Aust. J. Phys., 1992, **45**, 165–83

Relativistic Quantum Response of a Strongly Magnetised Plasma. II. Ultrarelativistic Pair Plasma*

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Sydney, N.S.W. 2006, Australia.

Abstract

Approximate analytic expressions are derived for the linear response 4-tensor of a strongly magnetised, ultrarelativistic thermal pair plasma. The response is obtained in terms of relativistic plasma dispersion functions known as Dnestrovskii functions. These functions allow for a relatively simple study of wave properties of the pair plasma without requiring complicated numerical schemes. The results obtained are valid in general for frequencies below the electron cyclotron frequency. It is believed that the results could be of importance in some models of radio pulsars and gamma-ray bursters.

1. Introduction

Electron–positron pair production is generally regarded as an exotic high-energy process and is usually only associated with particle physics. However, many astrophysical objects provide an environment in which electron–positron pairs are produced in large concentrations. Furthermore, these pairs can have a significant impact on both the dynamics of the source region and the characteristics of the radiation spectrum.

The distinctive mark of electron pairs is their annihilation radiation, which is generally observed as a spectral line centred at $0 \cdot 511$ MeV. This line has been directly observed in a number of objects, including the galactic centre (Riegler *et al.* 1981), gamma-ray burst sources (Teegarden and Cline 1980) and solar flares (Chupp *et al.* 1982). There are other objects where annihilation radiation has not been directly observed, but in which electron–positron pairs are expected to play an important role. These include active galactic nuclei (AGNs) and radio pulsars.

There is a number of papers that are concerned with the propagation of radiation in strongly magnetised pair plasmas. The properties of the natural wave modes of a pair plasma have been examined by several authors. Hardee and Rose (1976, 1978) examined the propagation of waves generated by the interaction of a relativistic electron–positron beam with a cool, electron–positron plasma. Melrose and Stoneham (1977) and Melrose (1979) examined the properties of the natural modes of a cold, relativistically streaming pair plasma, with a view to explaining the polarisation characteristics of pulsar radio emission. Arons and Barnard (1986) considered the propagation of normal modes well below the

* Part I, *Aust. J. Phys.* **45**, 1992, 131.

0004-9506/92/020165$05.00

cyclotron frequency in a relativistically streaming one-dimensional pair plasma with allowance made for momentum dispersion along the field lines. On the other hand, Volokitin *et al.* (1985) considered an arbitrary momentum distribution, other than taking it to be one-dimensional. Some papers have even examined the propagation of nonlinear Alfvén solitons in magnetised pair plasmas (Karpman and Washimi 1977; Sakai and Kawata 1980; Mikhailovskii *et al.* 1985; Stenflo *et al.* 1985), with possible applications to radio pulsars.

The basic quantity necessary for a determination of the wave properties of a plasma is the response tensor. As is discussed in Padden (1992; present issue p. 131, hereafter paper I), quantum and relativistic effects are important in the super-strong magnetic fields of neutron stars. In discussing the properties of wave modes in pair plasmas, most authors use the response of a classical magnetised plasma. Melrose and Stoneham (1977) calculated the response tensor using quantum plasmadynamics, as is done in paper I, but only in the long-wavelength limit and for a cold plasma. Also, in evaluating the wave properties they considered the regime where relativistic quantum effects could probably be neglected. In the present paper, the linear response 4-tensor of a strongly magnetised, pair plasma is evaluated within the framework of quantum plasmadynamics. The approach used is identical to that of paper I except that an ultrarelativistic Maxwellian distribution with a sharp momentum cutoff is employed. This is done primarily for mathematical convenience, allowing the approximation $p \gg m$ to be made. Such a distribution is also chosen with applications to gamma-ray bursters and radio pulsars in mind, where it is expected that the bulk of the electrons and positrons are highly relativistic. This work aims to extend the theory of wave propagation in pair plasmas to frequencies up to the cyclotron frequency, including both quantum and relativistic effects.

The layout of this paper is as follows. In Section 2, the linear response 4-tensor of a magnetised pair plasma with arbitrary distribution function is presented. These results give the 4-tensor generalisation of those given originally by Svetozarova and Tsytovich (1962). An ultrarelativistic Maxwellian distribution, which has a momentum cutoff, is introduced for the pairs in Section 3. This distribution leaves the fraction of electrons and positrons arbitrary. In Section 4, the resonant denominator is examined in detail. A correction to the ultrarelativistic approximation is used to locate the resonant momenta more accurately. In addition, the graphical technique introduced in paper I is used to determine the conditions under which the resonant momenta are valid, in a simple manner. In Section 5, the relativistic plasma dispersion function of Dnestrovskii is used to evaluate the momentum integrals of the linear response tensor. Specific results are presented for the response 4-tensor in Section 6, valid for all angles of propagation except those parallel to the magnetic field.

The notation used in this paper is the same as that employed in paper I unless otherwise stated.

2. Linear Response Tensor for a Strongly Magnetised Pair Plasma

In this section, expressions are presented for the components of the linear response 4-tensor of a magnetised pair plasma. As a starting point, since the expression given in equation (4) of paper I is to be employed, it is reproduced here for convenience:

$$\alpha^{\mu\nu}(k) = -\frac{e^3B}{2\pi}\sum_{n=0,n'=0}^{\infty}\int\frac{dp_{||}}{2\pi}\left\{\frac{N_q^+ - N_{q'}^+}{\omega - \epsilon_q + \epsilon_{q'} + i0}Q_+^{\mu\nu}(n',n)\right.$$
$$+\frac{N_q^+ + N_{q'}^-}{\omega - \epsilon_q - \epsilon_{q'} + i0}Q_-^{\mu\nu}(n',n) - \frac{N_{q'}^+ + N_q^-}{\omega + \epsilon_q + \epsilon_{q'} + i0}Q_-^{\mu\nu}(n',n)$$
$$\left. - \frac{N_q^- - N_{q'}^-}{\omega + \epsilon_q - \epsilon_{q'} + i0}\eta(\mu,\nu)Q_-^{\mu\nu}(n',n)\right\}, \tag{1}$$

with $\eta(\mu,\nu)$ given by equation (5) and the $Q_\pm^{\mu\nu}(n',n)$ by equations (6)–(15) in paper I. Since the magnetic field considered here is typically of the order $\gtrsim 0{\cdot}1B_{cr}$ and provided the photon frequencies of interest are of order the cyclotron frequency or less, the second and third terms on the right hand side of (1) can be ignored, as they are nonresonant. If the electron and positron distributions are now denoted by $f^+(\epsilon_q)$ and $f^-(\epsilon_q)$ respectively, then (1) is replaced by

$$\alpha_{RES}^{\mu\nu}(k) = -\frac{e^3B}{2\pi}\sum_{n=0,n'=0}^{\infty}\int\frac{dp_{||}}{2\pi}\left\{\frac{f^+(\epsilon_q) - f^+(\epsilon_{q'})}{\omega - \epsilon_q + \epsilon_{q'} + i0}Q_+^{\mu\nu}(n',n)\right.$$
$$\left. - \frac{f^-(\epsilon_q) - f^-(\epsilon_{q'})}{\omega + \epsilon_q - \epsilon_{q'} + i0}\eta(\mu,\nu)Q_+^{\mu\nu}(n',n)\right\}, \tag{2}$$

where RES denotes the resonant contribution to the response.

