm p'_{||}p_{||}}{\epsilon_{q'}\epsilon_{q'}}\right)(J^0_0)^2, \tag{23}$$

$$Q^{22}_{\pm}(1,0) = Q^{11}_{\pm}(1,0), \tag{24}$$

$$Q^{33}_{\pm}(1,0) = \tfrac{1}{2}\left(1 \mp \frac{m^2 \mp p'_{||}p_{||}}{\epsilon_{q'}\epsilon_{q'}}\right)(J^0_1)^2, \tag{25}$$

$$Q^{01}_{\pm}(1,0) = \mp\frac{(2eB)^{\frac{1}{2}}}{2\epsilon_{q'}}(J^0_1 J^0_0), \tag{26}$$

$$Q^{02}_{\pm}(1,0) = iQ^{01}_{\pm}(1,0), \tag{27}$$

$$Q^{03}_{\pm}(1,0) = \left(\frac{p'_{||}}{2\epsilon_{q'}} + \frac{p_{||}}{2\epsilon_q}\right)(J^0_1)^2, \tag{28}$$

$$Q^{12}_{\pm}(1,0) = iQ^{11}_{\pm}(1,0), \tag{29}$$

$$Q^{13}_{\pm}(1,0) = \mp\frac{p_{||}(2eB)^{\frac{1}{2}}}{2\epsilon_{q'}\epsilon_q}(J^0_1 J^0_0), \tag{30}$$

$$Q^{23}_{\pm}(1,0) = -Q^{13}_{\pm}(1,0), \tag{31}$$

with

$$J^0_0(u) = \exp(-\tfrac{1}{2}u), \quad J^0_1(u) = u^{\frac{1}{2}}\exp(-\tfrac{1}{2}u). \tag{32}$$

Consider now the energy denominators appearing in equation (19) in more detail. There are three distinct denominators appearing on the right hand side of (19), viz.

$$\omega + \epsilon_q - \epsilon_{q'}, \quad \omega - \epsilon_q - \epsilon_{q'}, \quad \omega + \epsilon_q + \epsilon_{q'}.$$

The vanishing of the first denominator corresponds to the process of cyclotron absorption. Provided magnetic fields are considered such that $\Omega_e \ll m$, which is a good approximation for X-ray pulsars and one is only interested in the response of the plasma to frequencies $\omega \lesssim \Omega_e$, then it can be seen that the remaining two denominators do not vanish since $\epsilon_q + \epsilon_{q'} \geq 2m \gg \omega$. Consequently, the second and third terms on the right hand side of equation (19) are nonresonant terms for cyclotron absorption and are insenstitive to finite temperature effects. Therefore, in these terms one may set

$$f(\epsilon_q) \simeq \frac{4\pi^2 N_e}{\epsilon_0 m \Omega_e}\delta(p_{||})\delta_{n0}, \quad f(\epsilon_{q'}) = 0. \tag{33}$$

Using (33) in the nonresonant terms of equation (19), it can be shown, with the aid of the sum rules (Sokolov and Ternov 1968)

$$\sum_{s'=0}^{\infty} J^s_{s'-s}(u) J^s_{s'-s''}(u) \;=\; \delta_{ss''}\,, \tag{34}$$

$$\sum_{s'=0}^{\infty} (s'-s)[J^{s'}_{s-s'}(u)]^2 \;=\; u\,, \tag{35}$$

that the contribution of these nonresonant terms to the linear response 4-tensor is

$$\alpha^{\mu\nu}_{nr}(k) = \tfrac{1}{2}\omega_p^2 f^{\mu\nu}\,, \tag{36}$$

where $\omega_p = (e^2 N_e/\epsilon_0 m)^{\frac{1}{2}}$ is the plasma frequency and

$$f^{\mu\nu} = \begin{cases} \dfrac{\omega^2}{4m^2} & \mu = 0,\ \nu = 0 \\ -g^{ij} & \mu = i,\ \nu = j \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Hence, only the first term on the right hand side of equation (19) needs to be treated in detail.

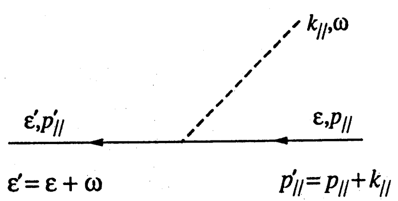

Fig. 1. Feynman diagram for cyclotron absorption.

The resonant part of the response tensor may be written

$$\alpha^{\mu\nu}_{RES}(k) = -\frac{e^2 m\Omega_e}{4\pi^2} \sum_{n=0,n'=0}^{\infty} \int dp_{||} \frac{f(\epsilon_{q'}) - f(\epsilon_q)}{\omega + \epsilon_q - \epsilon_{q'} + i0} Q^{\mu\nu}_{+}(n', n), \tag{38}$$

where the label *RES* has been attached to the response to indicate the resonant contribution. The response tensor is resonant when the denominator is zero; this is associated with the process of cyclotron absorption, which is represented by the Feynman diagram of Fig. 1. Time is ordered from right to left in the diagram. The kinematics of the related process of cyclotron emission are discussed in Melrose *et al.* (1982). It is a straightforward task to obtain corresponding results for cyclotron absorption by making the changes $s \to -s, k_{||} \to -k_{||}, \omega \to -\omega$ in the results of Melrose *et al.* (1982). One may proceed further with equations (38) by introducing functions Φ_1 and Φ_2 defined by

$$\Phi_1 = \omega + \epsilon_q - \epsilon_{q'}, \quad \Phi_2 = \omega + \epsilon_q + \epsilon_{q'}. \tag{39}$$

It can be shown that

$$\Phi_1 \Phi_2 \equiv (\epsilon_q + \omega)^2 - \epsilon^2_{q'}, = 2\gamma_{||} m \left[\omega \frac{1}{\gamma_{||}} \left(s\Omega_e - \frac{q^2}{2m} \right) - \frac{k_{||} p_{||}}{\gamma_{||} m} \right], \tag{40}$$

where $q^2 = \omega^2 - k^2_{||}$, $s = n' - n > 0$ and $\gamma_{||}$ is the Lorentz factor of the initial electron state given by

$$\gamma_{||} = \sqrt{1 + \frac{p^2}{m^2}}, \tag{41}$$

where

$$p^2 = p^2_{||} + p^2_n = p^2_{||} + 2neB. \tag{42}$$

Using equation (4) in (38) the response is now written as

$$\alpha^{\mu\nu}_{RES}(k) = -\frac{e^2 m\Omega_e}{4\pi^2} \sum_{n=0,n'=0}^{\infty} \int dp_{||} \frac{\Phi_2 [f(\epsilon_{q'}) - f(\epsilon_q)] Q^{\mu\nu}_{+}(n', n)}{2\gamma_{||} m \left[\omega - \frac{1}{\gamma_{||}} \left(s\Omega_e - \frac{q^2}{2m} \right) - \frac{k_{||} p_{||}}{\gamma_{||} m} \right]}. \tag{43}$$

3. Mildly Relativistic Particles

In this section, the 4-tensor response given in equation (43) is evaluated for a specific distribution function. At this point it is appropriate to make clear the simplifying assumptions to be used in evaluating the response, some of which have been briefly discussed in the previous section. The magnetic field is restricted to values such that $B/B_{cr} \lesssim 0{\cdot}1$, which are thought typical of X-ray pulsars. This means the cyclotron energy is small compared with the electron rest mass energy. As well, for the plasma densities thought to occur in X-ray pulsar accretion columns ($N_e \sim 10^{26}$–10^{30} m^{-3}) and for frequencies ω around the cyclotron frequency, one has $\omega_p^2/\omega^2 \ll 1$. The refractive index of the plasma differs only slightly from unity and one may restrict ω and $\mathbf{k}$ to real values with $\omega \simeq |\mathbf{k}|$. Furthermore, it is assumed that the waves satisfy $k_{||} < \omega$ and ω is taken to be positive. Also the initial distribution of electrons is taken to populate only the ground state ($n = 0$) and only absorption to the first excited Landau level ($n' = 1$) is considered. Finally, the electron temperature is assumed to be mildly relativistic. This includes temperatures in the range 1–100 keV, which are consistent with temperatures thought to exist in X-ray pulsar accretion columns 10–50 keV. By comparison, the electron rest mass corresponds to a temperature of 511 keV.

Since only mildly relativistic electrons in the ground state are considered, it is permissible to adopt a Maxwell Boltzmann distribution for motion along the magnetic field lines, i.e.

$$f(\epsilon_q) = \frac{4\pi^2 N_e}{\epsilon_0 m \Omega_e} \frac{\exp(-p_{||}^2/2mT)}{(2\pi m T)^{\frac{1}{2}}} \delta_{n0}\,, \tag{44}$$

where the Boltzmann constant is set to unity. Using the foregoing assumptions, one can write equation (43) as

$$\alpha_{RES}^{\mu\nu}(k) = \frac{\omega_p^2}{(2\pi m T)^{\frac{1}{2}}} \int dp_{||} \frac{\Phi_2 Q_+^{\mu\nu}(1,0)\exp(-p_{||}^2/2mT)}{2\gamma_{||}\left[\omega - \dfrac{1}{\gamma_{||}}\left(\Omega_e - \dfrac{q^2}{2m}\right) - \dfrac{k_{||}p_{||}}{\gamma_{||}m}\right]}\,, \tag{45}$$

with

$$\gamma_{||} = \sqrt{1 + \frac{p_{||}^2}{m^2}}\,,$$

and the $Q_+^{\mu\nu}$(1,0) are given by equations (22)–(31).

