Aust. J. Phys., 1996, **49**, 819–53

Vector Quantum Field Theory of Atoms: Nonlinear Atom Optics and Bose–Einstein Condensate*

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Abstract

The recent experimental progress in laser cooling and trapping of neutral atoms brings the atomic samples into the ultracold regime where the bosonic atoms and fermionic atoms are expected to have different dynamic behaviours in the laser fields. In this paper we systematically introduce the theoretical study of interaction of an ultracold atomic ensemble with a light wave in the frame of a vector quantum field theory. The many-body quantum correlation in the ultracold regime of atom optics is studied in terms of vector quantum field theory. A general formalism of nonlinear atom optics for a coherent atomic beam is developed.

1. Introduction

In this paper we introduce a new theoretical method to describe the interaction of atoms with photons in the frame of a vector quantum field theory. In order to understand the motivation and reason to develop such a field-theoretical description of atoms, we give a brief review of the development of the theory of interaction of matter with light in this section.

(*1a*) *Semiclassical Theory and Full Quantum Theory*

Atoms and photons have played the central roles in our study of the interaction of matter with light. In the semiclassical theory, the light field is treated as a classical quantity whereas the motion of the electrons of the atoms is treated by means of quantum theory. This theory successfully explains the discrete spectral lines of atoms and forms the basis of the semiclassical laser theory (Sargent *et al.* 1974). However, such a semiclassical theory fails to explain spontaneous emission by atoms, which has to be phenomenologically introduced. The successful treatment of spontaneous emission leads to the development of a fully quantum mechanical theory in which the light field is quantised so that the vacuum photon fluctuations can be taken into account. Such a fully quantum mechanical theory of light and matter not only exactly describes the quantum statistical properties of the laser, but also it results in the development of quantum optics (Haken 1970; Louisell 1973; Walls and Milburn 1994).

* Refereed paper based on a series of lectures presented to the Atom Optics Workshop, held at the Institute for Theoretical Physics, University of Adelaide, in September 1995.

0004-9506/96/040819$15.00

In quantum optics, the motion of internal electrons of atoms in the presence of a quantised light field is described by a density matrix master equation which is a result of a quantum mechanical ensemble average. In the theory, the centre-of-mass motion of the atoms is treated classically. The word 'classically' here has two implications: (1) the centre-of-mass motion of every individual atom is treated *classically*; and (2) the atoms in the sample are statistically considered to be *distinguishable*. The latter means a basic assumption that the response of a sample of atoms to a light field can be described statistically as an averaged cumulation of the individual contributions of every atom in the sample. In this sense, quantum optics effectively treats a single atom. Such a single-atom theory has been very successful in existing experimental conditions with low atomic densities and high temperatures. However, in recent years a new technique called laser cooling and trapping of atoms has been developed. By this technique, thermal motions of the atoms can be cooled down to a regime where a quantum mechanical description of the centre-of-mass motion of atoms is necessary. For such laser-cooled atoms, a research area called *atom optics* is emerging (Mlynek *et al.* 1992; Walls and Milburn 1994).

(*1b*) *Atom Optics*

Atom optics is an extension of quantum optics in which the centre-of-mass motion of atoms is quantised in terms of quantum mechanics. In atom optics, both the internal electronic motion and the external centre-of-mass motion of atoms are treated quantum mechanically. However, the existing atom optics theory still only treats a single atom, and the quantum statistical properties of identical atoms due to indistinguishability are ignored.

With the photon recoil limit broken, experimental research in laser cooling and trapping of neutral atoms has recently made rapid progress. Now it is no longer a theoretician's dream to generate an ultracold atomic source in a laboratory, where the atoms are expected to be *mutually indistinguishable* and obey different quantum statistics depending on whether they are bosonic atoms or fermionic atoms. The groups at the National Institute of Standards Technology and University of Colorado have recently successfully observed Bose–Einstein condensation in which the bosonic rubidium atoms (^{87}Rb) lost their individual distinguishability and degenerately condensed into a macroscopic single quantum state in the ultracold regime (Anderson *et al.* 1995). One month later, a group at Rice University also realised Bose–Einstein condensation of bosonic lithium atoms (^{7}Li) (Bradley *et al.* 1995). Hence it is now the right time to study such ultracold atomic samples. Evidently the single-atom theory is no longer valid to describe the interaction of the ultracold atomic sample with a light field. In the ultracold regime, atoms have a thermal de Broglie wavelength longer than the optical wavelength so that quantum correlations between atoms, due to many-body collective effects, and quantum statistics, due to the indistinguishability of atoms, are of crucial importance. To study the interaction of such atomic samples with a light field, new methods must be developed.

(*1c*) *Quantum Field Theory*

The purpose of this paper is to give a review of the development of a vector quantum field theory of the interaction of ultracold atoms with a light wave

(Zhang *et al.* 1993, 1994; Lenz *et al.* 1994; Lewenstein *et al.* 1994; Ruprecht *et al.* 1995). Quantum field theory is the most effective tool to study many-body systems. However, the existing treatment of a many-atom ensemble in quantum statistical mechanics employs a scalar quantum field theory in which the atoms are assumed to have no internal structure. Such a scalar quantum field theory is not valid to describe the atoms in the presence of a light field. To take into account the internal transitions of electronic states of atoms, we develop a vector quantum field theory. In Section 2 we systematically introduce the basic methods and techniques of such a vector quantum field theory. Both a two-level atomic ensemble and a multi-level atomic ensemble are studied in the frame of the vector quantum field theory. In Section 3 the vector quantum field theory is applied to study the many-body quantum correlation in atom optics with ultracold atoms. The effect of quantum statistics on the motion of atoms in the ultracold regime is shown. In Section 4 we develop a general formalism of nonlinear atom optics for a coherent atomic beam. Several typical nonlinear atom optics phenomena, such as an atomic soliton, nonlinear Bragg scattering and a nonlinear atomic cavity, are studied on the basis of nonlinear atom optics. The analogy with nonlinear optics for photons is discussed.

2. Vector Quantum Field Theory of Ultracold Atoms

We consider a physical system composed of N identical atoms confined to a space of volume V. In the so-called thermodynamic limit, N and V may go to infinity, but the density $n = N/V$ remains fixed at a preassigned value. For the case of laser cooling and trapping, both the number N of atoms and the volume V of the trapped atoms may be finite. However, they are still large enough that the study of their dynamic behaviour requires statistical mechanics. The statistical properties of a many-atom system depend on two important parameters: the thermal de Broglie wavelength $\lambda_{\mathrm{dB}} = \sqrt{2\pi\hbar^2/mk_{\mathrm{B}}T}$ and the average interatomic separation $\ell = 1/\sqrt[3]{n}$. The thermal de Broglie wavelength is a measure of the average spatial extent of the wave packets that represent the atoms and hence it represents a pure quantum effect. At high temperatures, where the spatial extent of atomic wave packets is small compared to the average interatomic separation ℓ, the atoms, though identical, are mutually distinguishable. In this case, classical statistical mechanics is enough to describe their statistical properties (or dynamics). However, at sufficiently low temperatures, the value λ_{dB} can be quite large compared to the interatom separation ℓ so that the atomic wave packets will overlap each other. To some extent, the atoms will lose their individual distinguishability and quantum statistics becomes important. For convenience, we call atoms satisfying the condition $\lambda_{\mathrm{dB}} \geq \ell$ ultracold atoms. To describe the overlapping of the atomic wave packets, or the indistinguishability of the ultracold atoms, the wavefunction of the atoms must be replaced by a quantised field in terms of the second quantisation rule.

(2a) From Scalar Quantum Field Theory to Vector Quantum Field Theory

In this section we consider an ensemble of N structureless, noninteracting and indistinguishable atoms. In terms of quantum mechanics, the centre-of-mass motion of the structureless atoms in this ensemble is described by the single-atom Schrödinger wave equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t}=\mathcal{H}\psi\,, \tag{2·1·1}$$

where $\mathcal{H}=-\hbar^2\nabla^2/2m+V_{\mathrm{ex}}(\boldsymbol{r})$ is the Hamiltonian of a single atom, $V_{\mathrm{ex}}(\boldsymbol{r})$ is the external potential of, for example, a trap potential, and $\psi(\boldsymbol{r})$ is the scalar wavefunction of the centre-of-mass motion of the structureless atoms. To study the many-atom quantum statistical properties of the ensemble, we follow the standard canonical quantisation technique to quantise the wave equation (2·1·1). At first, we consider the wavefunction $\psi(\boldsymbol{r})$ as a classical scalar field composed of the atoms and write down a Lagrangian for the ensemble such that the corresponding Lagrangian equation leads back to (2·1·1). Such a Lagrangian has the form

$$L=\int \mathrm{d}^3r\,\mathcal{L}(\psi,\psi^*)=\int \mathrm{d}^3r\,\psi^*(\boldsymbol{r})\left(\mathrm{i}\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2\nabla^2}{2m}\psi-V_{\mathrm{ex}}(\boldsymbol{r})\,\psi\right). \tag{2·1·2}$$

In terms of (2·1·2) we have the canonically conjugate momentum of the scalar field $\psi(\boldsymbol{r})$

$$\pi(x)=\frac{\delta L(\psi,\psi^*)}{\delta(\partial\psi/\partial t)}=\mathrm{i}\hbar\psi^*\,. \tag{2·1·3}$$

In terms of the rule of canonical quantisation, the canonically conjugate momentum π and the field $\psi(\boldsymbol{r})$ have to be considered as quantum operators which satisfy the usual equal-time commutators

$$[\psi(\boldsymbol{r}),\,\psi^+(\boldsymbol{r}')]_q=\delta(\boldsymbol{r}-\boldsymbol{r}')\,, \tag{2·1·4}$$

where we have used the notation $[A,B]_q=AB-qBA$ with $q=1$ corresponding to Bose–Einstein statistics and $q=-1$ to Fermi–Dirac statistics. The complex conjugate field ψ^* has been replaced by the Hermitian conjugate field operator ψ^+ in equation (2·1·4). The atoms which satisfy Bose–Einstein statistics are called bosonic atoms; otherwise we speak of fermionic atoms. Which particles will be bosonic atoms and which fermionic atoms depends on the total spin angular momentum of the atoms. For bosonic atoms, the total spin angular momentum is an integral multiple of $\hbar$ and for fermonic atoms, a half-odd integral multiple of $\hbar$. In terms of the canonically conjugate momentum and the field operator, the total Hamiltonian operator of the atomic ensemble has the form

$$H=\int \mathrm{d}^3r\left(\pi\frac{\partial\psi}{\partial t}-\mathcal{L}(\psi,\psi^*)\right)=\int \mathrm{d}^3r\,\psi^+(\boldsymbol{r})\,\mathcal{H}\psi(\boldsymbol{r})\,. \tag{2·1·5}$$

There are two ways to determine the quantum statistical properties of the many-atom ensemble. One is to work out directly the quantum state $|\phi\rangle$ of the ensemble in terms of the Schrödinger equation in its second quantisation form

$$\mathrm{i}\hbar\frac{\partial|\phi\rangle}{\partial t}=H|\phi\rangle\,. \tag{2·1·6}$$

Equation (2·1·6) is different from (2·1·1). Equation (2·1·1) just describes the quantum features of the centre-of-mass motion of a single atom. Equation (2·1·6) describes the quantum statistics of the ensemble composed of many atoms. Instead of using (2·1·6) one can determine the quantum statistics of the ensemble directly in terms of the Heisenberg equation of motion of the quantum field operator

$$i\hbar \frac{\partial \psi}{\partial t} = [\psi, H] . \qquad (2\cdot1\cdot7)$$

Equation (2·1·7) has an identical form to the single-atom Schrödinger equation (2·1·1), but the wavefunction ψ in (2·1·1) is now replaced by the field operator which obeys the commutators (2·1·4). Equations (2·1·6) and (2·1·7) are equivalent in physics, but the correct choice of the equations can bring great convenience in practical applications. Equation (2·1·6) or (2·1·7) with (2·1·4) and (2·1·5) completely determines the quantum statistics of an ensemble composed of noninteracting, indistinguishable and structureless atoms. However, in the presence of a light field, the atoms can no longer be considered structureless particles, since the internal transitions of electrons occur due to optical excitations. On the other hand, the noninteracting atoms are only an ideal case. In a realistic world, the interaction between atoms and atoms, or atoms and other particles such as vacuum photons due to spontaneous emission, cannot be avoided. Hence, we must develop an appropriate model to describe the interaction of an ensemble of ultracold atoms with a light field. In the following sections we will see that a vector quantum field theory is the most appropriate model to describe the ultracold atoms.