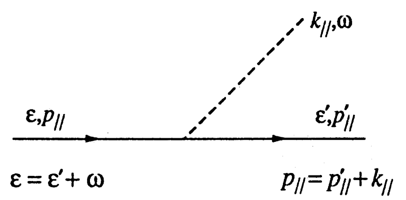

Fig. 1. Feynman diagram showing cyclotron absorption by a positron, along with the labelling of the momentum and energy in the initial and final states.

At this time it is necessary to mention a subtle point regarding equation (2). For an electron, the final state is denoted by the quantum numbers q', n' and energy eigenvalue $\epsilon_{q'}$. A positron, however, is interpreted as an electron propagating backward in time, so that the quantum numbers of the final state are denoted by q, n with corresponding energy eigenvalue ϵ_q and with primed quantities denoting the initial state, as is shown in Fig. 1. Thus for consistency, one should relabel the quantum numbers in the second term in (2) according to

$$q \to q',\ q' \to q,\ n \to n',\ n' \to n. \tag{3}$$

The denominators in (2) are now the same, however, the numerator of the positron term contains the function $Q_+^{\mu\nu}(n, n')$.

It is straightforward to show that (see Appendix A), using the relation

$$J_s^n(u) = (-1)^s J_{-s}^{n+s}(u) \tag{4}$$

for the $J^n_{n'}$ functions that appear in the expressions for $Q^{\mu\nu}_+(n',\, n)$, one has the following symmetry:

$$Q^{\mu\nu}_+(n,\, n') = \zeta(\mu,\, \nu) Q^{\mu\nu}_+(n',n)\,, \tag{5}$$

where

$$\zeta(\mu,\nu) = \begin{cases} +1, & \mu,\nu = 00,\ 11,\ 22,\ 33,\ 02,\ 03,\ 23 \\ -1, & \mu,\nu = 01,\ 12,\ 13\,. \end{cases} \tag{6}$$

Thus, combining (3) and (5) with (6), in (2), one can finally write

$$\begin{aligned} \alpha^{\mu\nu}_{RES}(k) &= -\frac{e^3 B}{2\pi} \sum_{n=0,n'=0}^{\infty} \int \frac{dp_{||}}{2\pi} Q^{\mu\nu}_+(n',\, n) \\ &\quad \times \left\{ \frac{[f^+(\epsilon_q) - f^+(\epsilon_{q'})] + \sigma(\mu,\nu)[f^-(\epsilon_q) - f^-(\epsilon_{q'})]}{\omega - \epsilon_q + \epsilon_{q'} + i0} \right\}, \end{aligned} \tag{7}$$

with

$$\sigma(\mu,\nu) = \begin{cases} +1, & \mu,\ \nu = 00,\ 11,\ 22,\ 33,\ 01,\ 03,\ 13 \\ -1, & \mu,\ \nu = 02,\ 12,\ 23\,. \end{cases} \tag{8}$$

Equation (8) shows that for a pair plasma in which there are equal numbers of electrons and positrons, the α^{02}, α^{12} and α^{23} components of the plasma vanish. In the case of the α^{02} component, this fact does not appear to have been mentioned in the literature.

As is discussed in paper I, if one is interested in cyclotron absorption rather than cyclotron emission, the primed quantities should be reinterpreted as initial states and *vice versa*. Then on interchanging primed and unprimed quantities, equation (7) is written

$$\begin{aligned} \alpha^{\mu\nu}_{RES}(k) &= -\frac{e^3 B}{2\pi} \sum_{n=0,n'=0}^{\infty} \int \frac{dp_{||}}{2\pi} Q^{\mu\nu}_+(n',\, n) \\ &\quad \times \frac{[f^+(\epsilon_{q'}) - f^+(\epsilon_q)] + \sigma(\mu,\, \nu)[f^-(\epsilon_{q'}) - f^-(\epsilon_q)]}{\omega + \epsilon_q - \epsilon_{q'} + i0} \\ &= -\frac{e^3 B}{2\pi} \sum_{n=0,n'=0}^{\infty} \int \frac{dp_{||}}{2\pi} \Phi_2 Q^{\mu\nu}_+(n',n) \\ &\quad \times \frac{[f^+(\epsilon_{q'}) - f^+(\epsilon_q)] + \sigma(\mu,\nu)[f^-(\epsilon_{q'}) - f^-(\epsilon_q)]}{2\gamma m \left[\sigma - \frac{1}{\gamma}\left(\Omega_e - \frac{q^2}{2m}\right) - \frac{k_{||}p_{||}}{\gamma m}\right]}, \end{aligned} \tag{9}$$

with $p'_{||} = p_{||} + k_{||}$ implicit. Also equations (39) and (40) of paper I are used.

In order to further simplify the linear response 4-tensor, two assumptions are introduced. Firstly, the distributions of electrons and positrons in the initial state, denoted $f^+(\epsilon_q)$ and $f^-(\epsilon_q)$ respectively, are restricted to the ground state while the distributions of electrons and positrons in the final state, denoted $f^+(\epsilon_{q'})$

and $f^{-}(\epsilon_{q'})$ respectively, are taken to be zero. This assumption is valid for magnetic fields of order the critical field, even for ultrarelativistic particles. This is due to the fact that the synchrotron decay rate (see equation (21) in paper I) greatly exceeds the collision rate, unless the plasma is very dense (typically $\gtrsim 10^{32}\,\mathrm{m}^{-3}$). This again assumes there is no significant flux of radiation near the cyclotron frequency. Secondly, only absorption of radiation from the ground state to the first excited state $n'=1$ is considered. Therefore, using these assumptions (9) becomes

$$\alpha^{\mu\nu}_{RES}(k) = \frac{e^3 B}{2\pi}\int \frac{dp_{||}}{2\pi}\frac{[f^{+}(\epsilon_q)+\sigma(\mu,\,\nu)f^{-}(\epsilon_q)]}{2\gamma m\left[\omega-\dfrac{1}{\gamma}\left(\Omega_e-\dfrac{q^2}{2m}\right)-\dfrac{k_{||}p_{||}}{\gamma m}\right]}\Phi_2 Q^{\mu\nu}_{+}(1,\,0)\,, \qquad (10)$$

with the $Q^{\mu\nu}_{+}(1,\,0)$ given by (22)–(31) in paper I.

3. Ultrarelativistic Particle Distribution

In this section, the linear response is evaluated for the case of a thermal, ultrarelativistic distribution of particles. This corresponds to temperatures of at least several MeV. It should be noted that some authors have raised doubts as regards the attainability of an ultrarelativistic Maxwellian distribution in super-strong magnetic fields. Their argument is that the magnetic field suppresses Coulomb collisions (see e.g. Storey and Melrose 1987), except those that result in the excitation of one or both particles to a higher Landau level. Thus, unless the particles have a relative velocity above the excitation threshold (which is roughly the cyclotron energy), scattering is ineffective in exchanging energy between the particles. Thus it is difficult to see how the particles could thermalise. Although these doubts are noted, it must be pointed out that Coulomb collisions are not the only interactions experienced by the electrons and that it is likely that interactions between the electrons and radiation field will determine the form of the distribution.