The primary interest here is to retain those relativistic effects which remain important even for mildly relativistic electrons. Thus as relativistic effects are crucial near the cyclotron frequency, $\gamma_{||}$ can be approximated by

$$\gamma_{||} \simeq 1 + \frac{p_{||}^2}{2m^2}\,, \tag{46}$$

in the resonant denominator, but is set to unity elsewhere in equation (45). The use of the so-called semirelativistic expansion given by equation (46) is valid provided that the angle of propagation θ is well away from $\theta = 0$. Kirk and Cramer (1985) have shown that for small angles of propagation the momenta of resonant electrons becomes large ($p_{||}^{res} \gg m$), thus violating the semirelativistic approximation.

Using the nonrelativistic approximation in equations (22)–(31) one finds

$$Q_+^{00}(1,0) \simeq (J_1^0)^2\,, \tag{47}$$

$$Q_{+}^{11}(1,0) \simeq \frac{\Omega_e}{2m}(J_0^0)^2\,, \tag{48}$$

$$Q_{+}^{22}(1,0) = Q_{\pm}^{11}(1,0)\,, \tag{49}$$

$$Q_{+}^{33}(1,0) \simeq \left\{\frac{\Omega_e}{2m} + \frac{p_{||}^2}{m^2}\right\}(J_1^0)^2\,, \tag{50}$$

$$Q_{+}^{01}(1,0) \simeq \frac{(2eB)^{\frac{1}{2}}}{2m}(J_1^0 J_0^0)\,, \tag{51}$$

$$Q_{+}^{02}(1,0) = iQ_{\pm}^{01}(1,0)\,, \tag{52}$$

$$Q_{+}^{03}(1,0) \simeq \left(\frac{p_{||}}{m}\right)(J_1^0)^2\,, \tag{53}$$

$$Q_{+}^{12}(1,0) = iQ_{\pm}^{11}(1,0)\,, \tag{54}$$

$$Q_{+}^{13}(1,0) \simeq -\frac{p_{||}(2eB)^{\frac{1}{2}}}{2m^2}(J_1^0 J_0^0)\,, \tag{55}$$

$$Q_{+}^{23}(1,0) = -iQ_{\pm}^{13}(1,0)\,. \tag{56}$$

In (47)–(56) terms of order $k_{||}^2/m^2$, $k_{||}p_{||}/m^2$ are neglected in comparison with terms of order $\Omega_e/2m$, $p_{||}^2/m^2$, and terms of order $k_{||}/m$ are neglected compared with terms of order $p_{||}/m$.

4. Roots of the Resonant Denominator

Consider now the approximation (46) in the resonant denominator given by equation (40); one has

$$\begin{aligned}\frac{\Phi_1\Phi_2}{2m} &\simeq \left(1+\frac{p_{||}^2}{2m^2}\right)\omega - \Omega_e + \frac{q^2}{2m} - \frac{k_{||}p_{||}}{m} \\ &= \frac{\omega p_{||}^2}{2m^2} - \frac{k_{||}p_{||}}{m} + (\omega - \Omega_e) + \frac{q^2}{2m}\,.\end{aligned} \tag{57}$$

Thus, the momenta of electrons occuring at resonance is found by solving the quadratic equation (57) for $p_{||}$. Before this is performed, recall that for a tenuous plasma one can assume $\omega \simeq |\mathbf{k}|$, so that the wavevector components are written

$$k_{||} \simeq \omega\cos\theta, \qquad k_{\perp} \simeq \omega\sin\theta\,, \tag{58}$$

and thus $q^2 \simeq \omega^2\sin^2\theta$. Using these results in equation (57), one obtains the quadratic equation

$$p_{||}^2 - 2mp_{||}\cos\theta + 2m^2(1-\Delta) + m\omega\sin^2\theta = 0\,, \tag{59}$$

with $\Delta = \Omega_e/\omega$. Equation (59) has solutions

$$p_{||} = p_{\pm} = m\,\cos\theta \pm \sqrt{m^2\,\cos^2\theta - 2m^2(1-\Delta) - m\omega\,\sin^2\theta}\,. \tag{60}$$

In order that the resonant momenta be real, the discriminant must be positive which leads to the following quadratic equation in ω:

$$\omega^2\,\sin^2\theta + m\omega(1+\sin^2\theta) - 2m\Omega_e \le 0\,. \tag{61}$$

If this is expressed in the form

$$(\omega-\omega_+)(\omega-\omega_-) \le 0\,, \tag{62}$$

where

$$\omega_{\pm} = \frac{-\frac{1}{2}m(1+\sin^2\theta) \pm m\sqrt{\frac{1}{4}(1+\sin^2\theta) + 2(\Omega_e/m)\,\sin^2\theta}}{\sin^2\theta}\,, \tag{63}$$

then (62) is satisfied for $0 < \omega < \omega_+$, or $\omega_- < \omega < 0$. The solution ω_- is a spurious one caused by introducing the function Φ_2 and is not physical for positive frequencies. Hence the solution

$$0 < \omega < \omega_+ = m\frac{\{\sqrt{\frac{1}{4}(1+\sin^2\theta) + 2(\Omega_e/m)\,\sin^2\theta} - \frac{1}{2}(1+\sin^2\theta)\}}{\sin^2\theta} \tag{64}$$

gives the conditions under which a photon can be absorbed by a plasma electron with momentum given by (60) and with ω_+ the cutoff frequency for resonant particles to exist. At frequencies greater than ω_+ there are no resonant particles, i.e. the resonant ellipse does not intersect a particle ellipse (Kirk and Cramer 1985; Melrose *et al.* 1982). The contour of integration in the complex $p_{||}$ plane is along the real axis in this case and the anti-hermitian part of the response tensor (which describes absorption) is identically zero (see Section 4*a*). The cutoff frequency ω_+ given in equation (64) corresponds to the result given in equation (30) of Herold *et al.* (1981). The approximate formula (64) becomes identical with the exact result given in equation (A3) in Appendix A, for $\theta = 90°$ and $n' = 1$. Herold *et al.* (1981) found that (64) remains a good approximation at least down to $\theta = 55°$.

For $0 < \omega < \omega_+$ one can express the momenta of resonant electrons as

$$\frac{p_{\pm}}{m} = \cos\,\theta \pm \left[\frac{(\omega_+ - \omega)(\omega - \omega_-)}{m\omega}\right]^{\frac{1}{2}}\,. \tag{65}$$

In terms of the resonant and particle ellipses introduced in Melrose *et al.* (1982) one can see that there are only two points of intersection between these ellipses, corresponding to the two values of the resonant momenta.

(a) Graphical Technique for Location of Resonant Roots

In this section, a simple graphical technique for locating the roots of the resonant denominator is presented. This technique is based on that used by

Batchelor and Goldfinger (1984) in examining fully relativistic classical plasmas. Firstly the case of the exact resonant denominator is treated, then results are presented for the semirelativistic resonant denominator.

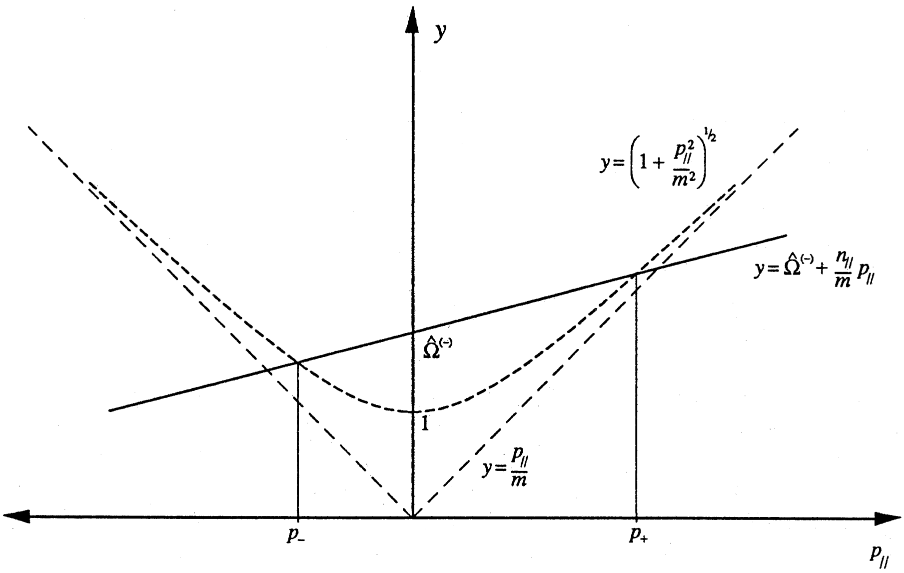

Fig. 2. Location of exact resonant momentum zeros.