(2b) *Vector Field Theory of a Two-level Atomic Ensemble*

In this section, we consider an ensemble composed of many two-level ultracold atoms interacting with a light field. The ideal two-level model is a simplified treatment for realistic atoms. Such an ideal model is quite a reasonable approximation to describe the internal electronic transitions of atoms in the near-resonant optical frequency regime, and has been a useful tool in studying the interaction of atoms with light fields. To initiate our discussion, we start with a single atom in the ensemble which couples to a monochromatic laser field with frequency ω_L and vector potential $\boldsymbol{A}$. In the dipole approximation, the single-atom Hamiltonian has the form

$$H = H_A + H_L + H_F + H_{A-L} + H_{A-F} ,$$

$$H_A = -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(\boldsymbol{r}) + \hbar\omega_a \sigma^+ \sigma ,$$

$$H_L = \tfrac{1}{2} \int d^3 r \left[\epsilon_0 \left(\frac{\partial \boldsymbol{A}}{\partial t} \right)^2 + \frac{1}{\mu_0} (\nabla \times \boldsymbol{A})^2 \right] ,$$

$$H_F = \sum_{k\lambda} \hbar\omega_k B^+_{k\lambda} B_{k\lambda} ,$$

$$H_{\mathrm{A-L}} = -\boldsymbol{J}\,.\,\boldsymbol{A} = -(\boldsymbol{J}_{12}\,\sigma + \boldsymbol{J}_{21}\,\sigma^{+})\,.\,\boldsymbol{A}\,,$$

$$H_{\mathrm{A-F}} = -\hbar\sum_{k\lambda}(g_{k\lambda})^{*}B_{k\lambda}^{+}\exp(-\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{r})(\sigma+\sigma^{+}) + \mathrm{h.c.}\,, \qquad (2\cdot2\cdot1)$$

where H_{A}, H_{L}, and H_{F} are, respectively, the free Hamiltonian of the atom confined in a external potential $V_{\mathrm{ex}}(\boldsymbol{r})$ with transition frequency ω_{a}, of the laser field with frequency ω_{L}, and of the vacuum electromagnetic field which is introduced to describe the effect of the spontaneous emission of the atom. The Hamiltonian $H_{\mathrm{A-L}}$ describes the interaction of the atom with the laser field, while $H_{\mathrm{A-F}}$ is the interaction Hamiltonian of the atom with the vacuum electromagnetic field. The vector $\boldsymbol{J}_{ij} = \mathrm{i}\omega_{\mathrm{L}}\,\boldsymbol{\mu}_{ij}$ $(i \neq j = 1, 2)$ is the matrix element of the transverse electric current operator $\boldsymbol{J}$ and $\boldsymbol{\mu}_{ij}$ is the matrix element of the dipole moment of the atom. Here we take $\boldsymbol{\mu}_{ij} = \boldsymbol{\mu}_{ji} = \boldsymbol{\mu}$ as a real number, while $B_{k\lambda}^{+}$ and $B_{k\lambda}$ are the bosonic creation and annihilation operators of the vacuum electromagnetic field. The coefficient $g_{k\lambda} = \mathrm{i}(2\pi\omega_k/\hbar V_e)^{\frac{1}{2}}\,\boldsymbol{\mu}\,.\,\boldsymbol{e}_{k\lambda}$ is the coupling strength of the atom to the vacuum electromagnetic field, with V_e denoting the quantisation volume and $\boldsymbol{e}_{k\lambda}$ the polarisation vectors of the vacuum fields. The atomic transition is described by the Pauli spin operators σ and σ^{+}. As in Section 2*a* we can write down the single-atom Schrödinger equation for the centre-of-mass motion of the atom

$$\mathrm{i}\hbar\,\frac{\partial\psi}{\partial t} = H\psi\,. \qquad (2\cdot2\cdot2)$$

Since the internal transitions of electrons of the atom are included in the Hamiltonian $(2\cdot2\cdot1)$, where the Pauli spin operators σ and σ^{+} appear, the centre-of-mass wavefunction of the atom is no longer a scalar and is instead a vector $\psi(\boldsymbol{r}) = \psi_1(\boldsymbol{r})\,|1\rangle + \psi_2(\boldsymbol{r})\,|2\rangle$, where the state vectors $|1\rangle$ and $|2\rangle$ denote the internal ground state and excited state of the atom, and $\psi_j(\boldsymbol{r})$ $(j = 1, 2)$ are the two components of the vector. To extend equation $(2\cdot2\cdot2)$ to include many-atom quantum statistics, we now interpret the vector wavefunction $\psi(\boldsymbol{r})$ as a vector field which describes the ensemble composed of many two-level atoms. The quantisation of such a vector field still follows the standard technique of canonical quantisation for a scalar field as shown in Section 2*a*. We will not repeat the procedures and only give the final results here. The total Hamiltonian of the atomic ensemble interacting with a laser field has the form, after the second quantisation (Zhang and Walls 1993*b*),

$$\begin{aligned} H_{\mathrm{sys}} = {} & \sum_{j=1}^{2}\int \mathrm{d}^3r\;\psi_j^{+}(\boldsymbol{r})\left[-\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{ex}}(\boldsymbol{r})\right]\psi_j(\boldsymbol{r}) + \int \mathrm{d}^3r\;\psi_2^{+}(\boldsymbol{r})\hbar\omega_{\mathrm{a}}\,\psi_2(\boldsymbol{r}) + H_{\mathrm{L}} + H_{\mathrm{F}} \\ & - \int \mathrm{d}^3r\;\boldsymbol{J}_{12}\,.\,\boldsymbol{A}\psi_1^{+}(\boldsymbol{r})\,\psi_2(\boldsymbol{r}) - \hbar\int \mathrm{d}^3r\;\sum_{k\lambda}(g_{k\lambda})^{*}B_{k\lambda}^{+}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{r}}\,\psi_2^{+}(\boldsymbol{r})\,\psi_1(\boldsymbol{r}) \\ & - \hbar\int \mathrm{d}^3r\;\sum_{k\lambda}(g_{k\lambda})^{*}B_{k\lambda}^{+}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{r}}\,\psi_1^{+}(\boldsymbol{r})\,\psi_2(\boldsymbol{r}) + \mathrm{h.c}\,. \end{aligned} \qquad (2\cdot2\cdot3)$$

The quantum field operators $\psi_j(\boldsymbol{r})$ and their Hermitian conjugate fields $\psi_j^{+}(\boldsymbol{r})$ satisfy the equal-time commutators

$$[\psi_i(\boldsymbol{r}), \psi_j(\boldsymbol{r}')]_q = [\psi_i^+(\boldsymbol{r}), \psi_j^+(\boldsymbol{r}')]_q = 0\,,$$

$$[\psi_i(\boldsymbol{r}), \psi_j^+(\boldsymbol{r}')]_q = \delta_{ij}\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,, \qquad (2\cdot2\cdot4)$$

where $q = 1$ corresponds to Bose–Einstein statistics and $q = -1$ to Fermi–Dirac statistics, which is the same as in the scalar case. On the other hand, the quantised laser field satisfies the commutation relations in the Coulomb gauge (Louisell 1973)

$$[A_i(\boldsymbol{r}),\, A_j(\boldsymbol{r}')] = [D_i(\boldsymbol{r}),\, D_j(\boldsymbol{r}')] = 0\,,$$

$$[D_i(\boldsymbol{r}),\, A_j(\boldsymbol{r}')] = \mathrm{i}\hbar\delta_{ij}^T(\boldsymbol{r}-\boldsymbol{r}') \qquad (i,\, j = x,\, y,\, z)\,, \qquad (2\cdot2\cdot5)$$

where $\boldsymbol{D} = -\epsilon_0\,\boldsymbol{E} = -\epsilon_0\,\partial\boldsymbol{A}/\partial t$ is the electric displacement vector, with $\boldsymbol{E}$ denoting the strength of the electric field and $\delta_{ij}^T(\boldsymbol{r}-\boldsymbol{r}')$ the transverse delta function. In terms of the total Hamiltonian (2·2·3) we can write down the Heisenberg equations of motion for the vector quantum field ψ

$$\mathrm{i}\hbar\,\frac{\partial\psi_j}{\partial t} = [\psi_j, H_{\mathrm{sys}}] \qquad (j = 1,\, 2)\,, \qquad (2\cdot2\cdot6)$$

which is the second-quantisation version of the single-atom Schrödinger equation (2·2·2). By using the commutation relations (2·2·4) and eliminating the vacuum electromagnetic fields $B_{\boldsymbol{k}\lambda}$ and $B_{\boldsymbol{k}\lambda}^+$ in the interaction picture, equations (2·2·6) can be reduced to the following forms:

$$\mathrm{i}\hbar\,\frac{\partial\psi_1}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{ex}}(\boldsymbol{r})\right)\psi_1 - \boldsymbol{\mu}\,.\,\boldsymbol{E}^{(-)}\psi_2$$
$$+\,\mathrm{i}\hbar\int \mathrm{d}^3r\; L(\boldsymbol{r}-\boldsymbol{r}')\,\psi_2^+(\boldsymbol{r}')\,\psi_1(\boldsymbol{r}')\psi_2(\boldsymbol{r}) + G_1(\boldsymbol{r},t)\,, \qquad (2\cdot2\cdot7\mathrm{a})$$

$$\mathrm{i}\hbar\,\frac{\partial\psi_2}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{ex}}(\boldsymbol{r})\right)\psi_2 - \hbar(\Delta + \mathrm{i}\gamma/2)\psi_2 - \boldsymbol{\mu}\,.\,\boldsymbol{E}^{(+)}\psi_1$$
$$-\,\mathrm{i}\hbar\int \mathrm{d}^3r\; L^*(\boldsymbol{r}-\boldsymbol{r}')\,\psi_1^+(\boldsymbol{r}')\,\psi_2(\boldsymbol{r}')\,\psi_1(\boldsymbol{r}) + G_2(\boldsymbol{r},t)\,, \qquad (2\cdot2\cdot7\mathrm{b})$$

where $\gamma = 4|\boldsymbol{\mu}|^2\omega_L^3/3\hbar c^3$ is the spontaneous emission rate of a single atom in the ensemble. The usual Lamb level shift of a single atom induced by the vacuum electromagnetic fields has been included in the detuning Δ in equation (2·2·7b). The nonlinear terms in equations (2·2·7) describe the two-body collective correlations or collisions between excited-state and ground-state atoms which originate from the exchange of vacuum photons between atoms due to spontaneous emission. The correlation coefficient has the definition (in the Markoff approximation)

$$L(\boldsymbol{r}-\boldsymbol{r}') \equiv \gamma[K(\boldsymbol{r}-\boldsymbol{r}')/2 - \mathrm{i}W(\boldsymbol{r}-\boldsymbol{r}')] \qquad (2\cdot2\cdot8\mathrm{a})$$

for

$$K(\boldsymbol{r}-\boldsymbol{r}') = \tfrac{3}{2}\left[\sin^2\theta\,\frac{\sin\xi}{\xi} + (1-3\cos^2\theta)\left(\frac{\cos\xi}{\xi^2} - \frac{\sin\xi}{\xi^3}\right)\right], \qquad (2\cdot2\cdot8\mathrm{b})$$

$$W(\boldsymbol{r}-\boldsymbol{r}') = \tfrac{3}{4}\left[-\sin^2\theta\,\frac{\cos\xi}{\xi} + (1-3\cos^2\theta)\left(\frac{\sin\xi}{\xi^2} + \frac{\cos\xi}{\xi^3}\right)\right], \qquad (2\cdot2\cdot8\mathrm{c})$$

where we define $\xi = k_{\mathrm{L}}|\boldsymbol{r}-\boldsymbol{r}'|$ and θ is the angle between the dipole moment $\boldsymbol{\mu}$ and the relative coordinate $\boldsymbol{r}-\boldsymbol{r}'$. The noise terms

$$G_j(\boldsymbol{r},t) \equiv -\hbar[\Gamma_j^+(\boldsymbol{r},t)\psi_l + \psi_l\,\Gamma_l(\boldsymbol{r},t)] \qquad (j\neq l=1,2) \qquad (2\cdot2\cdot9\mathrm{a})$$

in $(2\cdot2\cdot7)$ come from the effect of vacuum fluctuations on the atomic quantum field. The vacuum noise operators have the definitions

$$\Gamma_j(\boldsymbol{r},t) \equiv \sum_{\boldsymbol{k}\lambda} g_{\boldsymbol{k}\lambda}\,B_{\boldsymbol{k}\lambda}(t_0)\,\mathrm{e}^{-\mathrm{i}[\omega_k+(-1)^j\omega_{\mathrm{L}}]t+i\boldsymbol{k}\cdot\boldsymbol{r}} \qquad (j=1,\,2) \qquad (2\cdot2\cdot9\mathrm{b})$$

and satisfy the following statistical correlations:

$$\langle\Gamma_l(\boldsymbol{r},t)\,\Gamma_j(\boldsymbol{r}',t')\rangle = \langle\Gamma_l^+(\boldsymbol{r},t)\,\Gamma_j(\boldsymbol{r}',t')\rangle = 0,$$

$$\langle\Gamma_l(\boldsymbol{r},t)\,\Gamma_j^+(\boldsymbol{r}',t')\rangle = \delta_{jl}\sum_{\boldsymbol{k}\lambda}|g_{\boldsymbol{k}\lambda}|^2\,\mathrm{e}^{-\mathrm{i}[\omega_k+(-1)^l\omega_{\mathrm{L}}](t-t')+\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} \qquad (2\cdot2\cdot9\mathrm{c})$$

The brackets $\langle\ \rangle$ denote an average over the vacuum states of the free electromagnetic field. On the other hand, in terms of the Hamiltonian $(2\cdot2\cdot3)$ and the commutation relations $(2\cdot2\cdot5)$, the positive frequency and negative frequency parts of the quantised laser field $E^{(\pm)}$ satisfy the quantum propagation equations

$$\nabla^2\boldsymbol{E}^{(\pm)} - \frac{1}{c^2}\,\frac{\partial^2\boldsymbol{E}^{(\pm)}}{\partial t^2} = \mu_0\,\frac{\partial^2}{\partial t^2}\,\boldsymbol{P}^{(\pm)}, \qquad (2\cdot2\cdot10)$$

where $\boldsymbol{P}^{(+)} = [\boldsymbol{P}^{(-)}]^+ = \boldsymbol{\mu}\psi_1^+(\boldsymbol{r})\,\psi_2(\boldsymbol{r})$ defines the positive frequency part of the polarisation of the atomic ensemble. Equations $(2\cdot2\cdot7)$ are two coupled nonlinear stochastic Schrödinger equations, and determine the dynamics of ultracold atoms in the presence of a laser field together with equations $(2\cdot2\cdot10)$. So far, we have developed a general vector quantum field theory for a two-level ultracold atomic ensemble. However, realistic atoms always have more than two internal levels. Although the atoms can be approximately described by a two-level model in a single-frequency, near resonant laser field, the two-level model is not valid in the following cases: (1) if the light field is composed of two circularly polarised counterpropagating components; (2) if the light field has more than a single frequency component; or (3) if there exists an external magnetic field which results in the Zeeman splitting of the hyperfine structure of the atoms. Hence a more exact model should be developed to include the multi-sublevel structures of the atoms. For this purpose, in the following section we will extend the two-component vector quantum field theory to a multi-component vector quantum field theory.