To facilitate the calculation, additional assumptions are employed. Firstly, it is assumed that the plasma is tenuous, so the refractive index of the plasma is close to unity and the dispersion relation $\omega \simeq |\mathbf{k}|$ is a valid approximation. In addition, the photon frequency ω and wavevector $\mathbf{k}$ are considered to be real, which is valid if one neglects damping. Also the waves are assumed to satisfy $k_{||}^2 < \omega^2$. Secondly, the magnetic field is assumed to be about an order of magnitude smaller than the critical field, i.e. $B \lesssim 0{\cdot}1B_{cr}$, so the cyclotron energy is small compared with the electron rest energy, $\Omega_e \ll m$. Also the photon frequencies of interest are considered to be in the vicinity of the cyclotron resonance.

Consider the particle distribution function for a magnetised quantum plasma. In a magnetic field, the exact, thermal relativistic distribution function, which satisfies the normalisation condition

$$\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} dp_{||}\, g_n f(\epsilon_q) = \frac{4\pi^2 N}{\epsilon_0 m\Omega_e}\,, \qquad (11)$$

with $g_n = 2-\delta_{n0}$ the degeneracy factor for the Landau levels and N the number density of particles, is given by

$$f(\epsilon_q) = \frac{4\pi^2 N}{\epsilon_0 m \Omega_e} \frac{\exp(-\epsilon_q/T)}{2\sum_{n=0}^{\infty} g_n \epsilon_n^0 K_1(\epsilon_n^0/T)}, \tag{12}$$

where ϵ_n^0 is defined by

$$\epsilon_n^0 = (m^2 + 2neB)^{\frac{1}{2}} .$$

In equation (12) the Boltzmann constant is set to unity, $K_1(z)$ is the McDonald function of order 1 and $\epsilon_q = (m^2 + p_{||}^2 + 2neB)^{\frac{1}{2}}$ the particle energy. Thus for a distribution of particles confined to the ground state, (12) reduces to

$$f(\gamma) = \frac{4\pi^2 N}{\epsilon_0 m^2 \Omega_e} \frac{\exp(-\gamma/T^*)}{2K_1(1/T^*)}, \tag{13}$$

where $\gamma = (1+p_{||}^2/m^2)^{\frac{1}{2}}$ and $T^* = T/m$ is the dimensionless temperature. Now suppose that a fraction δ of the particles are electrons and that the remaining fraction $1-\delta$ are positrons. Then in the ultrarelativistic limit, which corresponds to

$$\gamma \simeq \frac{|p_{||}|}{m}, \tag{14}$$

the following distribution functions are introduced:

$$f^+(p_{||}) = \delta \frac{4\pi^2 N}{\epsilon_0 m^2 \Omega_e} \frac{\theta(|p_{||}| - p_c) \exp(-|p_{||}|/T)}{2K_1(1/T^*)}, \tag{15}$$

$$f^-(p_{||}) = (1-\delta) \frac{4\pi^2 N}{\epsilon_0 m^2 \Omega_e} \frac{\theta(|p_{||}| - p_c) \exp(-|p_{||}|/T)}{2K_1(1/T^*)}. \tag{16}$$

Here, N is the total number of particles, p_c is a momentum cutoff, with typically $p_c > 2m$ for an ultrarelativistic distribution, while the step function $\theta(z-a)$ which is defined by

$$\theta(z-a) = \begin{cases} 1, & z > a \\ 0, & z < a \end{cases} \tag{17}$$

ensures that no particle has momentum less than the cutoff.

Next the ultrarelativistic approximation is used to simplify the expressions for the $Q_+^{\mu,\nu}(1, 0)$ functions. Employing the ultrarelativistic approximation in equations (22)–(31) of paper I, one can show

$$Q_+^{00}(1, 0) \simeq [J_1^0]^2, \tag{18}$$

$$Q_+^{11}(1, 0) \simeq \frac{m\Omega_e}{(m^2 + p_{||}^2)} [J_0^0]^2, \tag{19}$$

$$
\begin{aligned}
Q_+^{22}(1,\,0) &= Q_+^{11}(1,\,0)\,, && (20)\\
Q_+^{33}(1,\,0) &\simeq \frac{p_{||}^2}{m^2+p_{||}^2}[J_1^0]^2\,, && (21)\\
Q_+^{01}(1,\,0) &\simeq \frac{(2eB)^{\frac{1}{2}}|p_{||}|}{2(m^2+p_{||}^2)}[J_0^0][J_1^0]\,, && (22)\\
Q_+^{02}(1,\,0) &= iQ_+^{01}(1,\,0)\,, && (23)\\
Q_+^{03}(1,\,0) &\simeq \frac{p_{||}|p_{||}|}{m^2+p_{||}^2}[J_1^0]^2\,, && (24)\\
Q_+^{12}(1,\,0) &= iQ_+^{11}(1,\,0)\,, && (25)\\
Q_+^{13}(1,\,0) &\simeq \frac{(2eB)^{\frac{1}{2}}p_{||}}{2(m^2+p_{||}^2)}[J_0^0][J_1^0]\,, && (26)\\
Q_+^{23}(1,\,0) &= iQ_+^{11}(1,\,0)\,. && (27)
\end{aligned}
$$

Strictly, a *quasi-ultrarelativistic* approximation is used in writing down the results (18)–(27), as the electron rest mass m is kept as a correction in the energy denominators. However, terms of order $k_{||}$, Ω_e are ignored, compared with terms of order m, since $k_{||} < \omega \leq \Omega_e \ll m$.

4. Roots of the Resonant Denominator

In this section, the resonant denominator is examined in detail within the ultrarelativistic approximation. Consider then the approximation (14) in the resonant denominator of equation (10), which one can write as

$$
g(p_{||}) = \frac{|p_{||}|}{m} - n_{||}\frac{p_{||}}{m} - \hat{\Omega}_{(-)}\,, \tag{28}
$$

where $n_{||} = k_{||}/\omega$ is the refractive index and

$$
\hat{\Omega}_{(-)} = \frac{\Omega_e}{\omega} - \frac{q^2}{2m\omega}\,. \tag{29}
$$

The roots of the resonant denominator are obtained by solving the equation $g(p_{||}) = 0$. In order to obtain more accurate results for the resonant momenta, the approach used here is to calculate the momenta, firstly solving (28), then secondly, obtaining a correction to the roots based on using the approximation

$$
\gamma \simeq \frac{|p_{||}|}{m}\left(1 + \frac{m^2}{p_{||}^2}\right). \tag{30}
$$

The reason that the approximation (3) is not used from the outset is explained shortly.

Consider the solutions of the equation $g(p_{||}) = 0$, with $g(p_{||})$ given by (28).