The conditions found above under which both roots of the resonant denominator are real are not sufficient to ensure that the roots are physical. The exact resonant denominator can be expressed as

$$g(p_{||}) = \left(1 + \frac{p_{||}^2}{m^2}\right)^{\frac{1}{2}} - n_{||}\frac{p_{||}}{m} - \hat{\Omega}_{(-)}\,, \tag{66}$$

where $n_{||}$ (the parallel refractive index) and $\hat{\Omega}_{(-)}$ are defined to be

$$n_{||} = \frac{k_{||}}{\omega}\,, \tag{67}$$

$$\hat{\Omega}_{(-)} = \frac{\Omega_e}{\omega} - \frac{q^2}{2m\omega}\,. \tag{68}$$

The roots of (66) may be visualised as the points of intersection of the hyperbola $y = (1 + p_{||}^2/m^2)^{\frac{1}{2}}$ with the straight line $y = \hat{\Omega}_{(-)} + n_{||}p_{||}/m$; see Fig. 2. Solving $g(p_{||}) = 0$ one obtains the two roots

$$\frac{p_{||}}{m} = \frac{p_{\pm}}{m} = \frac{n_{||}\hat{\Omega}_{(-)} \pm (\hat{\Omega}_{(-)}^2 + n_{||}^2 - 1)^{\frac{1}{2}}}{1 - n_{||}^2}\,. \tag{69}$$

One or both solutions of (69) may be superfluous. Inspection of Fig. 2 shows that the following additional conditions are needed:

1. $p_{||}$ must be real and the discriminant non-negative, i.e. $\hat{\Omega}^2_{(-)} + n^2_{||} - 1 \geq 0$.
2. The roots must lie on the top branch of the hyperbola, i.e. $\hat{\Omega}_{(-)} + n_{||}p_{||}/m > 0$.

These conditions can be readily understood with the help of Fig. 2. One can classify the resonant roots according to the values of $\hat{\Omega}_{(-)}$ (the y intercept of the line $y = \hat{\Omega}_{(-)} + n_{||}p_{||}/m$). Without loss of generality assume $k_{||} > 0$, $\omega > 0$ and hence, $n_{||} > 0$.

Case 1:

$$\hat{\Omega}_{(-)} \leq 0, \begin{cases} \text{no valid roots for } n_{||} \leq 1 \\ p_{-} \text{ is a valid root for } n_{||} > 1 \end{cases}$$

Comment: For the case $n_{||} > 1$, the point of intersection lies in the region $p_{||} > 0$ and since $1 - n^2_{||} < 0$ then $p_{-} > 0$, $p_{+} < 0$.

Case 2:

$$0 < \hat{\Omega}_{(-)} < 1, \begin{cases} \text{there are no valid roots for } n_{||} < n^{min}_{||} = (1 - \hat{\Omega}^2_{(-)})^{\frac{1}{2}} \\ \text{there are two valid roots for } n^{min}_{||} \leq n_{||} < 1 \\ p_{-} \text{ is the valid root for } n_{||} > 1 \end{cases}$$

Comment: The value $n^{min}_{||}$ defines the point of tangency between the hyperbola and the line; this corresponds to vanishing of the discriminant.

Case 3:

$$\hat{\Omega}_{(-)} = 1, \begin{cases} p_{-} = 0, p_{+} = 0 \text{ are valid roots for } n_{||} = 0 \\ p_{-} = 0, p_{+} > 0 \text{ are valid roots for } 0 \leq n_{||} < 1 \\ p_{-} = 0 \text{ is a valid root for } n_{||} > 1 \end{cases}$$

Case 4:

$$\hat{\Omega}_{(-)} > 1, \begin{cases} p_{-}, p_{+} \text{ are valid roots for } 0 \leq n_{||} < 1 \\ p_{-} \text{ is the valid root for } n_{||} > 1 \quad (p_{-} < 0) \end{cases}$$

The cases where resonants roots are invalid, imply there is no absorption.

Inspection of Fig. 2 reveals the asymmetric character of relativistic cyclotron absorption. This feature is present in both classical and quantum plasmas. For a quantum plasma it is seen that the asymmetry is due to the one-dimensional nature of the plasma in a strong magnetic field. An electron moving in the same direction as $k_{||}$ has a higher energy on absorbing the photon than a counter-propagating electron, which loses energy.

For the case of interest here, one has $\omega > k_{||}$ and thus $n_{||} < 1$. If each of the cases $\hat{\Omega}_{(-)} \leq 0$, $0 < \hat{\Omega}_{(-)} < 1$, $\hat{\Omega}_{(-)} = 1$ and $\hat{\Omega}_{(-)} > 1$ is examined separately, it is found that only in case 4, i.e. $\hat{\Omega}_{(-)} > 1$, is one in a frequency range relevant to cyclotron absorption. This follows from the definition given in (68), since

$\hat{\Omega}_{(-)} > 1$ corresponds to photon frequencies less than the cyclotron frequency. Thus for $n_{||} < 1$ it can be seen that both zeros of the resonant denominator contribute to cyclotron absorption.

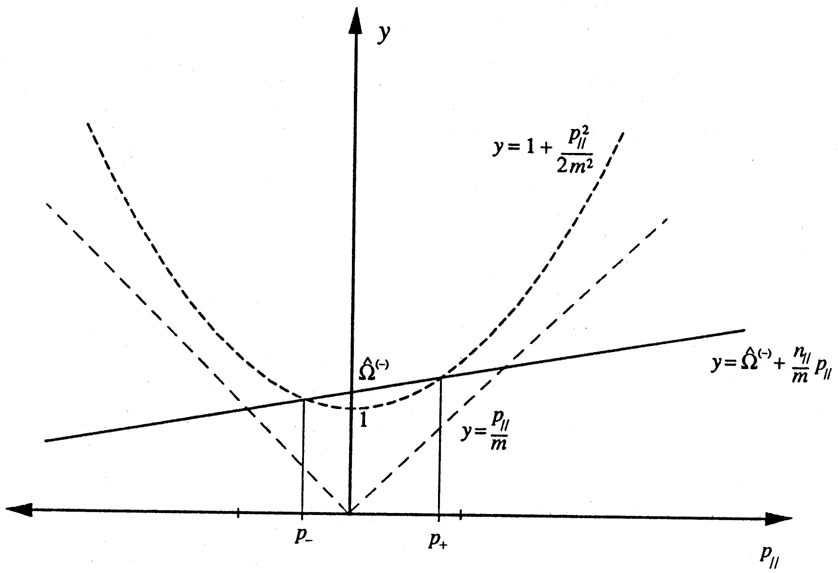

Fig. 3. Location of semirelativistic resonant momentum zeros.

Performing the same calculations as above for the semirelativistic resonant denominator, in place of (66) one now has

$$g(p_{||}) = \left(1 + \frac{p_{||}^2}{2m^2}\right) - n_{||}\frac{p_{||}}{m} - \hat{\Omega}_{(-)} \, . \tag{70}$$

Roots of this function can be visualised as the points of intersection of the parabola $y = (1 + p_{||}^2/2m^2)$ with the straight line $y = \hat{\Omega}_{(-)} + n_{||}p_{||}/m$, as shown in Fig. 3. Solving $g(p_{||}) = 0$ gives

$$\frac{p_{||}}{m} = \frac{p_{\pm}}{m} = n_{||} \pm \sqrt{n_{||}^2 + 2\hat{\Omega}_{(-)} - 2} \, . \tag{71}$$

Note that equation (71) is equivalent to (60). Conditions 1 and 2 given above still apply with a difference being that the discriminant in the semirelativistic case is now $n_{||}^2 + 2\hat{\Omega}_{(-)} - 2$.

The roots of the resonant denominator can be classified according to the values of $\hat{\Omega}_{(-)}$ as before, however, only the case where $\hat{\Omega}_{(-)} > 1$ is considered, as this is the one relevant to cyclotron absorption. In this case both momentum roots satisfy the auxilliary conditions for $n_{||} \geq 0$. However, some care should be exercised in using these solutions, since by making the semirelativistic approximation one limits the momentum zeros to values $< m$. This in turn leads to further restrictions on $n_{||}$ and $\hat{\Omega}_{(-)}$ as shown below.

The condition $\hat{\Omega}_{(-)} > 1$ corresponds to frequencies $\omega < \Omega_e$; the larger $\hat{\Omega}_{(-)}$ the further ω moves away from the cyclotron frequency. Thus, far away from the cyclotron resonance, it requires particles of much larger momentum to absorb the photons. Recall that cyclotron absorption is strongly peaked about the cyclotron frequency and the semirelativistic approximation only tends to broaden the resonance. Therefore, unless one is 'close' to the cyclotron frequency, the resonant momenta are large ($p^{res} > m$). For example $\hat{\Omega}_{(-)} = 1{\cdot}5$ corresponds to frequencies $\omega \simeq 2\Omega_e/3$, well below the cyclotron absorption frequency and one finds $p^{res} \sim m$. Thus, strictly, for the semirelativistic approximation to be valid one requires that $\hat{\Omega}_{(-)} \gtrsim 1$. To quantify this, first introduce a momentum cutoff p_c, above which the semirelativistic approximation is inaccurate. From equation (71) it is apparent that p_+ is the largest root, so that if one requires $p_+ < p_c$, this leads to the following restriction on $n_{||}$:

$$0 \le n_{||} \le \frac{p_c^2/m + 2 - 2\hat{\Omega}_{(-)}}{2p_c} \le \frac{p_c}{2}\,. \tag{72}$$

Equation (72) in turn can be used to determine the maximum allowed value of $\hat{\Omega}_{(-)}$ consistent with the semirelativistic approximation. This occurs at $n_{||} = 0$, i.e. for perpendicular propagation. In this case one finds

$$1 \le \hat{\Omega}_{(-)} \le 1 + p_c^2/2m^2\,. \tag{73}$$

It is assumed in the rest of this work that (72) and (73) are satisfied, so that one is in a regime where both resonant roots, as given by (71), contribute to the absorption of waves.