(2c) Vector Field Theory for a Multi-level Atomic Ensemble

Here we consider a more complicated model in which the Zeeman sublevels of the atoms are included. The multi-component vector quantum field can be expressed as

$$\psi(\boldsymbol{r}) = \sum_{m_g} \psi_{m_g} |J_g\, m_g\rangle + \sum_{m_e} \psi_{m_e} |J_e\, m_e\rangle\,, \tag{2·3·1}$$

where $|J_g\, m_g\rangle$ and $|J_e\, m_e\rangle$ are respectively the ground states and excited states of the atoms with J_β the angular momenta and $m_\beta = J_\beta, J_\beta - 1, \cdots, -J_\beta$ $(\beta = g, e)$ the magnetic quantum numbers. In terms of the atomic field (2·3·1) we can write down the total Hamiltonian for the ensemble as in Section 2*b*:

$$\begin{aligned} H_{\text{sys}} = & \sum_{m_g} \int \mathrm{d}^3 r\, \psi^+_{m_g}(\boldsymbol{r}) \Big(- \frac{\hbar^2 \nabla^2}{2m} + V_{\text{ex}}(\boldsymbol{r}) \Big) \psi_{m_g}(\boldsymbol{r}) \\ & + \sum_{m_e} \int \mathrm{d}^3 r\, \psi^+_{m_e}(\boldsymbol{r}) \Big(- \frac{\hbar^2 \nabla^2}{2m} + V_{\text{ex}}(\boldsymbol{r}) + \hbar\omega_{m_e} \Big) \psi_{m_e}(\boldsymbol{r}) \\ & - \sum_{m_g m_e} \int \mathrm{d}^3 r \, \{ \boldsymbol{J}_{m_g m_e} \cdot \boldsymbol{A}\, \psi^+_{m_g}(\boldsymbol{r})\, \psi_{m_e}(\boldsymbol{r}) + \hbar \Gamma^{(+)}_{m_g m_e}(\boldsymbol{r}, t)\, \psi^+_{m_g}(\boldsymbol{r})\, \psi_{m_e}(\boldsymbol{r}) \\ & + \hbar \Gamma^{(+)}_{m_e m_g}(\boldsymbol{r}, t)\, \psi^+_{m_e}(\boldsymbol{r})\, \psi_{m_g}(\boldsymbol{r}) + \text{h.c.} \}\,, \end{aligned}$$

$$\Gamma^{(+)}_{m_g m_e}(\boldsymbol{r}, t) = \mathrm{i} \boldsymbol{\mu}_{m_g m_e} \cdot \sum_{\boldsymbol{k}\lambda} \left(\frac{2\pi\omega_k}{\hbar V_e} \right)^{\frac{1}{2}} \boldsymbol{e}_{\boldsymbol{k}\lambda}\, B_{\boldsymbol{k}\lambda} \exp[\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r} - \mathrm{i}(\omega_k + \omega_{\text{L}})t]\,,$$

$$\Gamma^{(+)}_{m_e m_g}(\boldsymbol{r}, t) = \mathrm{i} \boldsymbol{\mu}_{m_e m_g} \cdot \sum_{\boldsymbol{k}\lambda} \left(\frac{2\pi\omega_k}{\hbar V_e} \right)^{\frac{1}{2}} \boldsymbol{e}_{\boldsymbol{k}\lambda}\, B_{\boldsymbol{k}\lambda} \exp[\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r} - \mathrm{i}(\omega_k - \omega_{\text{L}})t]\,, \tag{2·3·2}$$

where $\boldsymbol{J}_{m_g m_e} = \mathrm{i}\omega_{\text{L}}\, \boldsymbol{\mu}_{m_g m_e} = \mathrm{i}\omega_{\text{L}} \langle J_g\, m_g | \boldsymbol{\mu} | J_e\, m_e \rangle$ is the matrix element of the transverse currents of the atoms and $\omega_{m_e} = \omega_{\text{a}} + \delta\omega_{m_e}$ are the transition frequencies corresponding to different sublevels, with $\delta\omega_{m_e}$ the Zeeman splitting in the external magnetic field. The other parameters have the same definitions as in Section 2*b*. The equal-time commutators for the multi-component vector field have the forms

$$\begin{aligned} & [\psi_m(\boldsymbol{r}),\, \psi_{m'}(\boldsymbol{r}')]_q = [\psi^+_m(\boldsymbol{r}),\, \psi^+_{m'}(\boldsymbol{r}')]_q = 0, \\ & [\psi_m(\boldsymbol{r}),\, \psi^+_{m'}(\boldsymbol{r}')]_q = \delta_{mm'}\, \delta(\boldsymbol{r} - \boldsymbol{r}'), \qquad m, m' = \{m_g, m_e\}\,. \end{aligned} \tag{2·3·3}$$

As in Section 2*b* we can derive the Heisenberg equations of motion for the multi-component vector quantum field:

$$
\begin{aligned}
\mathrm{i}\hbar\frac{\partial\psi_{m_g}}{\partial t} &= \Big(-\frac{\hbar^2\nabla^2}{2m}+V_{\mathrm{ex}}(\boldsymbol{r})\Big)\psi_{m_g}-\sum_{m_e}\boldsymbol{\mu}_{m_g m_e}\cdot\boldsymbol{E}^{(-)}\psi_{m_e}\\
&\quad+\mathrm{i}\hbar\sum_{\{m_e m_g',m_e'\}}\int \mathrm{d}^3r'\, L_{m_g m_e m_g' m_e'}(\boldsymbol{r}-\boldsymbol{r}')\,\psi^+_{m_e'}(\boldsymbol{r}')\,\psi_{m_g'}(\boldsymbol{r}')\,\psi_{m_e}(\boldsymbol{r})\\
&\quad+G_{m_g}(\boldsymbol{r},t)\,,\\
\mathrm{i}\hbar\frac{\partial\psi_{m_e}}{\partial t} &= \Big(-\frac{\hbar^2\nabla^2}{2m}+V_{\mathrm{ex}}(\boldsymbol{r})\Big)\psi_{m_e}-\hbar(\Delta_{m_e}+\mathrm{i}\gamma/2)\psi_{m_e}\\
&\quad-\sum_{m_g}\boldsymbol{\mu}_{m_e m_g}\cdot E^{(+)}\psi_{m_g}\\
&\quad-\mathrm{i}\hbar\sum_{\{m_g,m_g',m_e'\}}\int \mathrm{d}^3r'\, L^*_{m_g m_e m_g' m_e'}(\boldsymbol{r}-\boldsymbol{r}')\,\psi^+_{m_g'}(\boldsymbol{r}')\psi_{m_e'}(\boldsymbol{r}')\,\psi_{m_g}(\boldsymbol{r})\\
&\quad+G_{m_e}(\boldsymbol{r},t)\,,
\end{aligned}
$$

$$
\begin{aligned}
G_{m_g}(\boldsymbol{r},t) &= -\hbar\sum_{m_e}\{\psi_{m_e}(\boldsymbol{r})\,\Gamma^{(+)}_{m_e m_g}(\boldsymbol{r},t)+\Gamma^{(-)+}_{m_g m_e}(\boldsymbol{r},t)\,\psi_{m_e}(\boldsymbol{r})\}\,,\\
G_{m_e}(\boldsymbol{r},t) &= -\hbar\sum_{m_g}\{\psi_{m_g}(\boldsymbol{r})\,\Gamma^{(-)}_{m_g m_e}(\boldsymbol{r},t)+\Gamma^{(+)+}_{m_g m_e}(\boldsymbol{r},t)\,\psi_{m_g}(\boldsymbol{r})\}\,, \qquad (2\cdot3\cdot4)
\end{aligned}
$$

where $\gamma=4|\langle J_g|\boldsymbol{\mu}|J_e\rangle|^2\,\omega_a^3/3\hbar c^3(2J_e+1)$ is the single-atom spontaneous emission rate and $\Delta_{m_e}=\omega_{\mathrm{L}}-\omega_{m_e}$ are the detunings of the laser field from atomic resonance. The nonlinear correlation coefficients now have the definitions

$$
L_{m_g m_e m_g' m_e'}(\boldsymbol{r}-\boldsymbol{r}')=\gamma[K_{m_g m_e m_g' m_e'}(\boldsymbol{r}-\boldsymbol{r}')/2-\mathrm{i}W_{m_g m_e m_g' m_e'}(\boldsymbol{r}-\boldsymbol{r}')] \qquad (2\cdot3\cdot5\mathrm{a})
$$

for

$$
K_{m_g m_e m_g' m_e'}(\boldsymbol{r}-\boldsymbol{r}')=\tfrac{3}{2}\Big[\alpha_{m_g m_e m_g' m_e'}\frac{\sin\xi}{\xi}+\beta_{m_g m_e m_g' m_e'}\Big(\frac{\cos\xi}{\xi^2}-\frac{\sin\xi}{\xi^3}\Big)\Big]\,, \qquad (2\cdot3\cdot5\mathrm{b})
$$

$$
W_{m_g m_e m_g' m_e'}(\boldsymbol{r}-\boldsymbol{r}')=\tfrac{3}{4}\Big[-\alpha_{m_g m_e m_g' m_e'}\frac{\cos\xi}{\xi}+\beta_{m_g m_e m_g' m_e'}\Big(\frac{\sin\xi}{\xi^2}+\frac{\cos\xi}{\xi^3}\Big)\Big]\,, \qquad (2\cdot3\cdot5\mathrm{c})
$$

$$
\begin{aligned}
\alpha_{m_g m_e m_g' m_e'} &= \frac{2J_e+1}{2J_g+1}\Big(\sum_{\nu=\{0,\pm1\}}C^{J_g m_g}_{J_e m_e 1\nu}\,C^{J_g m_g'}_{J_e m_e' 1\nu}\\
&\quad-\sum_{\{\nu,\nu'=0,\pm1\}}C^{J_g m_g}_{J_e m_e 1\nu'}\,C^{J_g m'_g}_{J_e m'_e 1\nu'}\,\frac{4\pi}{3}\,Y^*_{1\nu}(\theta,\varphi)\,Y_{1\nu'}(\theta,\varphi)\Big)\,,
\end{aligned}
$$

$$\beta_{m_g m_e m_g' m_e'} = \alpha_{m_g m_e m_g' m_e'} - \frac{2J_e+1}{2J_g+1}\left(\sum_{\{\nu,\nu'=0,\pm1\}} C^{J_g m_g}_{J_e m_e 1\nu} C^{J_g m_g'}_{J_e m_e' 1\nu'} \frac{8\pi}{3} Y^*_{1\nu}(\theta,\varphi) Y_{1\nu'}(\theta,\varphi)\right), \tag{2·3·5d}$$

where $C^{J_g m_g}_{J_e m_e 1\nu}$ are the Clebsch–Gordon coefficients and $Y_{1\nu}(\theta,\varphi)$ the spherical harmonics, with (θ,φ) giving the orientations of dipole moments of the atoms. On the other hand, from Hamiltonian (2·3·2) we have the quantum propagation equations for the quantised laser field:

$$\nabla^2 \boldsymbol{E}^{(\pm)} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}^{(\pm)}}{\partial t^2} = \mu_0 \frac{\partial^2}{\partial t^2}\boldsymbol{P}^{(\pm)}. \tag{2·3·6a}$$

Equations (2·3·6a) have the same forms as (2·2·10), but the total polarisation of the ensemble is now given by

$$P^{(+)} = [P^{(-)}]^+ = \sum_{\{m_g,m_e\}} \boldsymbol{\mu}_{m_g m_e} \psi^+_{m_g}(\boldsymbol{r})\,\psi_{m_e}(\boldsymbol{r}). \tag{2·3·6b}$$

In the absence of an external magnetic field, one can easily prove that when a linearly polarised laser field is used to excite the atomic transition $J_g = 0 \rightarrow J_e = 1$, equations (2·3·4)–(2·3·6) reduce to (2·2·7)–(2·2·10) for the two-level atoms. Hence the two-level model is only valid for an atom with transition $J_g = 0 \rightarrow J_e = 1$ coupled to a single-frequency linearly polarised laser field. In the following sections, we will assume for simplicity that this condition is always satisfied.

3. Many-body Quantum Correlation in Atom Optics

In Section 2 we introduced the general theoretical methods and techniques to study a many-atom ensemble interacting with a laser field. A vector quantum field theory was developed. In this section our purpose is to apply the vector quantum field theory to different physical systems where special conditions apply.