1. $p_{||} > 0$. In this case the solution is given by

$$\frac{p_+^0}{m} = \frac{\hat{\Omega}_{(-)}}{1 - n_{||}}. \tag{31}$$

2. $p_{||} < 0$. In this case the solution is given by

$$\frac{p_-^0}{m} = \frac{\hat{\Omega}_{(-)}}{1 + n_{||}}, \tag{32}$$

where the 0 superscript indicates these are the lowest order solutions. To obtain a correction to resonant roots, consider the equation

$$z + \frac{a}{2z} = b, \quad z \gg 1. \tag{33}$$

Then if one takes the lowest order solution to be given by $z_0 = b$, the next order solution is found by substituting the lowest order solution into the second term on the left hand side of (33), treating this term as a perturbation. Therefore, one has

$$z_1 + \frac{a}{2z_0} = b,$$

which implies

$$z_1 = b - \frac{a}{2z_0} = z_0\left(1 - \frac{a}{2z_0^2}\right). \tag{34}$$

Hence, using the solutions (31) and (32) in equation (34), one obtains

1. $p_{||} > 0$. In this case the solution is given by

$$\frac{p_+}{m} = \frac{\hat{\Omega}_{(-)}}{1 - n_{||}}\left\{1 - \frac{(1 - n_{||})^2}{2\hat{\Omega}_{(-)}^2}\right\}. \tag{35}$$

2. $p_{||} < 0$. In this case the solution is given by

$$\frac{p_-}{m} = -\frac{\hat{\Omega}_{(-)}}{1 + n_{||}}\left\{1 - \frac{(1 + n_{||})^2}{2\hat{\Omega}_{(-)}^2}\right\}, \tag{36}$$

where now the 0 superscript is omitted.

The above method is preferred to using equation (30) from the outset to calculate the resonant roots, as (30) leads to four solutions, of which at most two are valid. This occurs because (30) breaks down for small $p_{||}/m$ and thus solutions are obtained in the nonrelativistic regime, which are clearly not valid and thus must be discarded. Not only is this approach unnecessarily complicated, it does not result in much improvement in the values for the resonant roots when compared with (35) and (36).

It is interesting to note that a cutoff frequency is not predicted by the solutions (35) and (36) as they are always real. However, one can show they are consistent with the cutoff frequency given by (A3) of paper I, which shows for $n' = 1$ that the cutoff is below the cyclotron frequency. If the equation $g(p_{||}) = 0$ is treated as a function of frequency rather than momentum, then one may obtain the frequency of a photon that is absorbed by a resonant particle. This yields

$$\omega_{res}(\theta) = \frac{\sqrt{p_{||}^2(1 - cos\theta)^2 + 2m\Omega_e \sin^2\theta} - p_{||}(1 - \cos\theta)}{\sin^2\theta}, \quad p_{||} > 0, \tag{37}$$

$$\omega_{res}(\theta) = \frac{\sqrt{p_{||}^2(1 + \cos\theta)^2 + 2m\Omega_e \sin^2\theta} + p_{||}(1 + \cos\theta)}{\sin^2\theta}, \quad p_{||} < 0, \tag{38}$$

where the relation $k_{||} \simeq \omega \cos\theta$ is used. Equations (37) and (38) are only valid for resonant momenta that are large compared with the electron rest mass m. If they are used for a cold plasma, they imply an absorption frequency of $\sqrt{2m\Omega_e}/\sin\theta \gg \Omega_e$. Thus, for a particle with resonant momentum $p_{res} \gg m$ then (37) indicates the absorption takes place at a frequency

$$\begin{aligned} \omega_{res}(\theta) &\simeq \frac{\Omega_e}{p^*_{res}(1 - \cos\theta)}, \\ &\simeq \frac{\Omega_e}{\gamma(1 - \cos\theta)} \ll \Omega_e, \end{aligned}$$

where $p^*_{res} = p_{res}/m$ and γ is the Lorentz factor. Thus the absorption takes place well below the cyclotron frequency, with the highest absorption frequency

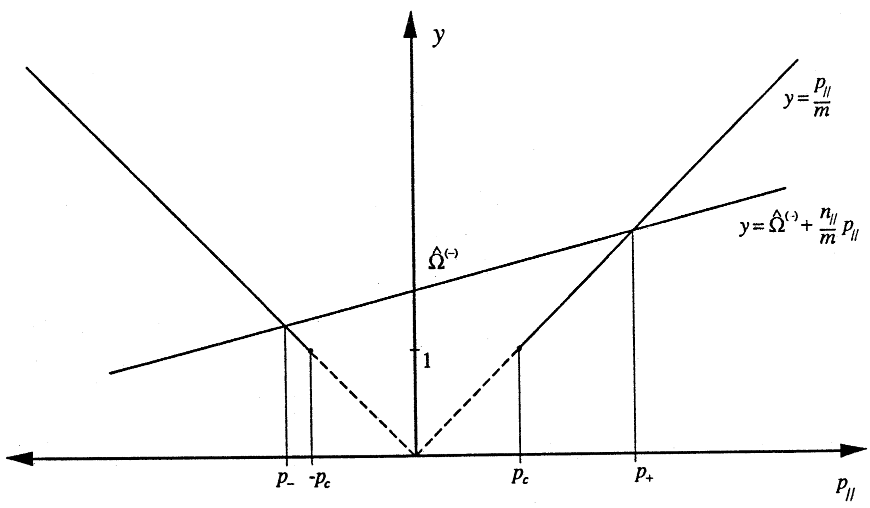

Fig. 2. Location of ultrarelativistic resonant momentum zeros.

being Ω_e/p_c^*. This is a consequence of the monotonic decrease in the absorption frequency as the particle momentum increases. Relativistically, as the momentum increases the energy eigenvalues ϵ_q, $\epsilon_{q'}$ of the initial and final states tend to merge and thus the frequency required to satisfy the resonance condition $\omega+\epsilon_q-\epsilon_{q'}=0$ decreases.

Returning to the expressions for the resonant momenta, one may use the graphical technique of Section 4*a* in paper I to visually locate the roots and classify them according to the values of $\hat{\Omega}_{(-)}$. This also allows one to determine the conditions under which the roots are valid and thus absorption takes place. The roots given by (35) and (36) can be visualised as the points of intersection of $y=|p_{||}|/m$ with the straight line $y=\hat{\Omega}_{(-)}+n_{||}p_{||}/m$; see Fig. 2. To simplify matters it is assumed, without loss of generality, that $k_{||}>0$ and $\omega>0$, hence, $n_{||}>0$. There are four separate cases to analyse depending on the values of $\hat{\Omega}_{(-)}$.