5. Introduction of Relativistic Plasma Dispersion Functions

In this section the integrals over parallel momentum occuring in the response tensor are performed in terms of relativistic plasma dispersion functions, whose analytic properties have been extensively discussed in Robinson (1986). Using the results of the preceding section, the response tensor in equation (45) can now be expressed in the form

$$\alpha^{\mu\nu}_{RES}(k) = \frac{2m^3\omega_p^2}{\omega(2\pi mT)^{\frac{1}{2}}}\int dp_{||}\frac{Q_+^{\mu\nu}(1,0)\exp(-p^2/2mT)}{(p_{||}-p_+)(p_{||}-p_-)}\,, \tag{74}$$

where Φ_2 is approximated by $2m$. The causal condition is invoked to determine how to integrate around the poles occuring at p_+,p_- in (74). One replaces ω by $\omega + i0$ under the square root sign in equation (65):

$$(\omega_+ - \omega - i0)(\omega - \omega_- + i0) = \omega(\omega_+ + \omega_-) - \omega^2 - \omega_+\omega_- + (\omega_+ + \omega_- - 2\omega)i0\,. \tag{75}$$

The sign of the infinitesimal imaginary part is therefore determined by the sign of $\omega_+ + \omega_- - 2\omega$. Noting that this can be written in the form $(\omega_+ - \omega) + (\omega_- - \omega)$, it is simple to see, using equation (63), that the above expression is negative. Hence,

$$(\omega_+ - \omega - i0)(\omega - \omega_- + i0) = (\omega_+ - \omega)(\omega - \omega_-) - i0\,, \tag{76}$$

which implies one must use the prescription

$$p_+ \to p_+ - i0, \qquad p_- \to p_- + i0. \tag{77}$$

This shows that in the complex $p_{||}$ plane, the pole p_- is displaced into the upper half-plane while the pole p_+ is displaced into the lower half-plane. The resonant denominator in equation (74) is then to be interpreted according to

$$(p_{||} - p_+)(p_{||} - p_-) \to (p_{||} - p_+)(p_{||} - p_-) + i0. \tag{78}$$

On examining the structure of the functions $Q_+^{\mu\nu}(1,0)$ in equations (47)–(56), it is seen that the class of integral that needs to be evaluated in equation (74) can be written as

$$S_\ell = \int_{-\infty}^{\infty} dp_{||} \frac{p_{||}^\ell \exp(-\beta p^2)}{p_{||}^2 - \sigma p_{||} + \xi}, \qquad \ell = 0, 1, 2 \tag{79}$$

where β, σ and ξ are defined to be

$$\beta = \frac{1}{2mT}, \tag{80}$$

$$\sigma = p_+ + p_- = 2m \cos\theta, \tag{81}$$

$$\begin{aligned} \xi &= p_+ p_- \\ &= m^2 \cos^2\theta - \frac{m}{\omega}[(\omega_+ - \omega)(\omega - \omega_-)]. \end{aligned} \tag{82}$$

These integrals may be performed with the aid of the following integral relationship (Robinson 1986):

$$\mathcal{F}_{1/2}(z,\, a) = \frac{1}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} du \frac{\exp(-u^2)}{u^2 - 2a^{\frac{1}{2}} u + z}, \tag{83}$$

where $\mathcal{F}_q(z,\, a)$ is a Shkarofsky function (Shkarofsky 1966; Robinson 1986) defined via

$$\mathcal{F}_q(z,\, a) = e^{z-2a} \int_1^{\infty} dt\, t^{-q} \exp\left[(a - z)t + \frac{a}{t}\right], \qquad \mathrm{Im}(z - a) > 0. \tag{84}$$

With the change of variables $u^2 = \beta p_{||}^2$, (83) becomes

$$\mathcal{F}_{1/2}(z,\, a) = \frac{1}{(\beta\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dp_{||} \frac{\exp(-\beta p_{||}^2)}{p_{||}^2 - 2\sqrt{\dfrac{a}{\beta}} p_{||} + \dfrac{z}{\beta}}. \tag{85}$$

Hence, on making the associations

$$\frac{z}{\beta} = \xi, \quad a = \frac{\beta}{4}\sigma^2 = \frac{m\cos^2\theta}{2T}, \tag{86}$$

one obtains the simple relation

$$S_0 = \sqrt{\beta\pi}\mathcal{F}_{1/2}(z,\ a)\,. \tag{87}$$

To evaluate the integral S_2 given by

$$S_2 = \int dp_{||}\frac{p_{||}^2\exp(-\beta p^2)}{p_{||}^2 - \sigma p_{||} + \xi}\,, \tag{88}$$

where it now implicit that $\mathrm{Im}(\xi) > 0$, note that it is possible to write

$$S_2 = -\frac{dS_0}{d\beta}\,.$$

Some care is required here, as the Shkarofsky function appearing in equation (87) depends on two parameters z and a which are in turn a function of β. Thus one must write

$$\begin{aligned}-S_2 = \frac{dS_0}{d\beta} &= \sqrt{\beta\pi}\left\{\frac{\partial\mathcal{F}_{1/2}(z(\beta),a(\beta))}{\partial z}\frac{dz(\beta)}{d\beta}+\frac{\partial\mathcal{F}_{1/2}(z(\beta),a(\beta))}{\partial a}\frac{da(\beta)}{d\beta}\right\}\\ &\quad -\tfrac{1}{2}\sqrt{\frac{\pi}{\beta}}\mathcal{F}_{1/2}(z,\ a)\,.\end{aligned} \tag{89}$$

Equation (86) implies that

$$\frac{dz(\beta)}{d\beta} = \xi, \qquad \frac{da(\beta)}{d\beta} = \frac{\sigma^2}{4}\,, \tag{90}$$

while Robinson (1986) obtained the differential equation

$$\frac{\partial\mathcal{F}_q(z,\ a)}{\partial z} = \mathcal{F}_q(z,\ a) - \mathcal{F}_{q-1}(z,\ a)\,. \tag{91}$$

It is straightforward to calculate the relation for $\partial\mathcal{F}_q(z,\ a)/\partial a$ from the integral relation in (84) to obtain

$$\frac{\partial\mathcal{F}_q(z,\ a)}{\partial a} = \mathcal{F}_{q-1}(z,\ a) - 2\mathcal{F}_q(z,\ a) + \mathcal{F}_{q+1}(z, a)\,. \tag{92}$$

Using (90), (91) and (92) in equation (89), one finally obtains

$$\begin{aligned}S_2 \;=\; &-\sqrt{\beta\pi}\Big\{\xi[\mathcal{F}_{1/2}(z,\ a) - \mathcal{F}_{-1/2}(z,\ a)] - \frac{\sigma^2}{4}[2\mathcal{F}_{1/2}(z,\ a)\\ &- \mathcal{F}_{-1/2}(z,\ a) - \mathcal{F}_{3/2}(z,\ a)]\Big\} - \tfrac{1}{2}\sqrt{\frac{\pi}{\beta}}\mathcal{F}_{1/2}(z,\ a)\,.\end{aligned} \tag{93}$$

If new quantities λ and τ are defined by

$$\lambda = \frac{\sigma^2}{2} - \xi = \tfrac{1}{2}[(p_+)^2 + (p_-)^2], \tag{94}$$

$$\frac{\tau^2}{4} = \frac{\sigma^2}{4} - \xi = \tfrac{1}{4}(p_+ - p_-)^2, \tag{95}$$

then equation (93) can be written as

$$S_2 = \sqrt{\beta\pi}\left[\left(\lambda - \frac{1}{2\beta}\right)\mathcal{F}_{1/2}(z,\, a) - \frac{\tau^2}{4}\mathcal{F}_{-1/2}(z,\, a) - \frac{\sigma^2}{4}\mathcal{F}_{3/2}(z,\, a)\right]. \tag{96}$$

To obtain an expression for S_1 the following identity is required:

$$p_{||} = -\frac{1}{\sigma}[p_{||}^2 - \sigma p_{||} + \xi - p_{||}^2 - \xi], \tag{97}$$

which allows one to write

$$\begin{aligned} S_1 &= -\frac{1}{\sigma}\int_{-\infty}^{\infty} dp_{||}\frac{[p_{||}^2 - \sigma p_{||} + \xi - p_{||}^2 - \xi]\exp(-\beta p^2)}{p_{||}^2 - \sigma p_{||} + \xi} \\ &= -\frac{1}{\sigma}\left\{\int dp_{||}\int_{-\infty}^{\infty} dp_{||}\exp(-\beta p^2) - S_2 - \xi S_0\right\} \\ &= -\frac{\sqrt{\beta\pi}}{\sigma}\left\{\frac{1}{\beta} - \left(\xi + \lambda - \frac{1}{2\beta}\right)\mathcal{F}_{1/2}(z,\, a) + \frac{\tau^2}{4}\mathcal{F}_{-1/2}(z,\, a) + \frac{\sigma^2}{4}\mathcal{F}_{3/2}(z,\, a)\right\}. \end{aligned} \tag{98}$$

It should be noted that the Shkarofsky functions of half-integer index can be re-expressed in terms of the more familiar plasma dispersion function of Fried and Conte (1961) (see e.g. Robinson 1986).