(3a) Bosonic and Fermionic Atoms in a Linear Beam Splitter

The most typical system studied in atom optics is an atomic beamsplitter which is composed of a standing-wave laser beam. When a monochromatic atomic beam passes through a standing-wave laser beam, the interaction of the atom with the laser beam results in the transfer of photon momenta to the atom. As a result, the atom will be 'kicked' by photons in two different directions along the standing-wave laser beam and the centre-of-mass motion of the atom in the incident beam is changed after interaction. In the standard atom optics language, we say that the atomic beam is diffracted by the standing-wave laser beam, which effectively acts as a grating for an atomic wave. The details of diffraction of the atoms by a standing-wave laser have been well studied in a single-atom theory of atom optics (Martin *et al.* 1988; Arimondo *et al.* 1981*a*, 1981*b*). As we have pointed out, the single-atom theory is valid only when all atoms in the incident beam are statistically distinguishable. In this section we study the

diffraction of an atomic beam composed of ultracold atoms which are mutually indistinguishable. The realisation of such an ultracold atomic beam requires a high degree of cooling and transverse collimation so that the thermal de Broglie wavelength of the atoms is comparable to the average interatomic separation in this beam. With the breaking of the photon recoil limit in laser cooling and the realisation of Bose–Einstein condensation of atoms in a confined space, an ultracold atomic beam will not be too far from reality. In order to study the diffraction of such an ultracold atomic beam by a standing-wave laser beam, the vector quantum field theory will need to be employed so that the quantum statistics of the atoms is taken into account. To initiate our discussion, we assume that the incident atomic beam propagates in the z direction and a plane-wave laser beam propagates in the y direction. The vector potential operator of the plane-wave laser beam can be expressed as $\boldsymbol{A} = \sqrt{\hbar/2\omega_{\mathrm{L}}\,\epsilon_0\,V}\,\boldsymbol{e}\cos(k_{\mathrm{L}}\,y)[b\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{L}}t} + b^{+}\mathrm{e}^{\mathrm{i}\omega_{\mathrm{L}}t}]$, with b and b^{+} the annihilation and creation operators for the laser photons. In addition we assume that the flight time of the atoms through the laser beam is short enough and the laser field weak enough that spontaneous emission can safely be neglected in this case. Under these assumptions the Hamiltonian $(2\cdot2\cdot3)$ is reduced in the interaction picture to the form

$$H = \sum_{j=1}^{2} \int \mathrm{d}^3 r\, \psi_j^{+}(\boldsymbol{r}) \Big(- \frac{\hbar^2 \nabla^2}{2m} \Big) \psi_j(\boldsymbol{r}) - \int \mathrm{d}^3 r\, \psi_2^{+}(\boldsymbol{r}) \hbar \Delta \psi_2(\boldsymbol{r})$$

$$- \tfrac{1}{2}\hbar g^{*} b^{+} \int \mathrm{d}^3 r\, \psi_1^{+}(\boldsymbol{r}) \cos(k_{\mathrm{L}}\, y)\, \psi_2(\boldsymbol{r}) + \mathrm{h.c.}, \qquad (3\cdot1\cdot1)$$

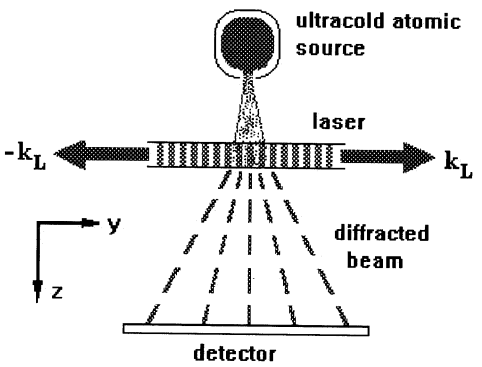

Fig. 1. Schematic diagram for diffraction of an atomic beam by a standing-wave laser beam.

where $g = \mathrm{i}\sqrt{2\pi\omega_{\mathrm{L}}/\hbar V}\,\boldsymbol{\mu}.\boldsymbol{e}$ is the single-photon Rabi frequency of the atoms. From equation $(3\cdot1\cdot1)$ we see that the interaction term has a periodic structure in the y direction which is the physical origin of the diffraction of the atomic beam. The diffraction of atoms will lead to the splitting of the atomic beam into many plane-wave components in the y direction (see Fig. 1). Hence in general the atomic quantum field can be expanded in terms of the plane-wave components (Zhang and Walls 1993*a*)

$$\psi_1(\boldsymbol{r}) = V^{-\frac{1}{2}} \sum_n a_n \exp[\mathrm{i}(\boldsymbol{p}_0 + n\hbar k_\mathrm{L}\, \boldsymbol{e}_y)\,.\,\boldsymbol{r}/\hbar\,,$$

$$\psi_2(\boldsymbol{r}) = V^{-\frac{1}{2}} \sum_n c_n \exp[\mathrm{i}(\boldsymbol{p}_0 + n\hbar k_\mathrm{L}\, \boldsymbol{e}_y)\,.\,\boldsymbol{r}/\hbar\,, \qquad (3\cdot1\cdot2)$$

where $\boldsymbol{p}_0 = |\boldsymbol{p}_0|\boldsymbol{e}_z$ is the momentum of the incident atomic beam; $n\hbar k_\mathrm{L}$ the momentum gained by atoms in the y direction in the possible $|n|$-photon interaction processes; $\boldsymbol{e}_z$ and $\boldsymbol{e}_y$ the unit vectors in the z and y direction, respectively; and V the quantisation volume. The operators a_n and c_n describe the annihilation of a ground-state and an excited-state atom in diffraction mode n. Their Hermitian conjugate operators a_n^+ and c_n^+ describe the corresponding creation processes of atoms. These operators satisfy the commutators

$$[a_n,\, a_{n'}^+]_q = [c_n.c_{n'}^+]_q = \delta_{nn'} \qquad (q = \pm 1)\,. \qquad (3\cdot1\cdot3)$$

Evidently equation $(3\cdot1\cdot2)$ leads to a multi-mode coupled Hamiltonian in the Hilbert space expanded by the plane-wave diffraction modes. For such a multi-mode quantum system, the exact solutions must depend on the numerical simulation. However, physically the multi-mode quantum system can be reduced to a system with a few modes. Here we consider a typical case in atom optics where the zeroth-order Doppleron resonance is satisfied. The idea to achieve the Doppleron resonance is to select an appropriate laser detuning so that only the diffraction modes whose photon recoil frequencies match the laser detuning are resonantly enhanced. The other diffraction modes off Doppleron resonance may be neglected. The condition for the nth-order Doppleron resonance is $\Delta = (2n+1)^2\omega_\mathrm{r}$ with $\hbar k_\mathrm{L}^2/2m$ the single-photon recoil frequency. For zeroth-order Doppleron resonance, only the diffraction modes with indices $n = 0, \pm1$ are important. In this case the standing-wave laser beam just acts as an atomic beamsplitter. For such an atomic beamsplitter, the simplified Hamiltonian for the diffraction of atoms has the form in the interaction picture

$$H_1 = -\frac{\sqrt{2}}{4}\,\hbar g b^+ a^+ c + \mathrm{h.c.}\,, \qquad (3\cdot1\cdot4)$$

where we have defined $a^+ = a_0^+$ and $c = (c_{-1} + c_1)/\sqrt{2}$. The combination operator c still satisfies the commutation relations $(3\cdot1\cdot3)$. To study the quantum statistics of atoms in diffraction, we need to find the time evolution of the quantum state $|\Phi\rangle$ of the system in terms of the Schrödinger equation

$$\mathrm{i}\hbar\,\frac{\partial|\Phi\rangle}{\partial t} = H_1|\Phi\rangle\,. \qquad (3\cdot1\cdot5)$$

We assume that the initial state of the photon–atom system is $|\Phi(0)\rangle = |\beta\rangle\otimes|\phi_a\rangle\otimes|0\rangle$, where $|\beta\rangle$ is a coherent state for the laser field, $|\phi_a\rangle$ is the state vector describing the initial quantum statistical properties of the incident ground-state atomic beam, and $|0\rangle$ is the vacuum state for the excited-state atomic beam which is initially assumed to be 'empty'. For a short flight time τ or interaction time of

the atoms through the laser beam, in terms of equation (3·1·5), the diffraction results in the evolution of the initial quantum state $|\Phi(0)\rangle$ into

$$|\Phi(\tau)\rangle = |\Phi(0)\rangle + \mathrm{i}\frac{g\beta\tau}{2\sqrt{2}}\, a|\beta\rangle \otimes |\phi_a\rangle \otimes |1\rangle - \frac{g^2\beta\tau^2}{16}[b^+a^+a|\Phi(0)\rangle$$
$$\times \sqrt{2}\,\beta a^2|\beta\rangle \otimes |\phi_a\rangle \otimes |2\rangle] + \mathrm{O}(\tau^3) \qquad (3\cdot1\cdot6\mathrm{a})$$

for bosonic atoms, and

$$|\Phi(\tau)\rangle = |\Phi(0)\rangle + \mathrm{i}\frac{g\beta\tau}{2\sqrt{2}}\, a|\beta\rangle \otimes |\phi_a\rangle \otimes |1\rangle + \frac{g^2\beta\tau^2}{16}\, b^+a^+a|\Phi(0)\rangle$$
$$+ \mathrm{O}(\tau^3) \qquad (3\cdot1\cdot6\mathrm{b})$$

for fermionic atoms. The second terms in (3·1·6a) and (3·1·6b) describe the annihilation of a ground-state atom and the creation of an excited-state atom. Hence they correspond to the usual single-atom diffraction which obviously has the meaning that diffraction modes $n = 1$ and -1 are populated by an excited-state atom after interaction. In atom optics language, this means that the incident atomic beam is split into two parts with momentum $\pm\hbar k_{\mathrm{L}}$ in the y direction. The third terms of (3·1·6a) and (3·1·6b) describe the self-exchange processes of the ground-state atoms, which do not lead to a change of the initial atomic state. The self-exchange processes evidently depend on the quantum statistics of the atoms and hence we see different signs appearing before the third terms of (3·1·6a) and (3·1·6b) for bosonic atoms and fermionic atoms. For bosonic atoms we have an additional term $\sqrt{2}\,\beta a^2|\beta\rangle \otimes |\phi_a\rangle \otimes |2\rangle$, which gives the probability that two ground-state atoms are excited and then diffracted into $n = \pm 1$ modes. A similar two-atom diffraction process does not happen for fermionic atoms because one cannot have two fermions occupying the same quantum state due to the Pauli exclusion principle. The interesting quantum statistical properties in the diffraction quantum state can be observed in the macroscopic dynamical behaviours of the atoms. To show the effect of quantum statistics on the macroscopic dynamics of atomic diffraction, we calculate the spread of the total transverse momentum of the atomic beam in the diffraction in terms of (3·1·6). For bosonic atoms we have

$$\Delta p \equiv \left(\left\langle \Phi(\tau)\middle|\sum_{j=1}^{2}\int \mathrm{d}^3r\ \psi_j(\boldsymbol{r})(\boldsymbol{p}-\boldsymbol{p}_0)^2\psi_j(\boldsymbol{r})\middle|\Phi(\tau)\right\rangle\right)^{\frac{1}{2}}$$
$$= \sqrt{N}\,\frac{\hbar k_{\mathrm{L}}\,\Omega\tau}{2\sqrt{2}}\sqrt{1 + NG^{(2)}(0)\,\Omega^2\tau^2/8} \qquad (3\cdot1\cdot7\mathrm{a})$$

and for fermionic atoms

$$\Delta p = \sqrt{N}\,\frac{\hbar k_{\mathrm{L}}\,\Omega\tau}{2\sqrt{2}}\,, \qquad (3\cdot1\cdot7\mathrm{b})$$

where $\Omega \equiv g|\beta|$ is the Rabi frequency, $N \equiv \langle\phi_a|a^+a|\phi_a\rangle$ is the average atomic number in the incident atomic beam, and $G^{(2)}(0) \equiv \langle\phi_a|a^{+2}a^2|\phi_a\rangle/N^2$ is the

second-order correlation degree of the ground-state atoms in the incident beam. We see that the spread of transverse momentum for bosonic atoms depends on both the average atomic number and the higher-order quantum correlation, but for fermionic atoms only on the average atomic number in the incident beam. For an incident single-atom beam with $|\phi_a\rangle = |1\rangle$, we find that bosonic atoms and fermionic atoms have the same spread of transverse momentum, which is identical to the result obtained in single-atom theory (Arimondo *et al.* 1981*a*, 1981*b*). Generally speaking an atomic beam composed of bosonic atoms has a larger spread of transverse momentum than that of a beam of fermionic atoms. This means that the atomic beam composed of bosonic atoms is diffracted by the standing-wave laser beam through a larger diffraction angle than both the single-atom beam and the beam composed of fermionic atoms. Physically this is easily understood since bosonic atoms statistically exhibit enhanced quantum collective behaviours.