Case 1:

$$\hat{\Omega}_{(-)}\le 0,\begin{cases}\text{no valid roots for } n_{||}\le 1\\ p_+ \text{ is a valid root for } 1<n_{||}<1+|\hat{\Omega}_{(-)}|/p_c^*\\ \text{no valid roots for } n_{||}>1+|\hat{\Omega}_{(-)}|/p_c^*\end{cases}$$

Case 2:

$$0<\hat{\Omega}_{(-)}\le p_c^*,\begin{cases}\text{there are no valid roots for } n_{||}<1-\hat{\Omega}_{(-)}/p_c^*\\ p_+ \text{ is a valid root for } 1-\hat{\Omega}_{(-)}/p_c^*<n_{||}<1\\ \text{there are no valid roots for } n_{||}>1\end{cases}$$

Case 3:

$$p_c^*<\hat{\Omega}_{(-)}<2p_c^*,\begin{cases}p_-,p_+ \text{ are valid roots for } 0\le n_{||}<\hat{\Omega}_{(-)}/p_c^*-1\\ p_+>0 \text{ is a valid root for } \hat{\Omega}_{(-)}/p_c^*-1<n_{||}<1\\ \text{there are no valid roots for } n_{||}>1\end{cases}$$

Case 4:

$$\hat{\Omega}_{(-)}>2p_c^*,\begin{cases}p_-,p_+ \text{ are valid roots for } 0\le n_{||}<1\\ p_-<0 \text{ is a valid root for } 1<n_{||}<\hat{\Omega}_{(-)}/p_c^*-1\\ \text{there are no valid roots for } n_{||}>\hat{\Omega}_{(-)}/p_c^*-1\end{cases}$$

For the above conditions, the dimensionless momentum cutoff $p_c^*=p_c/m$ is used.

If one is in a regime where absorption takes place, then for (35) and (36) to be consistent with the ultrarelativistic approximation, it is necessary that $p_c^*(1+n_{||})\lesssim\hat{\Omega}_{(-)}$. Now, for the case of interest here, one must have $n_{||}<1$. Inspection of case 3 and case 4 above shows both roots contribute to cyclotron absorption provided $0\le n_{||}<\hat{\Omega}_{(-)}/p_c^*-1$ for $p_c^*<\hat{\Omega}_{(-)}<2p_c^*$, or $0\le n_{||}<1$ for $\hat{\Omega}_{(-)}>2p_c^*$. Inspection of case 2 shows that absorption can also occur for $0<\hat{\Omega}_{(-)}<p_c^*$ provided $1-\hat{\Omega}_{(-)}/p_c^*<n_{||}<1$. In this case however, only one of the roots, p_+, contributes. Furthermore, to be consistent with the assumption of photon frequencies of order the cyclotron frequency or less, one must have the restriction $1 \lesssim \hat{\Omega}_{(-)}$, which is shown in paper I. Note that, in the parameter regime $p_c^* < \hat{\Omega}_{(-)}$, the solutions obtained in (35) and (36) are quite accurate when compared with the exact solutions given in equation (69) of paper I. For example, for $p_c^* = 2$ (approximately the smallest acceptable value of the cutoff) and $n_{||} = 0{\cdot}5$ there is only a 2% error between the two results.

5. Introduction of Relativistic Plasma Dispersion Functions

In this section, the integrals over parallel momentum occuring in the response tensor are performed in terms of relativistic plasma dispersion functions, known as Dnestrovskii functions.

The first step is to write down the linear response 4-tensor given by equation (10) using the results of Sections 3 and 4. Using equations (15), (16) and (28) in equation (10) one obtains

$$\begin{aligned} \alpha^{\mu\nu}_{RES}(k) \simeq\ & \frac{\omega_p^2}{2K_1(\beta^*)}\frac{m}{\omega}\int_{-\infty}^{\infty} dp_{||}^* \\ & \times \frac{[\delta + \sigma(\mu,\nu)(1-\delta)]\theta(|p_{||}^*| - p_c^*)|p_{||}^*|\exp(-\beta^*|p_{||}^*|)}{[|p_{||}^*| - n_{||}p_{||}^* - \hat{\Omega}_{(-)}]} Q_+^{\mu\nu}(1,0)\,, \end{aligned} \tag{39}$$

$\beta^* = 1/T^*$ and the * indicates the quantity is dimensionless, being written in units of m. Also the function $\Phi_2 = \omega + \epsilon_q + \epsilon_{q'}$ that appears in (10) is approximated by $\Phi_2 \simeq 2|p_{||}|$. Note that the casual condition has not yet been applied to the right hand side of (39); this is performed later.

If one refers to equations (18)–(27) for the $Q_+^{\mu\nu}(1,0)$ functions that appear in (39), then one can verify there are two classes of integral to evaluate, which can be written as

$$G(z) = \int_z^{\infty} dy \frac{y\exp(-\beta y)}{y-a}\,, \tag{40}$$

and

$$V_\ell(z) = \int_z^{\infty} dy \frac{y^\ell \exp(-\beta y)}{(1+y^2)(y-a)}\,, \quad \ell = 1,2,3\,. \tag{41}$$

If one uses the partial fraction decomposition

$$\frac{1}{(1+y^2)(y-a)} = -\frac{1}{2(1-ia)}\frac{1}{y+i} - \frac{1}{2(1+ia)}\frac{1}{y-i} + \frac{1}{(1+a^2)}\frac{1}{y-a}\,, \tag{42}$$

then (40) and (41) are reduced to the single class of integral

$$G_\ell(z) = \int_z^{\infty} dy \frac{y^\ell \exp(-\beta y)}{y-a}, \quad \ell = 0,1,2,3\,. \tag{43}$$

This integral for arbitrary ℓ may be evaluated in terms of the integral for $\ell = 0$ by using the identity

$$\int_z^\infty dy \frac{y^\ell \exp(-\beta y)}{y-a} = (-1)^\ell \frac{d^\ell}{d\beta^\ell} \int_z^\infty dy \frac{\exp(-\beta y)}{y-a} . \tag{44}$$

The integral $G_0(z)$ follows from equation (3.352.5) of Gradshteyn and Ryshik (1965), which yields

$$G_0(z) = -e^{-\beta a} Ei^\star(\beta a - \beta z), \quad a > z \tag{45}$$

$$= -e^{-\beta a} Ei(\beta a - \beta z), \quad a < z \tag{46}$$

where $Ei(x)$ is related to the standard exponential integral function $E_1(x)$ by $Ei(x) = -E_1(-x)$ and $\star$ denotes the complex conjugate. The result presented in (45) is used if absorption takes place. In this case the exponential integral is complex and the imaginary part is associated with the absorptive part of the response. The result (46) is the form used if absorption does not take place; in this case the exponential integral is real. In the work that follows it is assumed that (45) applies.

The exponential integral function is expressed in terms of the Dnestrovskii function $F_q(z)$ (Dnestrovskii *et al.* 1964) by making use of equation (28) of Robinson (1986), which gives

$$F_q(z) = e^z \int_z^\infty dy\, y^{-q} e^{-u} = e^z E_1(z), \quad q \geq 0 \tag{47}$$

for $Re(z) > 0$, with appropriate analytic continuation for $Re(z) \leq 0$. Therefore, $F_1(z) = e^z E_1(z) = -e^z Ei(-z)$ and (45) can be written

$$G_0(z) = e^{-\beta z} F_1^\star(\beta z - \beta a) . \tag{48}$$

Repeated application of (44) on (48), along with the differential equation

$$\frac{dF_q(z)}{dz} = \left(1 - \frac{1-q}{z}\right) F_q(z) - \frac{1}{z} ,$$

given in equation (20) of Robinson (1986), allows one to obtain the following:

$$G_1(z) = e^{-\beta z} \left[a F_1^\star(\beta z - \beta a) + \frac{1}{\beta} \right] , \tag{49}$$

$$G_2(z) = e^{-\beta z} \left[a^2 F_1^\star(\beta z - \beta a) + \frac{z+a}{\beta} + \frac{1}{\beta^2} \right] , \tag{50}$$

$$G_3(z) = e^{-\beta z} \left[a^3 F_1^\star(\beta z - \beta a) + \frac{z^2 + az + a^2}{\beta} + \frac{2z+a}{\beta^2} + \frac{2}{\beta^3} \right] . \tag{51}$$