6. Specific Results for the Linear Response 4-Tensor

Using results (87), (96) and (98) it is now possible to write down expressions for the linear response 4-tensor. This gives the following results

$$\alpha_{RES}^{00}(k) \simeq \omega_p^2\left(\frac{m^2}{\omega T}\right)[J_1^0(k_\perp^2/2eB)]^2\mathcal{F}_{1/2}(z,\, a), \tag{99}$$

$$\alpha_{RES}^{11}(k) \simeq \omega_p^2\left(\frac{m^2}{\omega T}\right)\left(\frac{\Omega_e}{2m}\right)[J_0^0(k_\perp^2/2eB)]^2\mathcal{F}_{1/2}(z,\, a), \tag{100}$$

$$\alpha_{RES}^{22}(k) = \alpha_{RES}^{11}(k), \tag{101}$$

$$
\begin{aligned}
\alpha^{33}_{RES}(k) \simeq\ & \omega_p^2\left(\frac{m^2}{\omega T}\right)[J^0_1(k^2_\perp/2eB)]^2\left\{\frac{\Omega_e}{2m}\mathcal{F}_{1/2}(z,\,a)\right.\\
&+\frac{1}{m^2}\left[\left(\lambda-\frac{1}{2\beta}\right)\mathcal{F}_{1/2}(z,\,a)\right.\\
&\left.\left.-\frac{\tau^2}{4}\mathcal{F}_{-1/2}(z,\,a)-\frac{\sigma^2}{4}\mathcal{F}_{3/2}(z,\,a)\right]\right\}, \qquad (102)
\end{aligned}
$$

$$
\alpha^{01}_{RES}(k) \simeq \omega_p^2\left(\frac{m^2}{\omega T}\right)\left(\frac{\Omega_e}{2m}\right)^{\frac{1}{2}}[J^0_0(k^2_\perp/2eB)][J^0_1(k^2_\perp/2eB)]\mathcal{F}_{1/2}(z,\,a)\,, \qquad (103)
$$

$$
\alpha^{02}_{RES}(k) = i\alpha^{01}_{RES}(k)\,, \qquad (104)
$$

$$
\begin{aligned}
\alpha^{03}_{RES}(k) \simeq\ & -\frac{\omega_p^2}{\sigma}\left(\frac{m^2}{\omega T}\right)[J^0_1(k^2_\perp/2eB)]^2\left\{\frac{1}{\beta}-\left(\xi+\lambda-\frac{1}{2\beta}\right)\mathcal{F}_{1/2}(z,\,a)\right.\\
&\left.+\frac{\tau^2}{4}\mathcal{F}_{-1/2}(z,\,a)+\frac{\sigma^2}{4}\mathcal{F}_{3/2}(z,\,a)\right\}, \qquad (105)
\end{aligned}
$$

$$
\alpha^{12}_{RES}(k) = i\alpha^{11}_{RES}(k)\,, \qquad (106)
$$

$$
\begin{aligned}
\alpha^{13}_{RES}(k) \simeq\ & -\frac{\omega_p^2}{m\sigma}\left(\frac{m^2}{\omega T}\right)\left(\frac{\Omega_e}{2m}\right)^{\frac{1}{2}}[J^0_0(k^2_\perp/2eB)][J^0_1(k^2_\perp/2eB)]\\
&\times\left\{\frac{1}{\beta}-\left(\xi+\lambda-\frac{1}{2\beta}\right)\mathcal{F}_{1/2}(z,\,a)+\frac{\tau^2}{4}\mathcal{F}_{-1/2}(z\ a)\right.\\
&\left.+\frac{\sigma^2}{4}\mathcal{F}_{3/2}(z,\,a)\right\}, \qquad (107)
\end{aligned}
$$

$$
\alpha^{23}_{RES}(k) = -\alpha^{13}_{RES}(k)\,. \qquad (108)
$$

The J^0_0 and J^0_1 functions which appear in the linear response tensor may be simplified by noting that their argument $k^2_\perp/2eB$ is small compared with unity under conditions appropriate to X-ray pulsars. To see this, note that one can write $k_\perp \simeq \omega\sin\theta$ so that

$$
u = \frac{k^2_\perp}{2eB} \simeq \frac{\omega^2}{2eB}\sin^2\theta\,. \qquad (109)
$$

Since the frequencies of interest lie in the vicinity of the cyclotron frequency, this allows one to write $\omega \simeq \Omega_e$ in (109). This gives the final result

$$
u \simeq \frac{B}{2B_{cr}}\sin^2\theta\,, \qquad (110)
$$

which for magnetic fields typical of X-ray pulsars, $B \sim 0{\cdot}1\ B_{cr}$, indicates $u \ll 1$. Using (32) one obtains to good accuracy

$$
\begin{aligned}
J_0^0(u) &\simeq 1\,, \\
J_1^0(u) &\simeq \left(\frac{B}{2B_{cr}}\right)^{\frac{1}{2}} \sin\theta\,.
\end{aligned} \tag{111}
$$

(a) Behaviour of the Response near the Cutoff

In the paper by Herold *et al.* (1981), it was noted that the response function of the plasma is singular at a certain critical frequency unless the finite lifetime of the first excited Landau level is included explicity. This is a unique feature of the quantum plasma. In classical plasmas, singularities in the response function are removed by including thermal effects. However, even with the inclusion of finite temperatures, Herold *et al.* (1981) found that the singular behaviour persisted. In this section it is shown how the singularities encountered by Herold *et al.* (1981) may be deduced from equations (99)–(108) directly.

The hermitian and antihermitian parts of the linear response tensor determine the dispersion and absorption of waves respectively. In equations (99)–(108), these are associated with the real and imaginary parts of the Shkarofsky functions. Before the imaginary part of the Shkarofsky function is examined, the following preliminaries are necessary. Using (80), (82) and (86) one can show that

$$
z = a - \frac{1}{2\omega T}[(\omega_+ - \omega)(\omega - \omega_-)]\,, \tag{112}
$$

and thus for frequencies above the cutoff ω_+, it can be seen from (112) that $z > a$. Robinson (1986) has shown that in this case the imaginary part of the Shkarofsky function vanishes. Hence, above the cutoff frequency the antihermitian part of the response vanishes and there is no absorption, just as one expects.

Now consider what happens in the case where the cutoff frequency is approached from below. In this case $z < a$ and Robinson (1986) obtained the following result for $\mathrm{Im}[\mathcal{F}_q(z,\, a)]$:

$$
\mathrm{Im}[\mathcal{F}_q(z,\, a)] = -\pi e^{z-2a}\left[\frac{(a-z)}{a}\right]^{(q-1)/2} I_{q-1}[2\sqrt{a(a-z)}], \quad z < a\,, \tag{113}
$$

where $I_q(x)$ is a modified Bessel function. The dominant behaviour for $\mathrm{Im}[\mathcal{F}_q(z,\, a)]$ as ω approaches the cutoff, comes from the term $[(a-z)/a]^{(q-1)/2}$. From (112), it is seen that

$$
\begin{aligned}
a - z &= \frac{1}{2\omega T}[(\omega_+ - \omega)(\omega - \omega_-)] \\
&\sim \zeta(\omega_+ - \omega)\,,
\end{aligned} \tag{114}
$$

near the cutoff, where ζ is roughly constant and the symbol $\sim$ denotes 'of the order'. Also using (95), one can write

$$\frac{\tau^2}{4} \sim \eta(\omega_+ - \omega)\,, \tag{115}$$

where η is also roughly constant. Using (113) with (114) and (115) in equations (99)–(108), it is then possible to write for the antihermitian part of the response

$$\begin{aligned}\alpha^{\mu\nu A}_{RES}(k) &= a(\mu,\, \nu)\mathrm{Im}[\mathcal{F}_{1/2}(z,\, a)] + b(\mu,\, \nu)\mathrm{Im}[\mathcal{F}_{-1/2}(z,\, a)] \\ &\quad + c(\mu,\, \nu)\mathrm{Im}[\mathcal{F}_{3/2}(z,\, a)] \\ &\sim a(\omega_+ - \omega)^{-\frac{1}{2}} + b(\omega_+ - \omega)(\omega_+ - \omega)^{-\frac{3}{2}} + c(\omega_+ - \omega)^{\frac{1}{2}} \\ &\sim (\omega_+ - \omega)^{-\frac{1}{2}}, \qquad \text{as } \omega \rightarrow (\omega_+)_-\,. \end{aligned} \tag{116}$$

The behaviour given in (116) confirms the result obtained by Kirk and Cramer (1985) in their equation (24). An equivalent result to (116) is obtainable for the hermitian part of the response for frequencies that approach the cutoff from above. Hence the mathematical origin of the singularities encountered by Herold *et al.* (1981) at the cutoff is now apparent.