(*3b*) *Quantum Pair Correlation of Ultracold Atoms in an Optical Crystal*

In the above section we studied the quantum statistics of atoms in a linear atomic beamsplitter composed of a standing-wave laser beam. In the example of the atomic beamsplitter, the atoms have a large velocity perpendicular to the propagation direction of the laser beam. The diffraction occurs in the direction parallel to the laser propagation direction. In this section we study another example where the atoms move along the standing-wave laser beam. In this case the standing-wave laser just acts as a one-dimensional 'optical crystal' for ultracold atoms. We show that under some special conditions the motion of ultracold atoms in the standing-wave laser beam is very similar to that of electrons in a realistic crystal or superconductor in solid physics. We start by seeking a solution of equation (2·2·7). In (2·2·7b) the excited state decays with time due to spontaneous emission. The spontaneous decay of the excited-state atoms will result in a loss of atoms from the standing-wave laser beam. To reduce the loss of the excited-state atoms via spontaneous emission, we choose the laser detuning to satisfy the conditions $\Delta \gg \gamma$, $\boldsymbol{p}^2/2m\hbar$, $\Omega \equiv 2\boldsymbol{\mu}\,.\,\boldsymbol{E}_0/\hbar$ so that the population of the excited state is small. In this case the usual adiabatic approximation can be used to eliminate the excited-state component of the atomic vector field in equations (2·2·7); this leads to a reduced scalar field equation for the ground-state atomic field (Zhang 1993):

$$\mathrm{i}\hbar\frac{\partial\psi_1}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V(\boldsymbol{r})\right)\psi_1 + \int \mathrm{d}^3r'\, Q(\boldsymbol{r},\boldsymbol{r}')\,\psi_1^+(\boldsymbol{r}')\,\psi_1(\boldsymbol{r}')\,\psi_1(\boldsymbol{r})\,, \quad (3\cdot2\cdot1\mathrm{a})$$

where $V(\boldsymbol{r})$ is the effective single-atom dipole potential and has the definition

$$V(\boldsymbol{r}) = \frac{\hbar|\Omega|^2}{4\Delta}\cos^2(\boldsymbol{k}_\mathrm{L}\,.\,\boldsymbol{r})\,. \quad (3\cdot2\cdot1\mathrm{b})$$

The nonlinear term in (3·2·1a) originates from the vacuum photon exchange between atoms and describes the light-induced dipole–dipole interaction. The light-induced dipole–dipole interaction potential in the standing-wave laser beam has the definition

$$Q(\boldsymbol{r},\boldsymbol{r}') = \frac{\hbar|\Omega|^2}{2\Delta^2}\,\gamma W(\boldsymbol{r}-\boldsymbol{r}')\cos(\boldsymbol{k}_{\mathrm{L}}\,.\,\boldsymbol{r})\cos(\boldsymbol{k}_{\mathrm{L}}\,.\,\boldsymbol{r}')\,. \qquad (3\cdot2\cdot1\mathrm{c})$$

The physics indicated in $(3\cdot2\cdot1)$ is very obvious. For ultracold atoms a standing-wave laser beam effectively acts as a periodic potential for individual atoms and the vacuum photon exchange induced between individual atoms by the excitation of the standing-wave laser leads to an effective two-body interatomic interaction potential which is a direct result of our vector quantum field theory. Since the interatomic interaction potential has the spatial exchange symmetry $Q(\boldsymbol{r},\boldsymbol{r}') = Q(\boldsymbol{r}',\boldsymbol{r})$, we can write down an effective Hamiltonian for the motion of the ultracold atoms in the standing-wave laser beam:

$$H_{\mathrm{eff}} = \int \mathrm{d}^3r\;\psi_1^+(\boldsymbol{r})\,H_0\,\psi_1(\boldsymbol{r}) + \tfrac{1}{2}\int \mathrm{d}^3r \int \mathrm{d}^3r'\;\psi_1^+(\boldsymbol{r})\,\psi_1^+(\boldsymbol{r}')\,Q(\boldsymbol{r},\boldsymbol{r}')\,\psi_1(\boldsymbol{r}')\,\psi_1(\boldsymbol{r})\,, \qquad (3\cdot2\cdot2)$$

where $H_0 \equiv -\hbar^2\nabla^2/2m + V(\boldsymbol{r})$ is the single-atom Hamiltonian. To determine the dynamic behaviour of the ultracold atoms in a standing-wave laser field, it is helpful to make a comparison of the motion of electrons in a crystal lattice (Haken 1976) with a lattice potential $V(\boldsymbol{r})$ and a phonon-induced many-electron interaction $Q(\boldsymbol{r},\boldsymbol{r}')$. Although the physics is totally different for the two cases, the mathematical forms of the systematic Hamiltonians are identical. Hence we apply the same methods used to study the motion of electrons in a crystal lattice to treat the motion of ultracold atoms in a standing-wave laser. The first step is to determine a complete set of eigenfunctions $\phi_K(\boldsymbol{r})$ for the single-atom Hamiltonian $H_0\,\phi_K = E_K\,\phi_K$. For simplicity we discuss only the one-dimensional case where the atoms move along the laser beam, which is assumed to propagate in the x direction. Since the single-atom potential has periodic structure in the x direction, in terms of the well-known Bloch theorem the eigenfunctions may further be written in the form $\phi_K(\boldsymbol{r}) = \exp(-\mathrm{i}Kx)\,u_K(\boldsymbol{r})$, with the periodic functions $u_K(\boldsymbol{r})$ satisfying $u_K(x+\lambda_{\mathrm{L}}/2,\,y,\,z) = u_K(x,\,y,\,z)$. Hence we see that the standing-wave laser beam effectively acts as a one-dimensional 'optical crystal' for the ultracold atoms with the lattice constant $\lambda_{\mathrm{L}}/2$ in the x direction, where λ_{L} is the laser wavelength. On the other hand, the periodicity of the potential leads to an energy-band structure of the eigenvalues E_K. The periodic functions $u_K(\boldsymbol{r})$ and the energy eigenvalues E_K can be calculated by numerical methods, but this is not the main task in this section. Here we assume that they are known. In terms of the eigenfunctions and the energy-band structure of the energy eigenvalues of the single-atom Hamiltonian, we now discuss the filling of the ultracold atoms into the energy bands. For convenience, we limit our discussion to bosonic atoms. At first we assume that the atoms are cooled down to a temperature which is close to the edge of the bottom energy band, where the energy eigenvalues may be expressed approximately as $E_K = E_0 + \hbar^2K^2/2m^*$. The value E_0 depends on the choice of the laser parameters and determines the lowest energy of the atoms in the 'optical lattice'. The effective mass of the atom in the 'optical lattice' is defined by $m^* = \hbar^2/(\partial^2E_K/\partial K^2)_0$, with the

subscript 0 denoting the edges of the energy bands ($K = 0$). Near the edges of the energy bands, the motion of the atoms in the 'optical lattice' is analogous to that of atoms with mass m^* in free space. Hence when ultracold atoms with a temperature below the edge of the bottom energy band are filled into the 'optical lattice', one expects that most bosonic atoms will condense at the edge of the bottom energy band and form a macroscopic condensate at the lowest energy state. However, with an increase of the filling number of the ultracold atoms near the edge of the bottom energy band, the interatomic interaction will become important. The interatomic interaction will result in collisions of atoms which will 'kick' atoms away from the condensate. If the number of atoms filled into the 'optical crystal' is finite, so that the effect of the interatomic interaction may be treated as a perturbation, the atom 'kicked' out by collisions will move like a free atom near the edge of the bottom energy band with an effective kinetic energy $\hbar^2 K^2/2m^*$ and an effective mass m^*. To exactly describe the collisional excitations of atoms from the lowest-energy condensate state to the 'free-moving' states near the edge of the bottom energy band, we expand the atomic quantum field in terms of the single-atom eigenfunctions

$$\psi_1(\boldsymbol{r}) = \sum_K a_K \, \phi_K(\boldsymbol{r}) \,. \tag{3·2·3}$$

Equation (3·2·3) transforms the Hamiltonian (3·2·2) into the form

$$H_{\text{eff}} = \sum_K E_K \, a_K^+ \, a_K + \tfrac{1}{2} \sum_{K_1, K_2, K} U(K_1, K_2, K) \, a_{K_1}^+ \, a_{K_2}^+ \, a_{K_2+K} \, a_{K_1-K} \,, \tag{3·2·4}$$

where the collisional integral is

$$U(K_1, K_2, K) = \int \mathrm{d}^3 r \int \mathrm{d}^3 r' \, \mathrm{e}^{\mathrm{i}K(x-x')} \, u_{K_1}(\boldsymbol{r})^* \, u_{K_2}(\boldsymbol{r}')^* \times Q(\boldsymbol{r}, \boldsymbol{r}') \, u_{K_2+K}(\boldsymbol{r}') \, u_{K_1-K}(\boldsymbol{r}) \,.$$

At low temperatures, where most of the bosonic atoms occupy the lowest energy state E_0 ($K = 0$), the Hamiltonian can be diagonalised into the form

$$H_{\text{eff}} = \sum_{K \neq 0} \epsilon(K) \, b_K^+ \, b_K + \epsilon_0 \,, \tag{3·2·5}$$

where $\epsilon(K) = \sqrt{E_K(E_K + 2NU_0)}$, $\epsilon_0 = NE_0 + N^2 U_0/2$ and the coefficient $U_0 = \int \mathrm{d}^3 r \int \mathrm{d}^3 r' \, |u_0(\boldsymbol{r})|^2 \, |u_0(\boldsymbol{r}')|^2 \, Q(\boldsymbol{r}, \boldsymbol{r}')$. In equation (3·2·5) the operators b_K and b_K^+ are determined by the well-known Bogoliubov transformation

$$b_K = \xi_K \, a_K - \eta_K \, a_{-K}^+, \qquad b_K^+ = \xi_K \, a_K^+ - \eta_K \, a_{-K} \,, \tag{3·2·6a}$$

where the transformation coefficients satisfy the relations

$$\xi_K^2 - \eta_K^2 = 1, \qquad \xi_K = \sqrt{[1 + (E_K + NU_0)/\epsilon(K)]/2} \,. \tag{3·2·6b}$$

Physically the operators b_K and b_K^+ describe the annihilation and creation of a 'quasi-particle'. Here the quasi-particle is composed of a pair of correlated ultracold atoms with momenta $\hbar K$ and $-\hbar K$ near the edge of the bottom energy band. Such pairs of correlated ultracold atoms represent the collisional excitations of atoms from the lowest-energy condensate state. For the light-induced dipole–dipole collisions discussed here, the interatomic dipole–dipole interaction is generally attractive. This leads to a negative collisional coefficient U_0 which corresponds to a negative scattering length. The negative collisional coefficient will result in an imaginary excitation energy $\epsilon(K)$ when the filling number of atoms increases to the value $N_c = E_K/2|U_0|$ for certain excitations $\pm\hbar K$. This means that the Bogoliubov transformation and the quasi-particle picture will no longer be valid to explain the excitations of the atoms near the edge of the bottom energy band by dipole–dipole collisons when the lowest energy state has a large population of atoms. In fact it is well known that the attractive interatomic interaction results in the so-called Bose instability in the case of a superfluid. Fortunately in our case, the filling number of atoms in the standing-wave laser is usually finite for the case of laser cooling and trapping. Hence in general the dipole–dipole interaction may be treated as a perturbation. In addition, for fermionic atoms, the discussion is similar to that for bosonic atoms. In particular, the quantum pair correlation of fermionic atoms will be a direct analogy to the well-known Cooper pairs of electrons in a superconductor (Haken 1976). Finally, we point out that the quantum pair correlation of atoms can be formed only at a very low temperature. With increasing temperatures, the thermal collisions will predominate over the attractive dipole–dipole interaction induced by vacuum photon exchange, at which point such quantum pairs of atoms will be destroyed by thermal collisions.

4. Nonlinear Atom Optics

Before the appearance of lasers, optical phenomena were almost always linear. Nonlinear optics for light waves was born just after the invention of the laser. The high degree of spatial and temporal coherence and the high intensity of a laser beam enables us to study nonlinear optical phenomena. Similarly, one may ask the question whether it is possible to study nonlinear effects of atomic de Broglie waves in atom optics. By analogy to nonlinear optics, the study of nonlinear atom optics relies on two conditions. One is that a coherent atomic source is necessary to provide a coherent atomic beam which has analogous properties to the laser beam in nonlinear optics. The other is that a nonlinear 'medium' for a coherent atomic de Broglie wave should exist to provide an atomic nonlinearity. Although an atomic source for a coherent atomic beam is not yet available in the laboratory, research towards the Bose–Einstein condensation of atoms in a confined space has recently made great progress. The groups at the National Institute of Standards Technology and University of Colorado, and also at Rice University, have observed Bose–Einstein condensation in an atomic trap. The realisation of Bose–Einstein condensation of atoms in a confined space was a big step towards a coherent atomic source or 'atom laser', where the bosonic atoms play a similar role to that of the photons in a conventional laser. Hence an 'atom laser' is no longer only an idea for theoreticians to study. In this section we will not discuss the mechanisms and principles of how to realise an 'atom laser'. Our

purpose is to develop a general nonlinear theory for atom optics which would apply if such an 'atom laser' were to be used in atom optics systems. We will show what the nonlinear 'medium' is for coherent atomic de Broglie waves.