Since the argument of the Dnestrovskii function which appears in the response tensor is negative when there is absorption (see Section 6), the integral appearing in equation (47) cannot be used, as it diverges. However, the integral of (47) may be explicitly evaluated to yield

$$E_1(z) = -\gamma - \ln(z) - \sum_{n=1}^{\infty} \frac{(-z)^n}{nn!}\,, \tag{52}$$

where $\gamma = 0\cdot 5772\ ...$ is Euler's constant. This function is analytic everywhere except $z = 0$ and thus can be used for $Re(z) < 0$. One is free to choose the branch cut for the logarithm anywhere in the complex plane and here it is convenient to choose the cut along

$$-\pi < \arg z \leq \pi\,. \tag{53}$$

Hence, using this definition one has analytically extended the definition of the Dnestrovskii function $F_1(z)$ to $Re(z) \leq 0$.

6. Specific Results for the Linear Response 4-Tensor

Using the results of the preceding section, it is now possible to write down explicit expressions for the linear response 4-tensor given in equation (39). Since some of the intermediate steps required in obtaining the final form of the response tensor are rather tedious, the details are left to Appendix B and only the final results are presented here. One finds:

$$\begin{aligned}\alpha^{00}_{RES}(k) \simeq{}& \frac{\omega_p^2 e^{-\beta^* p_c^*}}{2K_1(\beta^*)} \frac{m}{\omega} [J_1^0(k_\perp^2/2eB)]^2 \left\{ \frac{p_+^* F_1^\star[\beta(p_c^* - p_+^*)]}{1-\cos\theta} - \frac{p_-^* F_1^\star[\beta(p_c^* + p_-^*)]}{1+\cos\theta} \right.\\ &\left. + \frac{2}{\beta \sin^2\theta} \right\},\end{aligned} \tag{54}$$

$$\begin{aligned}\alpha^{11}_{RES}(k) \simeq{}& \frac{\omega_p^2 e^{-\beta^* p_c^*}}{4K_1(\beta^*)} \frac{\Omega_e}{m} [J_0^0(k_\perp^2/2eB)]^2 \\ &\times \left\{ -\frac{[-2p_+^* \mathrm{Re}\{F_1(\phi)\} - 2\mathrm{Im}\{F_1(\phi)\} + p_+^* F_1^\star(Z^+)]}{2(1-\cos\theta)[1+(p_+^*)^2]} \right.\\ &\left. + \frac{[2p_-^* \mathrm{Re}\{F_1(\phi)\} - 2\mathrm{Im}\{F_1(\phi)\} - p_-^* F_1^\star(Z^-)]}{2(1+\cos\theta)[1+(p_-^*)^2]} \right\},\end{aligned} \tag{55}$$

$$\alpha^{22}_{RES}(k) = \alpha^{11}_{RES}(k)\,, \tag{56}$$

$$\begin{aligned}\alpha^{33}_{RES}(k) \simeq{}& \frac{\omega_p^2 e^{-\beta^* p_c^*}}{2K_1(\beta^*)} \frac{m}{\omega} [J_1^0(k_\perp^2/2eB)]^2 \\ &\times \left\{ \frac{[2p_+^* \mathrm{Re}\{F_1(\phi)\} + 2\mathrm{Im}\{F_1(\phi)\} + (p_+^*)^3 F_1^\star(Z^+)]}{2(1-\cos\theta)[1+(p_+^*)^2]} \right.\\ &\left. + \frac{[-2p_-^* \mathrm{Re}\{F_1(\phi)\} + 2\mathrm{Im}\{F_1(\phi)\} - (p_-^*)^3 F_1^\star(Z^-)]}{2(1+\cos\theta)[1+(p_-^*)^2]} + \frac{1}{\beta\sin^2\theta} \right\},\end{aligned} \tag{57}$$

$$\alpha^{01}_{RES}(k) \simeq \frac{\omega_p^2 e^{-\beta^* p_c^*}}{2K_1(\beta^*)}\frac{m}{\omega}[J_0^0(k_\perp^2/2eB)][J_1^0(k_\perp^2/2eB)]\left(\frac{m\Omega_e}{2}\right)^{\frac{1}{2}}$$
$$\times\left\{\frac{[\mathrm{Re}\{F_1(\phi)\} - p_+^*\mathrm{Im}\{F_1(\phi)\} + (p_+^*)^2 F_1^\star(Z^+)]}{(1-\cos\theta)[1+(p_+^*)^2]}\right.$$
$$\left.+\frac{[\mathrm{Re}\{F_1(\phi)\} + p_-^*\mathrm{Im}\{F_1(\phi)\} + (p_-^*)^2 F_1^\star(Z^-)]}{(1+\cos\theta)[1+(p_-^*)^2]}\right\}, \tag{58}$$

$$\alpha^{02}_{RES}(k) = i(1-2\delta)\alpha^{01}_{RES}(k), \tag{59}$$

$$\alpha^{03}_{RES}(k) \simeq \frac{\omega_p^2 e^{-\beta^* p_c^*}}{2K_1(\beta^*)}\frac{m}{\omega}[J_1^0(k_\perp^2/2eB)]^2$$
$$\times\left\{\frac{[2p_+^*\mathrm{Re}\{F_1(\phi)\} + 2\mathrm{Im}\{F_1(\phi)\} + (p_+^*)^3 F_1^\star(Z^+)]}{2(1-\cos\theta)[1+(p_+^*)^2]}\right. \tag{60}$$
$$\left.-\frac{[-2p_-^*\mathrm{Re}\{F_1(\phi)\} + 2\mathrm{Im}\{F_1(\phi)\} - (p_-^*)^3 F_1^\star(Z^-)]}{2(1+\cos\theta)[1+(p_-^*)^2]} + \frac{\cos\theta}{\beta\sin^2\theta}\right\},$$

$$\alpha^{12}_{RES}(k) = i(1-2\delta)\alpha^{11}_{RES}(k), \tag{61}$$

$$\alpha^{13}_{RES}(k) \simeq \frac{\omega_p^2 e^{-\beta^* p_c^*}}{2K_1(\beta^*)}\frac{m}{\omega}[J_0^0(k_\perp^2/2eB)][J_1^0(k_\perp^2/2eB)]\left(\frac{m\Omega_e}{2}\right)^{\frac{1}{2}}$$
$$\times\left\{\frac{[\mathrm{Re}\{F_1(\phi)\} - p_+^*\mathrm{Im}\{F_1(\phi)\} + (p_+^*)^2 F_1^\star(Z^+)]}{(1-\cos\theta)[1+(p_+^*)^2]}\right.$$
$$\left.-\frac{[\mathrm{Re}\{F_1(\phi)\} + p_-^*\mathrm{Im}\{F_1(\phi)\} + (p_-^*)^2 F_1^\star(Z^-)]}{(1+\cos\theta)[1+(p_-^*)^2]}\right\}, \tag{62}$$