(b) Perpendicular Propagation

The results obtained in Section 5 apply to waves propagating obliquely to the magnetic field. In this section results are presented for the special case of perpendicular propagation, $\theta = 90°$ and thus $k_{\|} = 0$. From (86) it is seen that $a = 0$ in this limit. This leads one to introduce the Dnestrovskii function (Dnestrovskii *et al.* 1964), defined by Robinson (1986) as

$$F_q(z) = \mathcal{F}_q(z,\, 0)\,. \tag{117}$$

Therefore the result of (87) is replaced by

$$\lim_{\theta \rightarrow \pi/2} S_0 = \sqrt{\beta\pi} F_{1/2}(z)\,. \tag{118}$$

From (81), (82), (86), (94) and (95) for perpendicular propagation the following results hold:

$$\frac{\tau^2}{4}, \quad \lambda = -\frac{z}{\beta}, \quad \sigma = 0\,. \tag{119}$$

Hence, one obtains from (96) that

$$\lim_{\theta \rightarrow \pi/2} S_2 = \sqrt{\beta\pi}\Big\{ -\frac{z}{\beta}F_{1/2}(z) - \frac{1}{2\beta}F_{1/2}(z) + \frac{z}{\beta}F_{-1/2}(z)\Big\}\,. \tag{120}$$

Further, using the recurrence relations

$$F_{1/2}(z) - F_{-1/2}(z) = -\left(\frac{1}{2z} - 1\right) F_{1/2}(z) - \frac{1}{z},$$

$$(q-1)F_q(z) = 1 - zF_{q-1}(z),$$

equation (120) reduces to

$$\lim_{\theta \to \pi/2} S_2 = \sqrt{\frac{\pi}{4\beta}} F_{3/2}(z). \tag{121}$$

For S_1 the result (98) is no longer valid, as $\sigma = 0$ now. However, noting that $p_- = -p_+$ for perpendicular propagation, then in (79) with $\ell = 1$, the numerator is an odd function of $p_{||}$ while the denominator is an even function of $p_{||}$, hence, this integral vanishes by symmetry, so that $S_1 = 0$. This allows one to immediately write down the results for the components of the linear response tensor, viz.

$$\alpha^{00}_{RES}(k) \simeq \omega_p^2 \left(\frac{m^2}{\omega T}\right) [J_1^0(k_\perp^2/2eB)]^2 F_{1/2}(z,\, a), \tag{122}$$

$$\alpha^{11}_{RES}(k) \simeq \omega_p^2 \left(\frac{m^2}{\omega T}\right) \left(\frac{\Omega_e}{2m}\right) [J_0^0(k_\perp^2/2eB)]^2 F_{1/2}(z,\, a), \tag{123}$$

$$\alpha^{22}_{RES}(k) = \alpha^{11}_{RES}(k), \tag{124}$$

$$\alpha^{33}_{RES}(k) \simeq \omega_p^2 \left(\frac{m}{\omega}\right) [J_1^0(k_\perp^2/2eB)]^2 \left\{\frac{\Omega_e}{2T} F_{1/2}(z,\, a) + F_{3/2}(z,\, a)\right\}, \tag{125}$$

$$\alpha^{01}_{RES}(k) \simeq \omega_p^2 \left(\frac{m^2}{\omega T}\right) \left(\frac{\Omega_e}{2m}\right)^{\frac{1}{2}} [J_0^0(k_\perp^2/2eB)][J_1^0(k_\perp^2/2eB)] F_{1/2}(z,\, a), \tag{126}$$

$$\alpha^{02}_{RES}(k) = i\alpha^{01}_{RES}(k), \tag{127}$$

$$\alpha^{03}_{RES}(k) = 0, \tag{128}$$

$$\alpha^{12}_{RES}(k) = i\alpha^{11}_{RES}(k), \tag{129}$$

$$\alpha^{13}_{RES}(k) = 0, \tag{130}$$

$$\alpha^{23}_{RES}(k) = 0. \tag{131}$$

7. Summary

In this paper we present a new approach to the derivation of the linear response 4-tensor of a strongly magnetised electron gas. The calculations are performed within the framework of QPD, which is a manifestly covariant and gauge invariant theory. Through the use of QPD, the magnetic field has been included exactly, by employing eigenfunctions of the Dirac equation in a magnetic field, and also quantum and relativistic effects are included. The linear response 4-tensor is evaluated in the case of a nonrelativistic Maxwellian distribution in the ground state Landau level. A semirelativistic approximation is employed in the resonant denominator so that important relativistic effects are retained near the cyclotron resonance.

The roots of the resonant denominator are also examined in detail, showing that there is no absorption above a critical cutoff frequency. In addition, a simple graphical technique is introduced that allows one to visually locate the resonant electron momenta, as well as determine the conditions under which both roots are valid and contribute to absorption. The results for the linear response 4-tensor are expressed in terms of relativistic plasma dispersion functions known as Shkarofsky functions. The analytic properties of these functions, which have been extensively studied by Robinson (1986), make them extremely useful for studying the wave properties of the plasma, such as dispersion and absorption, without resorting to complicated numerical schemes. The work presented here thus extends previous analytic approaches which have either employed nonrelativistic approximations to the resonant denominator, thus neglecting crucial relativistic effects, or have calculated only the refractive index of the plasma directly. In addition, the result obtained for the behaviour of the response tensor near the cutoff gives an independent verification of the result derived by Kirk and Cramer (1985). Although no applications are given in the present work, the response is derived using parameters typical of X-ray pulsar accretion columns, for which it is thought that the present results are of importance in the calculation of physical processes occuring there.

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Appendix A: Cutoff and Absorption Frequencies

In this appendix details are presented for obtaining the exact cutoff and absorption frequencies for photons absorbed by resonant electrons. From Melrose *et al.* (1982) one can show that the kinematic condition for cyclotron absorption to occur is given by

$$(\epsilon_n - \epsilon_{n+s})^2 \geq (\omega^2 - k_{\|}^2)\,. \tag{A1}$$

Using this condition one may derive the cutoff photon frequency for cyclotron absorption or emission to occur. The discussion is restricted to the case $0 < q^2 < (\epsilon_n - \epsilon_{n'})^2$. Also if one considers a tenuous plasma, so that the photon dispersion relation is $\omega \simeq |\mathbf{k}|$, thus $k_{\|} \simeq \omega \cos\theta$, where θ is the angle between the magnetic field and photon wavevector, then equation (A1) is rewritten as

$$\omega^2 \sin^2\theta \leq (\sqrt{m^2 + 2n'eB} - \sqrt{m^2 + 2neB})^2$$

or

$$\omega \leq m\frac{[\sqrt{1 + 2n'(\Omega/m)} - \sqrt{1 + 2n(\Omega/m)}]}{\sin\theta}\,. \tag{A2}$$

The largest frequency for which absorption takes place, occurs when $n = 0$, i.e. for absorption from the ground state, hence,

$$\omega \leq \omega_{n'0} = m\frac{[\sqrt{1 + 2n'(\Omega/m)} - 1]}{\sin\theta}\,. \tag{A3}$$

Above $\omega_{n'0}$ there is no absorption by the plasma electrons. The result (A3) gives rise to the characteristic saw-tooth pattern in the absorption coefficient (see e.g. Fig. 3 of Kirk and Cramer 1985). Another important feature of the frequency $\omega_{n'0}$ is that at this point the two resonant momentum zeros coincide and thus on integrating the response over the momentum, an infinite value is obtained. This is the point at which Herold *et al.* (1981) encountered the singularity in the response function.

One can also derive the frequency of a photon actually absorbed by an electron, as follows. Firstly, the energy of the electron in the excited state is given by

$$\begin{aligned}\epsilon(n',p'_{||}) &= [m^2+(p_{||}+k_{||})^2+2n'\Omega m]^{\frac{1}{2}} \\ &\simeq [m^2+(p_{||}+\omega\cos\theta)^2+2n'\Omega m]^{\frac{1}{2}}\,. \end{aligned} \tag{A4}$$

If one squares both sides of the energy conservation equation

$$\epsilon'(n',p'_{||}) = \epsilon(n,p_{||})+\omega\,, \tag{A5}$$

and substitutes (A4) in the left hand side, it is found that

$$\omega^2\sin^2\theta+2\omega(\epsilon_q-p_{||}\cos\theta)-2seB=0\,, \tag{A6}$$

where for simplicity ϵ_q denotes $\epsilon(n,p_{||})$. If instead equation (A6) is considered as a function of frequency rather than momentum, one obtains the solutions

$$\omega=\omega_{\pm}=\frac{-\epsilon_q+p_{||}\cos\theta\pm\sqrt{(\epsilon_q-p_{||}\cos\theta)^2+2seB\sin^2\theta}}{\sin^2\theta}\,. \tag{A7}$$

For the case of cyclotron absorption the positive root is the correct choice. The other root is spurious; it is introduced by squaring equation (A5). Hence, the quantity defined by

$$\omega_{res}(\theta)=\frac{-\epsilon_q+p_{||}\cos\theta+\sqrt{(\epsilon_q-p_{||}\cos\theta)^2+2seB\sin^2\theta}}{\sin^2\theta} \tag{A8}$$

gives the frequency of a photon that an electron of energy ϵ_q and momentum $p_{||}$ actually absorbs on making a transition to the excited state with $n'=n+s$.