(4a) General Formalism for Nonlinear Atom Optics

The optical nonlinearity is caused when an intense laser beam propagates through an optical medium. Such a nonlinearity is usually represented by an intensity-dependent polarisation of the medium. In this section we show that a light field will act as a 'nonlinear medium' for a coherent atomic beam which is analogous to the laser for photons. A coherent atomic beam would fulfill at least the following two conditions: (1) the atomic beam should have a high single-mode degeneracy of bosonic atoms; and (2) the propagation of the atomic beam in space will not affect such a single-mode degeneracy. These conditions require that the atomic beam has a high degree of macroscopic coherence. The single-mode degeneracy of bosonic atoms in a coherent atomic beam is similar to the intensity of a single-mode laser beam in coherent optics. Hence we may also refer to the single-mode bosonic degeneracy as the intensity of a coherent atomic beam. By the above arguments, a single-mode coherent atomic beam may be considered to be just a propagating spatially uniform Bose–Einstein condensate. In general, it is difficult to produce a spatially uniform Bose–Einstein condensate by atomic cooling and trapping techniques. Hence in this paper we consider the general case. To describe the propagation of a coherent atomic beam in a laser beam, the vector quantum field theory is necessary. Since the coherent atomic beam is composed of a Bose–Einstein condensate which has a high bosonic degeneracy, the vector field operators $\psi_j(\boldsymbol{r})$ may approximately be replaced by the macroscopic coherent wavefunctions $\phi_j(\boldsymbol{r}) = \langle\psi_j(\boldsymbol{r})\rangle$. The difference caused by such a replacement depends on the bosonic degeneracy of the atomic beam and may be neglected for a 'high-intensity' coherent atomic beam. With such an assumption, the nonlinear Schrödinger equations (2·2·7) for a quantum field can now be reinterpreted as nonlinear Schrödinger equations for the macroscopic wavefunctions of a coherent atomic beam. Due to the presence of optical excitations, the atoms in a laser field are either in the excited state or in the ground state. However, only the atoms which remain in the ground state after the interaction can transport spatial coherence at large distances (Chebotayev *et al.* 1985). This condition is realised when the interaction of atoms with a laser field is in the adiabatic regime, where the laser detuning Δ is larger than other characteristic frequencies such as the Rabi frequency $|\Omega| \equiv |2\boldsymbol{\mu}\,.\,\boldsymbol{E}/\hbar|$ and the spontaneous emission rate γ. In this case the excited-state component of the macroscopic atomic wavefunction has the adiabatic solution (Zhang and Walls 1993*b*)

$$\phi_2(\boldsymbol{r}) \approx -\frac{\Omega^{(+)}}{2(\Delta+\mathrm{i}\gamma/2)}\,\phi_1(\boldsymbol{r}) - \frac{\mathrm{i}}{\Delta+\mathrm{i}\gamma/2}\int \mathrm{d}^3r\, L(\boldsymbol{r}-\boldsymbol{r}')^*\,\phi_1^*(\boldsymbol{r}')\,\phi_2(\boldsymbol{r}')\,\phi_1(\boldsymbol{r}) + \frac{G_2}{\hbar(\Delta+\mathrm{i}\gamma/2)}, \qquad (4\cdot1\cdot1)$$

where we have defined the Rabi frequencies $\Omega^{(\pm)} \equiv 2\boldsymbol{\mu}\,.\,\boldsymbol{E}^{(\pm)}/\hbar$. In the adiabatic regime the propagation of a coherent atomic beam is determined by a scalar nonlinear

Schrödinger equation of the ground-state atomic macroscopic wavefunction in terms of equations (2·2·7) and (4·1·1):

$$i\hbar\frac{\partial\phi_1}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V(\boldsymbol{r}) + V_R(\boldsymbol{r})\right)\phi_1 + \int d^3r'\,\chi(\boldsymbol{r},\boldsymbol{r}')\,\phi_1^*(\boldsymbol{r}')\,\phi_1(\boldsymbol{r}')\,\phi_1(\boldsymbol{r}) + \int d^3r' \int d^3r''\,\eta(\boldsymbol{r},\boldsymbol{r}',\boldsymbol{r}'')\,\phi_1^*(\boldsymbol{r}'')\,\phi_1^*(\boldsymbol{r}')\,\phi_1(\boldsymbol{r}'')\,\phi_1(\boldsymbol{r}')\,\phi_1(\boldsymbol{r}) + \cdots . \tag{4·1·2}$$

The first term in (4·1·2) describes the propagation of a single-atom beam in linear atom optics. The effective single-atom dipole potential has the definition

$$V(\boldsymbol{r}) = \frac{\hbar|\Omega^{(+)}(\boldsymbol{r})|^2}{4(\Delta + i\gamma/2}, \tag{4·1·3a}$$

and an additional random potential

$$V_R(\boldsymbol{r},t) = \{\hbar\Omega^{(-)}[\Gamma_1(\boldsymbol{r},t) + \Gamma_2^+(\boldsymbol{r},t)] + \text{h.c.}\}/2(\Delta + i\gamma/2) \tag{4·1·3b}$$

describes the effects of spontaneous emission by which the atoms are scattered to other incoherent channels. The second and third terms respectively correspond to the two-body and three-body collisions of atoms which are induced by the vacuum photon exchanges. In analogy to conventional nonlinear optics, we recognise that the two- and three-body interactions correspond to a third- and fifth-order nonlinearity for the coherent atomic beam. Hence the physics implicit in (4·1·2) is that when the coherent atomic beam propagates in a laser beam, the optical excitations result in the vacuum photon exchange between atoms which leads to a nonlinearity for the atomic wave. In other words, the laser beam effectively acts as a nonlinear 'medium' for a coherent atomic beam. The third-order atomic nonlinear coefficient has the definition

$$\chi(\boldsymbol{r},\boldsymbol{r}') = \frac{i\hbar(\gamma/2 + i\Delta)}{4(\Delta^2 + \gamma^2/4)^2}[(\gamma/2 - i\Delta)L(\boldsymbol{r}-\boldsymbol{r}')\,\Omega^{(+)}(\boldsymbol{r})\,\Omega^{(-)}(\boldsymbol{r}') - |L(\boldsymbol{r}-\boldsymbol{r}')|^2|\Omega^{(+)}(\boldsymbol{r}')|^2 + \text{h.c.}]\,. \tag{4·1·4}$$

In general the fifth-order atomic nonlinearity $\eta(\boldsymbol{r},\boldsymbol{r}',\boldsymbol{r}'')$, or the three-body interaction of atoms in the laser beam, is of order $\sim\gamma^3/(\Delta^2+\gamma^2/2)^{3/2}$ and may be neglected in the adiabatic regime where $\Delta \gg \gamma$.

(4b) Atomic Soliton in a Laser Beam

In the above section we developed a general formalism for nonlinear atom optics of a coherent atomic beam. In this section we apply the formalism to study the propagation of a coherent atomic wave packet in a travelling-wave laser beam. A coherent atomic wave packet is composed of a spatial non-uniform Bose–Einstein condensate, which is analogous to a coherent laser pulse for photons. The schematic diagram in Fig. 2 shows the geometry used in our discussion. An

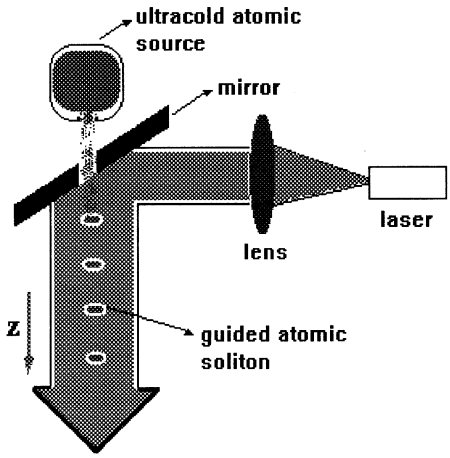

Fig. 2. Schematic diagram of atomic solitons guided by a laser beam.

ultracold atomic source is used to load a coherent atomic wave packet into the laser beam. The ultracold atomic source is assumed to achieve the condition for Bose–Einstein condensation. The atomic wave packet propagates along the direction of the laser propagation, which is specified in the z direction. The atomic wave packet is assumed to have a centre wave vector K_0 and a corresponding kinetic energy E_a, so that we have

$$\phi_1(\boldsymbol{r},t) = \phi(\boldsymbol{r},t)\,\mathrm{e}^{\mathrm{i}K_0 z - \mathrm{i}E_a t}\,. \tag{4·2·1}$$

In terms of (4·1·2)–(4·1·4), we have the nonlinear Schrödinger equation for the ground-state atomic wave packet (Zhang *et al.* 1994)

$$\mathrm{i}\hbar\left(\frac{\partial\phi}{\partial t} + v_g\frac{\partial\phi}{\partial z}\right) = -\frac{\hbar\nabla_T^2}{2m}\phi - \frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial z^2} + \frac{\hbar|\Omega^{(+)}|^2}{4\delta}\phi + \int \mathrm{d}^3r'\,\chi(\boldsymbol{r},\boldsymbol{r}')\,\phi^*(\boldsymbol{r}')\,\phi(\boldsymbol{r}')\,\phi(\boldsymbol{r})\,, \tag{4·2·2}$$

where we have neglected the loss term and the random potential in the adiabatic regime. The effective detuning has the definition $\delta = \Delta - k_{\mathrm{L}}\,v_{\mathrm{g}} - k_{\mathrm{L}}\,v_{\mathrm{r}}/2$ with $v_{\mathrm{g}} = \hbar K_0/m$ denoting the group velocity of the atomic wave packet and $v_{\mathrm{r}} = \hbar k_{\mathrm{L}}/m$ denoting the photon recoil velocity. The third-order nonlinear coefficient for atoms interacting with a travelling-wave laser beam has the form

$$\chi(\boldsymbol{r},\boldsymbol{r}') = \frac{\hbar\gamma|\Omega^{(+)}(\boldsymbol{r})|^2}{4\delta^2}[\chi_{\mathrm{s}}(\boldsymbol{r},\boldsymbol{r}') + \chi_{\mathrm{a}}(\boldsymbol{r},\boldsymbol{r}')]\,,$$

$$\chi_{\mathrm{s}}(\boldsymbol{r},\boldsymbol{r}') = 2W(\boldsymbol{r}-\boldsymbol{r}')\cos[k_{\mathrm{L}}(z-z')]\,,$$

$$\chi_{\mathrm{a}}(\boldsymbol{r},\boldsymbol{r}') = -K(\boldsymbol{r}-\boldsymbol{r}')\sin[k_{\mathrm{L}}(z-z')]\,. \tag{4·2·3}$$

The nonlinear coefficient $\chi(\boldsymbol{r},\boldsymbol{r}')$ is composed of an exchange symmetric part $\chi_{\rm s}(\boldsymbol{r},\boldsymbol{r}') = \chi_{\rm s}(\boldsymbol{r}',\boldsymbol{r})$ and an exchange asymmetric part $\chi_{\rm a}(\boldsymbol{r},\boldsymbol{r}') = -\chi_{\rm a}(\boldsymbol{r}',\boldsymbol{r})$. The asymmetrical part is caused by the many-atom spontaneous emission in the travelling-wave laser beam, which results in the breaking of spatial geometric symmetry. For a standing-wave laser beam, the nonlinear coefficient is always exchange-symmetric. Equation (4·2·2) is a three-dimensional nonlinear Schrödinger equation. In general it is difficult to find its exact solutions in 3D space. However, physically we can simplify the equation by introducing some restrictions. At first we assume that the travelling-wave laser beam has a Gaussian transverse profile so that

$$\Omega^{(+)}(\boldsymbol{r}) = \Omega_0 \exp[-(x^2+y^2)/2W_{\rm L}^2]\,,$$

where Ω_0 is the peak Rabi frequency and $W_{\rm L}$ is the transverse width of the laser beam. Second, we assume that the transverse width of the loaded atomic wave packet is much narrower than that of the laser beam. By means of these assumptions, we can seek an approximately transverse–longitudinal separated solution of equation (4·2·2) with the form $\phi(\boldsymbol{r},t) \approx u(x,y)\,\Phi(z,t)\exp(-{\rm i}E_T\,t/\hbar)$. The transverse motion of the atomic wave packet satisfies the Helmholtz equation

$$[\nabla_T^2 + k_T^2\, n_{\rm eff}^2(x,y)]u(x,y) = 0\,, \qquad (4\cdot2\cdot4{\rm a})$$

where

$$n_{\rm eff}(x,y) \approx \left[1 - \frac{m\Omega_0^2}{2\delta\hbar k_T^2}\left(1 - \frac{x^2+y^2}{W_{\rm L}^2}\right)\right]^{\frac{1}{2}} \qquad (4\cdot2\cdot4{\rm b})$$

is the effective refractive index for the atomic wave propagating along the laser beam. In (4·2·4b) we have assumed that the transverse width of the atomic wave packet is narrower than that of the laser beam so that the Gaussian transverse profile of the laser beam can be expanded to first order. For a laser frequency detuned below the atomic resonance, equation (4·2·4a) has an identical form to the Helmholtz equation describing the propagation of a light wave in a parabolic dielectric waveguide. Hence we conclude that a travelling-wave laser beam with a negative detuning just acts as an atomic waveguide. Such an atomic waveguide has the bounded transverse Hermite–Gaussian eigenmodes

$$u_{nm}(x,y) = \frac{(2^{n+m}n!\,m!)^{-\frac{1}{2}}}{W_a\sqrt{\pi}} \exp\left(-\frac{x^2+y^2}{2W_a^2}\right) H_n\left(\frac{x}{W_a}\right) H_m\left(\frac{y}{W_a}\right), \qquad (4\cdot2\cdot5)$$

with $W_a = (2\hbar|\delta|W_{\rm L}^2/m\Omega_0^2)^{\frac{1}{4}}$ the transverse width of fundamental mode $u_{00}(x,y)$, which is determined by the laser parameters and the mass of the atom. By choosing the appropriate parameters for the laser beam and the atom, we can realise the propagation of the atomic wave with the fundamental mode. This case is similar to that in waveguide optics where one can realise a single transverse mode for a guided light wave by choosing an appropriate core diameter for an optical waveguide such as a fibre. Here we assume that the transverse profile

of an initial loaded atomic wave packet matches the spatial distribution of the bounded fundamental mode. In this case, we have the propagation equation for the longitudinal envelope $\Phi(z,t)$ of the atomic wave packet

$$\mathrm{i}\hbar\left(\frac{\partial\Phi}{\partial t}+v_g\frac{\partial\Phi}{\partial z}\right)=-\frac{\hbar^2}{2m}\frac{\partial^2\Phi}{\partial z^2}-\hbar\int \mathrm{d}z'\,\bar{\chi}(z,z')\,|\Phi(z')|^2\,\Phi(z)\,, \quad (4\cdot2\cdot6)$$

where the nonlinear coefficient has the definition

$$\bar{\chi}(z,z')=\int \mathrm{d}x\int \mathrm{d}y\int \mathrm{d}x'\int \mathrm{d}y'\,|u_{00}(x',y')|^2\,|u_{00}(x,y)|^2\chi(\boldsymbol{r},\boldsymbol{r}')\,.$$