$$\alpha^{23}_{RES}(k) = -i(1-2\delta)\alpha^{13}_{RES}(k). \tag{63}$$

In equations (54)–(63), the following quantities are introduced:

$$\phi = \beta^*(p_c^* + i), \tag{64}$$

$$Z^\pm = \beta^*(p_c^* \mp p_\pm^* + i0). \tag{65}$$

In writing down these results it is assumed that both resonant momenta contribute to absorption. In this case, one can see from (65) that $Z^\pm$ has negative real argument, since the resonant momenta satisfy $|p_\pm| > p_c$. This is the reason the expression in (47) is analytically continued. The small and positive, imaginary part of $Z^\pm$, which arises from the causal condition, ensures that $Z^\pm$ lies on the branch defined by equation (53). It is a simple matter to extend these results to frequencies where, either one, or both roots do not contribute to the absorption. Firstly, ascertain which of the roots do not contribute, from the values of $n_{||}$ and $\hat{\Omega}_{(-)}$. Then for the Dnestrovskii function which contains this root in its argument, one removes the complex conjugation sign as the function is now real valued. The terms for which this is performed only contribute to the dispersion of the waves.

The symmetry property $F_q^\star(z) = F_q(z^\star)$, where q is an integer, given in equation (57a) of Robinson (1986), is used to write

$$F_1(\phi) + F_1(\phi^\star) = 2\mathrm{Re}\{F_1(\phi)\}\,, \tag{66}$$

$$F_1(\phi) - F_1(\phi^\star) = 2i\mathrm{Im}\{F_1(\phi)\}\,. \tag{67}$$

The resonant momenta $p_\pm$ appearing in the linear response 4-tensor have values given by (35) and (36). Equations (54)–(63) are valid for all angles of propagation except $\theta = 0$ as here $n_{||} = 1$ and the response diverges. It is simple to verify that in the case of perpendicular propagation, only the 03, 13 and 23 components of the response vanish, as is true in the semirelativistic approximation. If the response vanishes in one frame, it must vanish in all frames.

Before closing this section there is an interesting question that arises, which although of direct relevance to this work, is a separate problem in its own right and would need to be examined in more detail in any further work. Basically the question pertains to the importance of the momentum cutoff p_c. The results presented above are valid in a frame in which all the particles of the pair plasma are ultrarelativistic. Suppose, however, that another (primed) frame is chosen, moving parallel to the magnetic field in which $p_c' = 0$, so that now one has nonrelativistic particles present as well. In order, to quantify the contribution to the response of these nonrelativistic particles, consider the integral I given by

$$I = \int_0^\infty dpF(p)g(p)\,,$$

where p denotes the parallel momentum. Then I represents the typical integral appearing in the response, with $F(p)$ representing the distribution function. Also it is convenient to introduce the integrals I_{NR}, I_{UR} and I_{NR}' where

$$\begin{aligned} I_{NR} &= \int_0^{mc} dpF(p)g(p)\,, \\ I_{UR} &= \int_{mc}^\infty dpF(p)g(p)\,, \\ I_{NR}' &= \int_0^{mc} dp[F(p)g(p)]_{UR}\,. \end{aligned}$$

Then one has

$$I = I_{NR} + I_{UR}\,,$$

while I_{NR}' measures the error obtained in extrapolating the ultrarelativistic approximation into the nonrelativistic regime. If one finds that

$$I_{NR} \ll I_{UR}, \quad I_{NR}' \ll I_{UR}\,,$$

then I_{UR} is a good approximation to I, otherwise one would need to take into account the contribution of nonrelativistic particles to the response. It must be stressed that this issue is not crucial in the context of the work presented here. The results derived above for an ultrarelativistic pair plasma are still valid, so it is a matter of determining whether any other contribution to the response is required. If this is the case, the results presented in paper I for a nonrelativistic plasma may be used in evaluating the total response.

7. Summary

In this paper, the linear response 4-tensor of a strongly magnetised electron–positron plasma is derived. The calculations are performed using the methodology of quantum plasmadynamics as is the case in paper I. The response 4-tensor is evaluated in the case of an ultrarelativistic thermal distribution of pairs in the ground state, with a momentum cutoff. The distribution is also chosen to allow the relative fractions of positrons and electrons to be varied. This allows one to see how the wave properties change as the plasma goes from pure electron gas through pair gas to pure positron gas. It is also easy to include streaming motion into the distribution, as well as modifying it so the pairs move in either the same direction, or opposite directions. This could be of importance in some models of radio pulsars and gamma-ray bursters.

Although the ultrarelativistic approximation is employed, corrections are included in the resonant denominator to more accurately locate the resonant momenta. For particles with energy much larger than the electron rest mass energy, it is found that absorption takes place well below the cyclotron frequency. A graphical technique is used to enable one not only to classify the resonant momenta according to the value of a certain parameter, but also to determine the conditions required to ensure the roots are consistent with the ultrarelativistic approximation.

Explicit analytical results are obtained for the linear response 4-tensor in terms of a relativistic plasma dispersion function known as a Dnestrovskii function. As is the case with the Shkarofsky functions introduced in paper I, the analytic properties of the Dnestrovskii functions have been extensively studied (Robinson 1986). They are particularly useful for studying the wave properties of the plasma, such as dispersion, absorption and polarisation, without resorting to complicated numerical schemes. The results obtained for the response are valid for all angles of propagation except $\theta = 0°$ and for frequencies up to approximately Ω_e/p_c^*, although these results can be extended to frequencies $\omega \simeq \Omega_e$.

Acknowledgments

The author would like to thank Don Melrose and Peter Robinson for many useful discussions during the undertaking of this paper. Also I thank a referee for useful comments.

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Appendix A: Symmetry Property of $Q_{+}^{\mu\nu}(n',n)$

In this appendix it is shown how the result in equation (5) is obtained, by performing the calculation in detail for the 01 component, as this is the most tedious. From equation (10) in paper I, one has

$$\begin{aligned} Q_{\pm}^{01}(n,n') &= -\frac{p_{n'}}{2\epsilon_{q'}}[(J_{n-n'}^{n'-1}J_{n-n'-1}^{n'}) + (J_{n-n'}^{n'}J_{n-n'+1}^{n'-1})] \\ &\quad -\frac{p_n}{2\epsilon_q}[(J_{n-n'}^{n'-1}J_{n-n'+1}^{n'-1}) + (J_{n-n'}^{n'}J_{n-n'-1}^{n'})]\,. \end{aligned} \tag{A1}$$

The symmetry relation given by equation (4) implies

$$\begin{aligned} J_{n-n'}^{n'-1} &= (-1)^{n-n'}J_{n'-n}^{n-1}, \quad J_{n-n'-1}^{n'} = (-1)^{n-n'-1}J_{n'-n+1}^{n-1}, \\ J_{n-n'}^{n'} &= (-1)^{n-n'}J_{n'-n}^{n}, \quad J_{n-n'+1}^{n'-1} = (-1)^{n-n'+1}J_{n'-n-1}^{n}\,. \end{aligned} \tag{A2}$$