In the case of a cold plasma ($p_{||}=0$) and for electrons initially in the ground state $\omega_{res}(\theta)$ becomes

$$\omega^0_{res}(\theta)=m\frac{[\sqrt{1+2s(\Omega/m)\sin^2\theta}-1]}{\sin^2\theta}\,. \tag{A9}$$

For magnetic fields such that $\Omega/m\ll 1$ one may Taylor expand equation (A9) to obtain

$$\omega^0_{res}(\theta)\simeq s\Omega\left(1-\tfrac{1}{2}\frac{s\Omega}{m}\sin^2\theta+\cdots\right). \tag{A10}$$

It can be seen from this result that there is a small angle-dependent, frequency down-shift from the cyclotron frequency $s\Omega$. This is due to momentum transfer from the photon to the plasma electron. By way of comparison, one can perform the same calculation in the nonrelativistic limit. The energy of a nonrelativistic electron in a magnetic field is given by

$$\epsilon_q = m + \frac{p_{||}^2}{2m} + n\Omega\,. \tag{A11}$$

Using (A11) in equation (A5) one obtains in place of equation (A6)

$$\omega^2 \cos^2\theta - 2\omega(m - p_{||}\cos\theta) + 2seB = 0\,, \tag{A12}$$

which has the solutions

$$\omega = \omega_\pm = \frac{m - p_{||}\cos\theta \pm \sqrt{(m - p_{||}\cos\theta)^2 - 2seB\cos^2\theta}}{\cos^2\theta}\,. \tag{A13}$$

In this case the appropriate solution is given by the negative root and, hence, the nonrelativistic resonance frequency is then

$$\omega_{res}^{NR}(\theta) = \frac{m - p_{||}\cos\theta - \sqrt{(m - p_{||}\cos\theta)^2 - 2seB\cos^2\theta}}{\cos^2\theta}\,. \tag{A14}$$

Again, taking the limit of a cold plasma initially in the ground state gives

$$\omega_{res}^{NR}(\theta)_0 = m\frac{(1 - \sqrt{1 - 2s(eB/m)\cos^2\theta})}{\cos^2\theta}\,, \tag{A15}$$

which in the case of $\Omega/m \ll 1$ implies

$$\omega_{res}^{NR}(\theta)_0 \simeq s\Omega\left(1 + \tfrac{1}{2}\frac{s\Omega}{m} - \tfrac{1}{2}\frac{s\Omega}{m}\sin^2\theta + \cdots\right). \tag{A16}$$

In this limit, if one compares (A16) with (A10), it is seen that there is an additional frequency upshift that is independent of angle. This is a reflection of the difference between the relativistic excitation and the nonrelativistic excitation of the Landau levels.

Appendix B: 4-Tensor Generalisation of the Polarisability of Pavlov *et al.*

In this appendix results are given for the 4-tensor generalisation of the quantum response of a strongly magnetised electron gas as derived by Pavlov *et al.* (1980). The results of Pavlov *et al.* (1980) were obtained using the general expressions derived for the susceptibility (termed polarisability by Pavlov *et al.*) of an electron positron plasma by Svetozarova and Tsytovich (1962), which is identical to the three tensor part of equation (4). Pavlov *et al.* (1980) evaluated the susceptibility tensor in the nonrelativistic limit, so that near the cyclotron harmonics the results do not take into account important relativistic effects. In the work presented here, the methodology of Pavlov *et al.* (1980) is employed, with the exception that SI, rather than Gaussian units are used. As well, expressions are given here for the response tensor rather than the susceptibility tensor that Pavlov *et al.* (1980) quoted. The 3-tensors are related by $\alpha_{ij}^{(\mathrm{pav})} = \alpha_{ij}/\omega^2$, where $\alpha_{ij}^{(\mathrm{pav})}$ denotes the polarisability employed by Pavlov *et al.* (1980), α_{ij} denotes the linear response tensor employed in this paper and ω is the photon frequency.

The distribution employed by Pavlov *et al.* (1980) is given by

$$f(\epsilon_q) = \frac{4\pi^2 N_e}{\epsilon_0 m\Omega_e}\frac{\tanh(\lambda/2)}{(2\pi mT)^{\frac{1}{2}}}\exp(-p_{||}^2/2mT - n\lambda)\,, \tag{B1}$$

where $\lambda = \Omega_e/T$. The distribution of particles in the final state is

$$f(\epsilon_{q'}) = \frac{4\pi^2 N_e}{\epsilon_0 m\Omega_e}\frac{\tanh(\lambda/2)}{(2\pi mT)^{\frac{1}{2}}}\exp(-p_{||}'^2/2mT - n'\lambda)\,, \tag{B2}$$

with $p_{||}' = p_{||} + k_{||}$.

Using the expression for the energy eigenvalue given in equation (A2), if one Taylor expands the resonant denominator $\omega + \epsilon_q - \epsilon_{q'}$, it is found that

$$\begin{aligned}\omega + \epsilon_q - \epsilon_{q'} \;\simeq\; & \omega - s\Omega - \frac{p_{||}k_{||}}{m} - \frac{k_{||}^2}{2m} + s(n' + n)\frac{\Omega^2}{2m} \\ & + s\Omega\frac{(p_{||}^2 + p_{||}'^2)}{4m^2}\,, \end{aligned} \tag{B3}$$

with $s = n' - n$. The third term in (B3) takes into account the linear Doppler effect, the fourth term represents the effects of quantum recoil due to the electron–photon interaction, the fifth term arises from the anharmonicity of the Landau levels and the sixth term represents the quadratic Doppler effect. Pavlov *et al.* (1980) argued that, for $|\mathbf{k}| \sim \omega \approx \Omega$ one can neglect the quadratic Doppler term in comparison with the linear Doppler term provided $|\cos\theta| \gg \beta$, where θ is the angle of photon propagation with respect to the field and $\beta = (2T/m)^{\frac{1}{2}} \ll 1$. It was also argued that the anharmonicity term is only comparable with the linear Doppler term for $\lambda \gtrsim |\cos\theta|/\beta \gg 1$, when only the lowest Landau levels are populated ($n = 0$ or $n' = 0$). Therefore, they put $n' + n = s$ in the fifth term of (B3), which leaves the approximation

$$\omega + \epsilon_q - \epsilon_{q'} \simeq \omega - s\Omega - \frac{p_{||}k_{||}}{m} - \frac{k_{||}^2}{2m} + s|s|\frac{\Omega^2}{2m}\,. \tag{B4}$$

The nonresonant terms of the response tensor may be treated in the same manner as in Section 2, except in place of (33) one should use

$$f(\epsilon_q) \simeq \frac{4\pi^2 N_e}{\epsilon_0 m\Omega_e}\delta(p_{||})\delta_{n0}, \quad f(\epsilon_{q'}) \simeq \frac{4\pi^2 N_e}{\epsilon_0 m\Omega_e}\delta(p_{||}')\delta_{n'0}\,.$$

This leads to the result

$$\alpha_{nr}^{\mu\nu}(k) = \omega_p^2 f^{\mu\nu}\,, \tag{B5}$$

where

$$f^{\mu\nu} = \begin{cases} \dfrac{\omega^2}{4m^2} & \mu = 0,\ \nu = 0 \\ -g^{ij} & \mu = i,\ \nu = j \\ 0 & \text{otherwise}. \end{cases} \tag{B6}$$

The resonant part of the response 4-tensor that is evaluated here can be written

$$\begin{aligned} \alpha^{\mu\nu}_{RES}(k) &= \frac{e^2 m\Omega_e}{4\pi^2} \sum_{n=0,n'=0}^{\infty} \int dp_{\parallel} Q^{\mu\nu}_{+}(n',\, n) \frac{f(\epsilon_{q'}) - f(\epsilon_q)}{\omega + \epsilon_q - \epsilon_{q'} + i0} \\ &= -\frac{e^2 N_e}{\epsilon_0} \frac{\tanh(\lambda/2)}{(2\pi m T)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \sum_{s=-\infty}^{\infty} \int_{-\infty}^{\infty} dp_{\parallel} Q^{\mu\nu}_{+}(s,\, n) \\ &\quad \times \frac{\exp[-p'^2_{\parallel}/2mT - (n+s)\lambda] - \exp(-p^2_{\parallel}/2mT - n\lambda)}{\omega - s\Omega - \dfrac{p_{\parallel}k_{\parallel}}{m} - \dfrac{k^2_{\parallel}}{2m} + s|s|\dfrac{\Omega^2}{2m} + i0}\,. \end{aligned} \tag{B7}$$

In equation (B7) the summation over n' has been changed to a summation over $s = n' - n$, thus everywhere one replaces n' by $n + s$. In the nonrelativistic approximation the functions $Q^{\mu\nu}_{+}(s,\, n)$ are given by

$$Q^{00}_{+}(s,\, n) = \left[1 - (s+2n)\frac{\Omega}{2m}\right][(J^{n-1}_s)^2 + (J^n_s)^2] + \frac{2\Omega}{m}\sqrt{(n+s)n}(J^{n-1}_s J^n_s)\,, \tag{B8}$$

$$Q^{11}_{+}(s,\, n) = (s+2n)\frac{\Omega}{2m} + \frac{2\Omega}{m}\sqrt{(n+s)n}(J^n_{s-1}J^{n-1}_{s+1})\,, \tag{B9}$$

$$Q^{22}_{+}(s,\, n) = Q^{11}_{\pm}(s,\, n) - \frac{\Omega}{m}\sqrt{(s+n)n}(J^n_{s-1}J^{n-1}_{s+1})\,, \tag{B10}$$

$$\begin{aligned} Q^{33}_{+}(s,\, n) &= \left[(s+2n)\frac{\Omega}{2m} + \frac{(p_{\parallel} + k_{\parallel}/2)^2}{m^2}\right][(J^{n-1}_s)^2 + (J^n_s)^2] \\ &\quad - \frac{2\Omega}{m}\sqrt{(n+s)n}(J^{n-1}_s J^n_s)\,, \end{aligned} \tag{B11}$$