Equation $(4\cdot2\cdot6)$ is a one-dimensional, nonlocal, nonlinear Schrödinger equation. In analogy to the propagation of an optical pulse in an optical Kerr-type nonlinear medium, the function $\bar{\chi}(z,z')$ is similar to the third-order nonlinear optical response function in nonlinear optics (Agrawal 1989). The only difference is that the nonlinear optical response function of an optical medium to an optical pulse occurs in the time domain. In this sense, the function $\bar{\chi}(z,z')$ may be considered as the nonlinear spatial response function of a 'light medium' to a coherent atomic wave packet. Physically such a nonlinear spatial response function describes the light-induced elastic collisional strength of atoms. In nonlinear optics, the nonlinear optical response function of an optical medium to an optical pulse may be replaced by a delta function in the time domain, since the material response is usually faster than the duration of the propagation pulses (Agrawal 1989). The equivalent question in nonlinear atom optics is whether the function $\bar{\chi}(z,z')$ [or $\chi(\boldsymbol{r},\boldsymbol{r}')$] can effectively be treated as a delta function potential in space so that $(4\cdot2\cdot6)$ can approximately be replaced by a local nonlinear Schrödinger equation where the soliton solution is apparent. In general this is not true, and we must treat the nonlocal nonlinear Schrödinger equation. However, if the two-body interaction $\chi(\boldsymbol{r},\boldsymbol{r}')$ is weak enough that no two-body bound states exist, and the incident atomic wave packet has a width larger than the region where the two-body potential varies, the usual pseudopotential approximation can hold (Pathria 1972). For the present case, the function $\chi(\boldsymbol{r},\boldsymbol{r}')$ depends on the functions $W(\boldsymbol{r}-\boldsymbol{r}')$ and $K(\boldsymbol{r}-\boldsymbol{r}')$, which have sharp peaks in the region of the atomic absorption wavelength λ_{L}. Hence if the width of the atomic wave packet is larger than the optical wavelength, we can make the pseudopotential approximation, or 'shape-independent' potential approximation, by the replacement (Pathria 1972)

$$\chi(\boldsymbol{r},\boldsymbol{r}')\rightarrow\frac{4\pi a\hbar^2}{m}\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,,$$

where a is the s-wave scattering length corresponding to the actual dipole–dipole interaction potential. In the Born approximation we have

$$a=\frac{m}{4\pi\hbar^2}\int \mathrm{d}^3r'\,\chi(\boldsymbol{r},\boldsymbol{r}')\exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{r}')\,.$$

The scattering length a is negative for an attractive potential and positive for a repulsive potential. In the pseudopotential approximation equation (4·2·6) has the form

$$\mathrm{i}\hbar\frac{\partial\bar{\Phi}}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\bar{\Phi}}{\partial\zeta^2} - \hbar\chi_0|\bar{\Phi}|^2\bar{\Phi}, \qquad (4\cdot2\cdot7)$$

where we have made the transformations

$$\zeta = z - v_g t, \qquad t = t, \qquad \Phi = \sqrt{N}\,\bar{\Phi}$$

where N is the total number of atoms in the wave packet and the nonlinear coefficient now has the definition

$$\chi_0 = N\frac{4\pi\hbar^2|a|}{m}\int \mathrm{d}x \int \mathrm{d}y\,|u_{00}(x,y)|^4,$$

where we have considered the negative scattering length for the light-induced dipole–dipole interaction which is an attractive potential. It is well known that (4·2·7) has a family of soliton solutions of the form (Yajima 1974)

$$\bar{\Phi} = A\,\mathrm{sech}\left(\frac{m\chi_0}{2\hbar}(\zeta - Bt)\right)\exp[\mathrm{i}\chi_0 A^2 t/2 - \mathrm{i}(mB^2/2\hbar)t + \mathrm{i}mB\zeta/\hbar],$$

with soliton amplitude $A = \sqrt{m\chi_0/4\hbar}$. So far we have proved that when a coherent atomic wave packet propagates along a travelling-wave laser beam, the travelling-wave laser beam acts just as a nonlinear atomic waveguide which allows soliton propagation of an atomic wave under appropriate conditions. The light-induced dipole–dipole interaction is the origin of the atomic nonlinearity in such a waveguide. It is valuable to point out that if the characteristic region where the nonlinear coefficient varies is comparable to or less than the spatial soliton width, the nonlocal nonlinear Schrödinger equation (4·2·6) still allows the persistence of the robust soliton even in the presence of nonlocality (Moloney 1992).

(*4c*) *Nonlinear Bragg Scattering of a Coherent Atomic Beam*

In this section we study another nonlinear atom optics phenomenon. We consider a coherent atomic beam which passes through a standing-wave laser beam as shown in Fig. 3. The structure shown is similar to the case discussed in Section 3*a*. However, there we neglected the vacuum photon exchange between atoms due to spontaneous emission, which may be justified if a high-Q single-mode resonant cavity or a microwave field is used to act as an atomic beamsplitter. In general, when the spontaneous emission is included, only the ground-state atomic wave can transport the atomic coherence for a long distance. As a result, the propagation of a coherent atomic beam is described by (4·1·2) in the adiabatic regime. To simplify our discussion, we assume that the incident atomic beam is parallel to the yz plane, with a central momentum vector $(p_x = 0, p_y = \hbar K_{0y}, p_z = \hbar K_{0z})$ and a kinetic energy E. In the off-resonance adiabatic regime, we neglect the

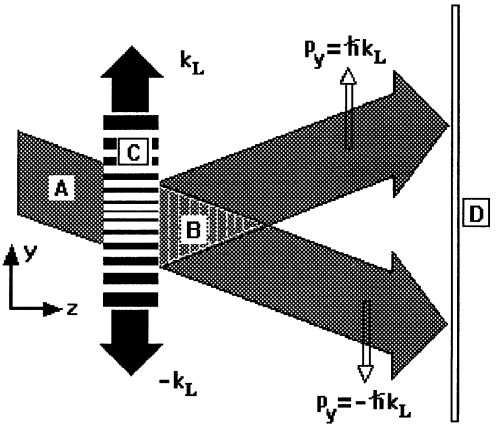

Fig. 3. Schematic diagram for Bragg scattering of a coherent atomic beam by a standing-wave laser beam. Here A denotes the incident atomic beam, B the near-field interference region, C the standing-wave laser beam, and D the detection screen.

effect of atomic absorption on the profile of the laser beam (Zhang and Walls 1993*b*) and have the Rabi frequency

$$|\Omega^{(+)}(\boldsymbol{r})|^2 \approx \Omega_0^2 \exp\left(-\frac{x^2+z^2}{W_L^2}\right)\cos^2(k_L\, y)\,, \qquad (4\cdot3\cdot1)$$

where the transverse Gaussian profile of the laser beam has been taken into account. In terms of $(4\cdot1\cdot2)$ we have the following nonlinear Schrödinger equation for the propagation of a coherent atomic beam passing through a standing-wave laser beam:

$$\mathrm{i}\hbar\frac{\partial\phi_1}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + \frac{\hbar\Omega_0^2}{4\Delta}\exp\left(-\frac{x^2+z^2}{W_L^2}\right)\cos^2(k_L\, y)\right]\phi_1 + \int \mathrm{d}^3r'\,\chi(\boldsymbol{r},\boldsymbol{r}')|\,\phi_1(\boldsymbol{r}')|^2\,\phi_1(\boldsymbol{r})\,,$$

$$\chi(\boldsymbol{r},\boldsymbol{r}') = \frac{\hbar\gamma\Omega_0^2}{2\Delta^2}\,W(\boldsymbol{r}-\boldsymbol{r}')\exp\left(-\frac{x^2+z^2+x'^2+z'^2}{2W_L^2}\right)\cos(k_L\, y)\cos(k_L\, y')\,. \qquad (4\cdot3\cdot2)$$

In $(4\cdot3\cdot2)$ the single-atom potential has a periodic structure in the y direction which results in the diffraction of the atomic beam when it passes through the laser beam. In addition, the single-atom potential is also a function of the coordinates x and z due to the transverse Gaussian profile of the laser beam. However, if the width of the atomic beam in the x direction is narrower than that of the laser beam, we can ignore the effect of the propagation of the atomic beam in the x direction. For stationary propagation the macroscopic wavefunction describing the coherent atomic beam can be expanded in terms of the diffraction modes

$$\phi_1(\boldsymbol{r}) = \left(\sum_{n=-\infty}^{\infty}\Phi_n(y,z)\exp[\mathrm{i}(K_{0y}+2nk_L)y]\right)\exp(\mathrm{i}K_{0z}\,z - \mathrm{i}Et/\hbar)\,, \qquad (4\cdot3\cdot3)$$

where $\Phi_n(y,z)$ is the spatial slowly varying envelope of the diffracted atomic beams. Substituting $(4\cdot3\cdot3)$ into $(4\cdot3\cdot2)$ and neglecting the second derivatives of the diffracted atomic wavefunctions $\Phi_n(y,z)$ in the slowly varying envelope

approximation, we obtain the multi-mode coupled wave equations for the diffracted beams:

$$\mathrm{i}\left(\frac{\partial \Phi_n}{\partial \tau} + v_n \frac{\partial \Phi_n}{\partial y}\right) = \omega_n \Phi_n + g(\tau)(2\Phi_n + \Phi_{n-1} + \Phi_{N+1}) + \sum_{jlq} \eta_{njlq} \Phi_j^* \Phi_l \Phi_q \,, \tag{4·3·4}$$

where $\tau = z/v_\mathrm{g}$ defines the effective time variable, $v_\mathrm{g} = \hbar K_{0z}/m$ is the group velocity of the atomic beam in the z direction, $\omega_n = (n_0 + 2n)^2\omega_\mathrm{r}$ is the de Broglie frequency for the nth diffraction mode with $\omega_\mathrm{r} = \hbar k_\mathrm{L}^2/2m$ defining the single-photon recoil frequency, and $v_n = (n_0 + 2n)v_\mathrm{r}$ is the group velocity of the nth diffraction mode in the y direction with $v_\mathrm{r} = \hbar k_\mathrm{L}/m$ defining the single-photon recoil velocity. We have denoted $n_0 = K_{0y}/k_\mathrm{L}$ and generally n_0 need not be an integer. The linear coupling coefficient between different diffracted beams has the definition $g(\tau) = \Omega_0^2 \exp(-\tau^2/\tau_0^2)/16\Delta$ with $\tau_0 = W_\mathrm{L}/v_\mathrm{g}$ defining the flight time of atoms through the laser beam. The linear coupling coefficient is effectively a time-dependent function due to the transverse Gaussian profile of the laser beam. If the nonlinear terms do not exist, then equations (4·3·4) are identical to those in the single-atom diffraction theory for a low-density atomic beam. For a coherent atom beam, the light-induced dipole–dipole interaction leads to a nonlinear mixing of the diffracted modes. The nonlinear coupling coefficients in the slowly varying envelope approximation have the forms (Zhang and Walls 1993*b*)

$$\eta_{njlq}(\tau) = -\beta g(\tau)(\delta_{j,l+q-n+1} + \delta_{j,l+q-n-1} + 2\delta_{jl+q-n}) \,, \tag{4·3·5}$$

where $\beta = 2\gamma|\int W(\boldsymbol{r})\,\mathrm{d}^3r|/\Delta$. If we ignore the physics implicit in (4·3·4), mathematically the equations are identical to the nonlinear coupled-wave equations describing the propagation of light waves in a nonlinear periodic medium. Hence we conclude that for a coherent atomic beam, a standing-wave laser beam acts as a nonlinear periodic medium for the atomic waves. Equations (4·3·4) can be solved by numerical techniques. Here we consider a simple example where the diffraction of atoms occurs in the Bragg resonance regime (Martin *et al.* 1988). The diffraction of a single-atom beam in the Bragg resonance regime has been experimentally demonstrated (Martin *et al.* 1988). To satisfy the Bragg resonance condition, the momentum component of the incident atomic beam in the y direction must be arranged to match one single-photon recoil momentum. Experimentally this can be realised by controlling the angle between the atomic beam and the standing-wave nodes so that the incident momentum $p_{y0} = -m\hbar k_\mathrm{L}$. The integer m denotes the diffraction order. For $m = 1$ and $n_0 = -1$, one has the first-order Bragg scattering. The higher-order Bragg scattering has similar characteristics to the first order. Therefore we need only consider the first-order Bragg scattering. In this case, only two diffraction modes with indices $n = 0$ and 1 can resonantly couple with each other. Physically, the first-order Bragg resonance corresponds to an absorption and stimulated emission process from the undiffracted ($n_0 + 2n = -1$) to the diffracted ($n_0 + 2n = 1$) momentum eigenmode. Hence for the first-order Bragg scattering, we can neglect

the higher-order diffraction modes with indices $n \geq 2$ in the numerical simulation. In this case, the standing-wave laser beam simply acts as a nonlinear atomic beamsplitter and the macroscopic atomic wavefunction in (4·3·3) now has the simple form in the Bragg resonance regime

$$\phi_1(\boldsymbol{r}) = [\Phi_0(y,z)\exp(-\mathrm{i}k_{\mathrm{L}}\,y) + \Phi_1(y,z)\exp(\mathrm{i}k_{\mathrm{L}}y)]\exp(\mathrm{i}K_{0z}\,z - \mathrm{i}Et/\hbar)\,. \quad (4\cdot3\cdot6)$$