Using the relations (A2) in (A1), one has

$$\begin{aligned} Q_{\pm}^{01}(n,n') &= -\frac{p_{n'}}{2\epsilon_{q'}}[(-1)(J_{n'-n}^{n-1}J_{n'-n+1}^{n-1}) + (-1)(J_{n'-n}^{n}J_{n'-n-1}^{n})] \\ &\quad -\frac{p_n}{2\epsilon_q}[(-1)(J_{n-n'}^{n-1}J_{n'-n-1}^{n}) + (-1)(J_{n'-n}^{n}J_{n'-n+1}^{n-1})] \\ &= \frac{p_n}{2\epsilon_q}[(J_{n'-n}^{n-1}J_{n'-n-1}^{n}) + (J_{n'-n}^{n}J_{n'-n+1}^{n-1})] \\ &\quad +\frac{p_{n'}}{2\epsilon_{q'}}[(J_{n'-n}^{n-1}J_{n'-n+1}^{n-1}) + J_{n'-n}^{n}J_{n'-n-1}^{n})] \end{aligned}$$

$$= -Q_+^{01}(n', n).$$

The results for the other components follow similarly.

Appendix B: Derivation of Linear Response 4-Tensor

In this appendix some of the details leading to the results of equations (54)–(63) are presented. The calculation is performed in detail for one particular component. Consider then the 00 component of the linear response 4-tensor given by equation (39):

$$\alpha_{RES}^{00}(k) = \frac{\omega_p^2}{2K_1(\beta^*)}\frac{m}{\omega}\int_{-\infty}^{\infty} dp_{||}^* \frac{[\delta+(1-\delta)]\theta(|p_{||}^*|-p_c^*)|p_{||}^*|\exp(-\beta^*|p_{||}^*|)}{[|p_{||}^*|-n_{||}p_{||}^*-\hat{\Omega}_{(-)}]}Q_+^{00}(1, 0). \tag{B1}$$

From (18) one has

$$Q_+^{00}(1, 0) = [J_1^0(k_\perp^2/2eB)]^2,$$

so that

$$\begin{aligned}
\alpha_{RES}^{00}(k) &\simeq \frac{\omega_p^2}{2K_1(\beta^*)}\frac{m}{\omega}[J_1^0(k_\perp^2/2eB)]^2\int_{-\infty}^{\infty} dp_{||}^*\frac{\theta(|p_{||}^*|-p_c^*)|\exp(-\beta^*|p_{||}^*|)}{[|p_{||}^*|-n_{||}p_{||}^*-\hat{\Omega}_{(-)}]}\\
&= \frac{\omega_p^2}{2K_1(\beta^*)}\frac{m}{\omega}[J_1^0(k_\perp^2/2eB)]^2\left\{\int_{-\infty}^{-p_c^*} dp_{||}^*\frac{-p_{||}^*\exp(\beta^*p_{||}^*)}{-p_{||}^*(1+n_{||})-\hat{\Omega}_{(-)}}\right.\\
&\quad\left.+\int_{p_c^*}^{\infty} dp_{||}^*\frac{p_{||}^*\exp(-\beta^*p_{||}^*)}{p_{||}^*(1-n_{||})-\hat{\Omega}_{(-)}}\right\}\\
&= \frac{\omega_p^2}{2K_1(\beta^*)}\frac{m}{\omega}[J_1^0(k_\perp^2/2eB)]^2\left\{\int_{-\infty}^{-p_c^*} dp_{||}^*\frac{-p_{||}^*\exp(\beta^*p_{||}^*)}{(1+n_{||})(-p_{||}^*+p_-^*+i0)}\right.\\
&\quad\left.+\int_{p_c^*}^{\infty} dp_{||}^*\frac{p_{||}^*\exp(-\beta^*p_{||}^*)}{(1-n_{||})(p_{||}^*-p_+^*+i0)}\right\},
\end{aligned} \tag{B2}$$

where in (B2) the division into positive and negative momentum parts arises from the modulus function and the resonant momenta $p_\pm^*$ are now understood to be given by equations (35) and (36). Also, the causal condition as given in equation (77) of paper I, is applied to the denominators in (B2). Next, the first integral in (B2) is written over positive momentum, so that one has

$$\alpha_{RES}^{00}(k) \simeq \frac{\omega_p^2}{2K_1(\beta^*)}\frac{m}{\omega}[J_1^0(k_\perp^2/2eB)]^2\left\{\int_{\infty}^{p_c^*} -dp_{||}^*\frac{p_{||}^*\exp(-\beta^*p_{||}^*)}{(1+n_{||})(p_{||}^*+p_-^*+i0)}\right.$$

$$+\int_{p_c^*}^{\infty} dp_{||}^* \frac{p_{||}^* \exp(-\beta^* p_{||}^*)}{(1-n_{||})(p_{||}^* - p_+^* + i0)}\Bigg\}$$

$$= \frac{\omega_p^2}{2K_1(\beta^*)}\frac{m}{\omega}[J_1^0(k_\perp^2/2eB)]^2 \int_{p_c^*}^{\infty} dp_{||}^* p_{||}^* \exp(-\beta^* p_{||}^*)\Bigg\{\frac{1}{(1+n_{||})(p_{||}^* + p_-^* + i0)}$$

$$+\frac{1}{(1-n_{||})(p_{||}^* - p_+^* + i0)}\Bigg\}. \tag{B3}$$

The integral over $p_{||}^*$ may be performed using the result (49) for $G_1(z)$, if one makes the identification $z = p_c^*$ along with $a = -p_-^*$ in the first term in (B3) and $a = p_+^*$ in the second term. This yields

$$\alpha_{RES}^{00}(k) \simeq \frac{\omega_p^2}{2K_1(\beta^*)}\frac{m}{\omega}[J_1^0(k_\perp^2/2eB)]^2 e^{-\beta^* p_c^*}\Bigg\{\frac{p_+^* F_1^\star[\beta(p_c^* - p_+^*)] + 1/\beta^*}{1-\cos\theta}$$

$$-\frac{p_-^* F_1^\star[\beta(p_c^* + p_-^*)] + 1/\beta^*}{1+\cos\theta}\Bigg\},$$

$$= \frac{\omega_p^2}{2K_1(\beta^*)}\frac{m}{\omega}[J_1^0(k_\perp^2/2eB)]^2 e^{-\beta^* p_c^*}\Bigg\{\frac{p_+^* F_1^\star[\beta(p_c^* - p_+^*)]}{1-\cos\theta}$$

$$-\frac{p_-^* F_1^\star[\beta(p_c^* + p_-^*)]}{1+\cos\theta} + \frac{2}{\beta \sin^2\theta}\Bigg\}, \tag{B4}$$

where $n_{||} = k_{||}/\omega \approx \cos\theta$ is used and it is to be understood that the arguments $Z^{\pm} = \beta(p_c^* \mp p_{\pm}^*)$ of the Dnestrovskii function have an infinitesimal, positive imaginary part, arising from the causal condition. The other components are treated similarly, however, one needs to use (42) in addition to (49), (50) and (51) as well as (66) and (67).

Manuscript received 6 November 1991, accepted 12 March 1992