$$\begin{aligned} Q^{01}_{+}(s,\, n) &= -\left(\frac{\Omega}{2m}\right)^{\frac{1}{2}}[\sqrt{n}(J^{n-1}_s J^n_{s-1}) + (J^n_s J^{n-1}_{s+1}) \\ &\quad + \sqrt{n+s}(J^{n-1}_s J^{n-1}_{s+1}) + (J^n_s J^n_{s-1})]\,, \end{aligned} \tag{B12}$$

$$\begin{aligned} Q^{02}_{+}(s,\, n) &= i\left(\frac{\Omega}{2m}\right)^{\frac{1}{2}}[\sqrt{n}(J^n_s J^{n-1}_{s+1}) - (J^{n-1}_s J^n_{s-1}) \\ &\quad + \sqrt{n+s}(J^{n-1}_s J^{n-1}_{s+1}) - (J^n_s J^n_{s-1})]\,, \end{aligned} \tag{B13}$$

$$Q_+^{03}(s,\,n) \;=\; \left(\frac{2p_{\|}+k_{\|}}{2}\right)[(J_s^{n-1})^2+(J_s^n)^2]\,, \tag{B14}$$

$$Q_+^{12}(s,\,n) \;=\; i(s+2n)\frac{\Omega}{2m}[(J_{s-1}^n)^2-(J_{s+1}^{n-1})^2]\,, \tag{B15}$$

$$\begin{aligned} Q_+^{13}(s,\,n) \;=&\; -\frac{1}{m}\frac{\Omega}{2m}\{p'_{\|}\sqrt{n}[(J_{s-1}^n J_s^{n-1})+(J_{s+1}^{n-1}J_s^n)] \\ &+p_{\|}\sqrt{n+s}[(J_{s-1}^n J_s^n)+(J_{s+1}^{n-1}J_s^{n-1})]\}\,, \end{aligned} \tag{B16}$$

$$\begin{aligned} Q_+^{23}(s,\,n) \;=&\; i\frac{1}{m}\frac{\Omega}{sm}\{p'_{\|}\sqrt{n}[(J_{s-1}^n J_s^{n-1})-(J_{s+1}^{n-1}J_s^n)] \\ &+p_{\|}\sqrt{n+s}[(J_{s-1}^n J_s^n)-(J_{s+1}^{n-1}J_s^{n-1})]\}\,, \end{aligned} \tag{B17}$$

The momentum integrals that appear in the response are of the type

$$H_\ell = \int_{-\infty}^{\infty} dp_{\|}\frac{p_{\|}^\ell e^{-(p_{\|}+a)^2/2mT}}{b-cp}, \qquad \ell=0,\,1,\,2\,. \tag{B18}$$

These integrals can be evaluated in terms of the plasma dispersion function, which here is given by the definition

$$\begin{aligned} W(z) \;&=\; 2i\pi^{-1/2}e^{-z^2}\int_{i\infty}^{z} d\tau\, e^{\tau^2} \\ &=\; -\frac{i}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} d\tau\frac{e^{-\tau^2}}{\tau-z+i0}\,, \end{aligned} \tag{B19}$$

and the differential equation

$$\frac{dW(z)}{dz} = 2i\pi^{-1/2}[1+i\pi^{\frac{1}{2}}zW(z)]\,. \tag{B20}$$

In order to perform the sums over n in (B7), the following special sum is required for the J_ℓ^n functions

$$\sum_{n=0}^{\infty}[J_\alpha^n(y)]^2 z^n = \frac{e^{-y}z^{-\alpha/2}}{1-z}\exp\left(-\frac{2yz}{1-z}\right)I_\alpha\left(\frac{2y\sqrt{z}}{1-z}\right), \tag{B21}$$

where $I_\alpha(x)$ is a modified Bessel function. Also one needs to use the recursion relations for J_ℓ^n given in equations (A4a)–(A7) of Melrose and Parle (1983*a*). Finally the recursion relations

$$I_{\alpha-1}(z)-I_{\alpha+1}(z) \;=\; \frac{2\alpha}{z}I_\alpha(z)\,, \tag{B22}$$

$$I_{\alpha-1}(z) + I_{\alpha+1}(z) = 2\frac{dI_\alpha(z)}{dz}, \tag{B23}$$

for the modified Bessel functions are required.

Using the results presented above it is now possible to obtain the following results for the resonant contribution to the linear response 4-tensor:

$$\alpha^{00}_{RES}(k) \simeq \frac{\omega_p^2}{2\kappa\beta^2}\zeta \sum_{s=-\infty}^{\infty} \{4I_s(\chi) - \beta^2\lambda\tanh(\lambda/2)[\chi I_{s-1}(\chi) - sI_s(\chi)]\}G_s, \tag{B24}$$

$$\alpha^{11}_{RES}(k) \simeq \frac{\omega_p^2}{2\kappa}\frac{\lambda}{\sinh(\lambda/2)}\zeta \sum_{s=-\infty}^{\infty} \frac{s^2 I_s(\chi)}{\chi}G_s, \tag{B25}$$

$$\alpha^{22}_{RES}(k) = \alpha^{11}_{RES}(k) - \frac{4\omega_p^2}{\kappa}\frac{\lambda}{\sinh(\lambda)}\zeta\chi \sum_{s=-\infty}^{\infty} [I'_s(\chi) - \cosh(\lambda/2)I_s(\chi)]G_s, \tag{B26}$$

$$\alpha^{33}_{RES}(k) \simeq \frac{\omega_p^2}{\kappa}\zeta \sum_{s=-\infty}^{\infty} \left[2I_s(\chi)G_s^{(2)} + \chi I'_s(\chi)\frac{\lambda}{2}\tanh(\lambda/2)G_s\right], \tag{B27}$$

$$\alpha^{01}_{RES}(k) \simeq \frac{\omega_p^2}{2\beta}\frac{\tan\theta}{\cosh(\lambda/2)}\zeta \sum_{s=-\infty}^{\infty} I'_s(\chi)G_s, \tag{B28}$$

$$\alpha^{02}_{RES}(k) \simeq i\frac{\omega_p^2}{\beta}\frac{\tan\theta}{\sinh\lambda}\zeta \sum_{s=-\infty}^{\infty} \frac{sI_s(\chi)}{\chi^2}G_s, \tag{B29}$$

$$\alpha^{03}_{RES}(k) \simeq \frac{2\omega_p^2}{\kappa\beta}\zeta \sum_{s=-\infty}^{\infty} I_s(\chi)G_s^{(1)}, \tag{B30}$$

$$\alpha^{12}_{RES}(k) \simeq -i\frac{\omega_p^2}{\kappa}\frac{\lambda}{\sin\lambda}\zeta \sum_{s=-\infty}^{\infty} s[\cosh(\lambda/2)I'_s(\chi) - I_s(\chi)]G_s, \tag{B31}$$

$$\alpha^{13}_{RES}(k) \simeq \omega_p^2\frac{\tan\theta}{2\sinh(\lambda/2)}\zeta \sum_{s=-\infty}^{\infty} \left[\frac{sI_s(\chi)}{\chi}G_s^{(1)} - \frac{\kappa}{4}\tanh(\lambda/2)I'_s(\chi)G_s\right], \tag{B32}$$

$$\alpha^{23}_{RES}(k) \simeq i\omega_p^2\frac{\tan\theta}{2\sinh(\lambda/2)}\zeta \sum_{s=-\infty}^{\infty} \left\{[\cosh(\lambda/2)I'_s(\chi) - I_s(\chi)]\frac{G_s^{(1)}}{\cosh(\lambda/2)} - \frac{\kappa}{4}\tanh(\lambda/2)\frac{sI_s(\chi)}{\chi}G_s\right\}. \tag{B33}$$

In equations (B24)–(B33) the following definitions are employed:

$$\zeta = e^{-u\coth(\lambda/2)}, \tag{B34}$$

$$\kappa = \omega\left(\frac{2}{mT}\right)^{\frac{1}{2}}\cos\theta, \tag{B35}$$

$$u \equiv \frac{k_\perp^2}{2m\Omega} \simeq \frac{\omega^2}{2m\Omega}\sin^2\theta\,, \tag{B36}$$

$$\chi = \frac{u}{\sinh(\lambda/2)}\,, \tag{B37}$$

$$I'_s(\chi) = \frac{dI_s(\chi)}{d\chi}\,, \tag{B38}$$

$$G_s = i\pi^{\frac{1}{2}}\left[e^{-s\lambda/2}W\left(z_s+\frac{\kappa}{2}\right) - e^{s\lambda/2}W(z_s)\right], \tag{B39}$$

$$G_s^{(1)} = \left(z_s+\frac{\kappa}{4}\right)G_s - 2\sinh(s\lambda/2)\,, \tag{B40}$$

$$G_s^{(2)} = \left(z_s+\frac{\kappa}{4}\right)G_s^{(1)} - \frac{\kappa}{2}\cosh(s\lambda/2)\,, \tag{B41}$$

$$z_s = \frac{x - s\lambda + s|s|\beta^2/4}{\kappa} - \frac{\kappa}{4}\,, \tag{B42}$$

$$x = \frac{\omega}{T}\,. \tag{B43}$$

It should be noted that these equations apply in the frame where

$$\mathbf{k} = (k_\perp,\ 0,\ k_{||})\,. \tag{B44}$$

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