To numerically simulate the nonlinear Bragg scattering of the coherent atomic beam, we assume that the incident atomic beam has a Gaussian density profile with wave vector in the y direction matching the single-photon recoil momentum

$$\phi_{\mathrm{in}}(y,-\infty) = \sqrt{\rho_0}\exp\left(-\frac{y^2}{2w_y^2} - \mathrm{i}k_{\mathrm{L}}\,y\right)\mathrm{e}^{\mathrm{i}K_{0z}z-\mathrm{i}Et/\hbar}\,, \quad (4\cdot3\cdot7)$$

where w_y is the beam width in the y direction and $\rho_0 = J_0/v_{\mathrm{g}}$ is the density of the atomic beam which is provided by an ultracold atomic source localised at $z = -\infty$. Fig. 3 above is a schematic diagram for Bragg scattering of atoms. The atomic source continuously releases the ultracold atomic flow with the rate J_0 which has the equivalent meaning to the light intensity in conventional coherent optics. In terms of (4·3·6) and (4·3·7) we have the following initial conditions for equations (4·3·4):

$$\Phi_0(y,-\infty) = \sqrt{\rho_0}\exp\left(-\frac{y^2}{2w_y^2}\right), \qquad \Phi_1(y,-\infty) = 0\,. \quad (4\cdot3\cdot8)$$

Under conditions (4·3·8) we numerically simulate the spatial propagation of the incident atomic beam in terms of (4·3·4) and (4·3·8). The spatial density distribution is shown in Figs 4 and 5. For comparison, in Fig. 4 we give the results for a low-density atomic beam with a negligible nonlinearity. The laser parameters in Fig. 4 are chosen to satisfy the condition $g_0 = \int_{-\infty}^{\infty} g(\tau)\,\mathrm{d}\tau = \pi/4$ so that the incident atomic beam splits into two components with identical density profiles. In terms of the result in Fig. 4*a*, when the incident atomic beam passes through the laser region, the interaction of atoms with the laser beam results in the splitting of the initial atomic beam into two coherent beams with group velocities $\pm v_{\mathrm{R}}$ respectively in the y direction. In the near-field region the two coherent components overlap and we see density oscillations due to the interference of the two coherent beams (see Fig. 4*b*). With an increase in propagation distance, the two coherent beams gradually separate and the interference vanishes in the far-field region where we have two fully separated beams with Gaussian profiles, as shown in Figs 4*c* and 4*d* and also in Fig. 4*a*. The splitting of the atomic beam is due to the exchange of atoms between two diffraction modes. Such an exchange means that some atoms in the incident beam change their momentum in the y direction. The change of atomic momentum is caused by the mechanical effect induced by the standing-wave laser beam. The number of exchanged atoms between two diffraction modes depends on the linear coupling strength g_0. When the laser parameters are chosen to give $g_0 = \pi$, the exchange of atoms between two diffraction modes experiences a cycle and the atoms are completely diffracted back to the initial incident beam when

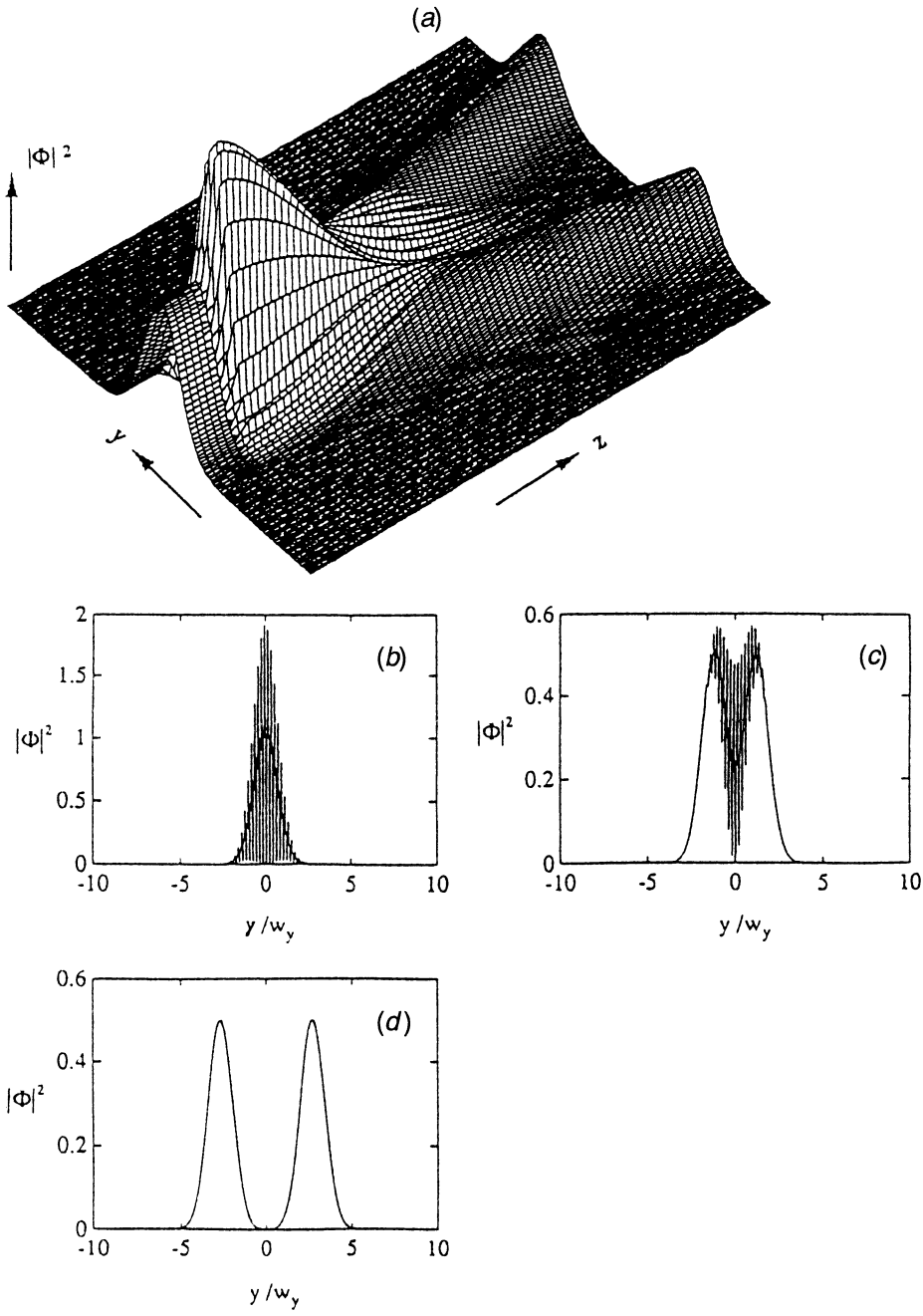

Fig. 4. Density distribution of the atomic beam for linear Bragg scattering. The parameters chosen are $g_0 = \pi/4$ and $\beta\rho_0 = 0{\cdot}001$.

the nonlinear coupling is negligible. In this case, the incident beam remains unchanged after interaction with the laser. However, for a coherent atomic beam with a high density, the nonlinearity is important and the diffraction of atoms will

be affected by it. To see how the nonlinearity affects the diffraction of the coherent atomic beam, we choose a typical linear coupling strength $g_0 = \pi$ which does not result in any change of the incident atomic beam in the low-density linear case.

In Fig. 5 we give the numerical results for the nonlinear beamsplitter when $g_0 = \pi$. We see that in the presence of atomic nonlinearity, the incident atomic beam coherently splits into two components and one of the split components exhibits two-peaked structure in far-field region. Evidently, the splitting of the atomic beam is caused by the nonlinear coupling terms. Physically the nonlinearity in $(4\cdot3\cdot4)$ is analogous to a Kerr-type nonlinearity in nonlinear optics. Such an atomic Kerr-type nonlinearity results in self-phase and cross-phase modulation of the diffracted atomic beams. Due to the self-phase and cross-phase modulation in the presence of the atomic nonlinearity, a density-dependent phase shift is added to the linear phase shift in the usual linear beam-splitter. As a result, we see the difference between linear Bragg scattering and nonlinear Bragg scattering of an atomic beam. In summary, we again see that the light-induced dipole–dipole interaction plays an important role in the propagation of a coherent atomic beam in a laser beam. Such an interatomic interaction can be employed to design different devices in coherent atom optics, just as the optical nonlinearity can be used to make different optical devices in coherent optics.

(4d) Nonlinear Atomic Cavity: Manipulating a Bose–Einstein Condensate

To further show the possible applications of the atomic nonlinearity to coherent atom optics, in this section we study a nonlinear atomic cavity which yields modulation and compression of a coherent atomic wave packet composed of a spatially non-uniform Bose–Einstein condensate. The atomic cavity is formed by two spatially separated atomic mirrors in the longitudinal x direction of the atomic centre-of-mass motion, and a harmonic trap in the transverse direction. In addition a laser beam perpendicular to the cavity axis induces an effective nonlinear atomic interaction via photon exchange and absorption. A schematic diagram of this arrangement is shown in Fig. 6. The cavity is perpendicular to the direction of the Earth's gravitational field, so that its longitudinal modes are not affected by gravity. The purpose of the harmonic trap in the transverse direction is to prevent the atoms from falling out of the cavity due to gravity. The transverse confinement could be provided by, for example, a trapping magnetic field or microwave field. Before the laser field is turned on, a coherent atomic wave packet is loaded into this cavity from an ultracold atomic source which is assumed to achieve the condition of Bose–Einstein condensation. The nonlinear atomic cavity is analogous to the nonlinear optical cavity in coherent optics. Here the laser beam acts as a nonlinear crystal for the intracavity atomic wave. To simplify our discussions, we consider a simple case where the cavity is set to the operation of a single transverse mode. This is the case if the atoms initially occupy the ground transverse mode of the transverse trap prior to application of the laser field. In addition, to obtain a nonlinear interaction for an extended time, we choose the incident laser beam width to fill the atomic cavity so that the transverse distribution of the laser beam can be considered uniform in the cavity. Under these assumptions, we have a reduced one-dimensional nonlinear Schrödinger equation for the longitudinal-envelope atomic wavefunction $\phi(x,t)$ in terms of the general formalism of nonlinear atom optics given in Section 4*a* and Zhang *et al.* (1995):

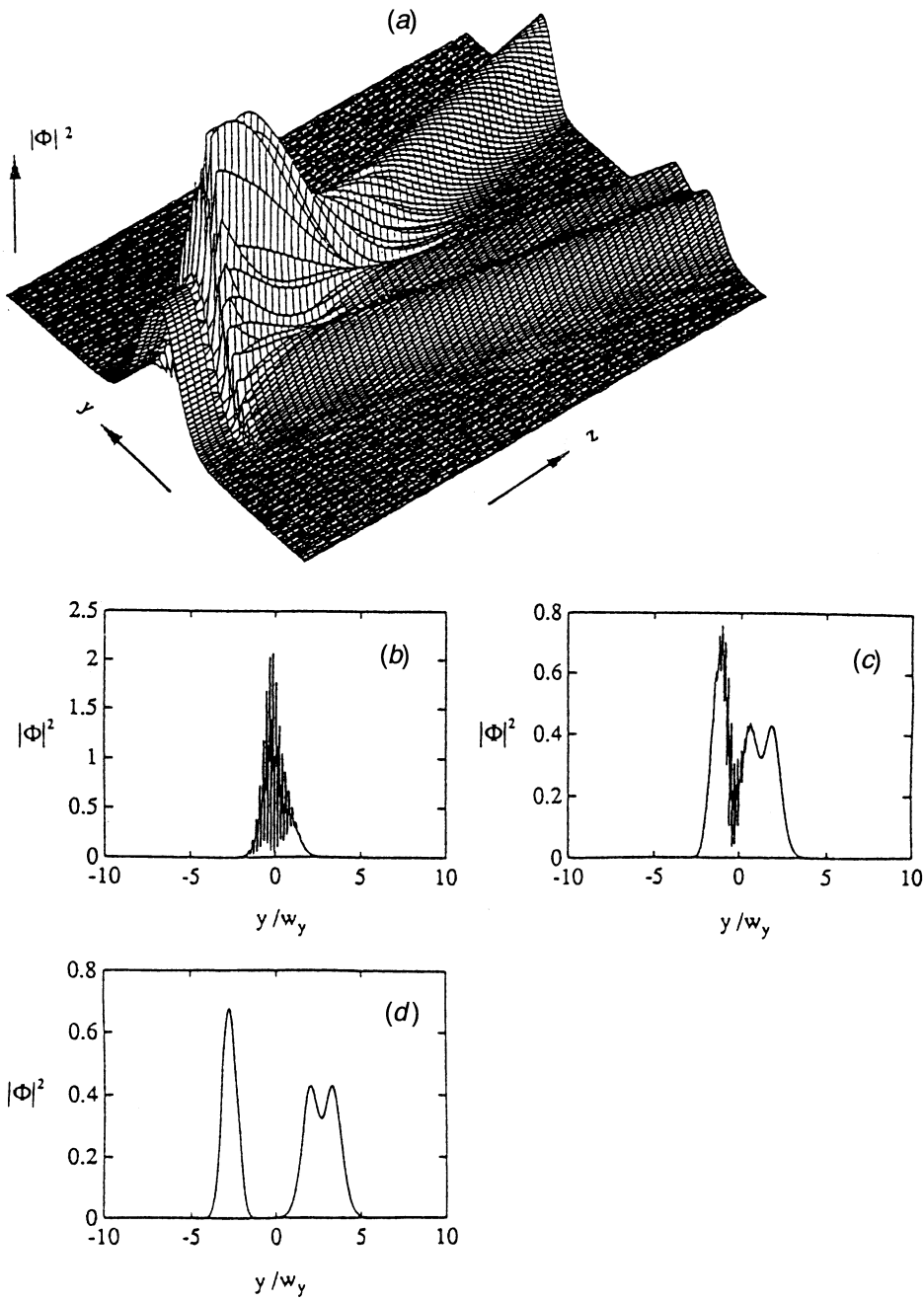

Fig. 5. Nonlinear Bragg scattering of a coherent atomic beam. The parameters chosen are $g_0 = \pi$ and $\beta\rho_0 = 0{\cdot}3$.

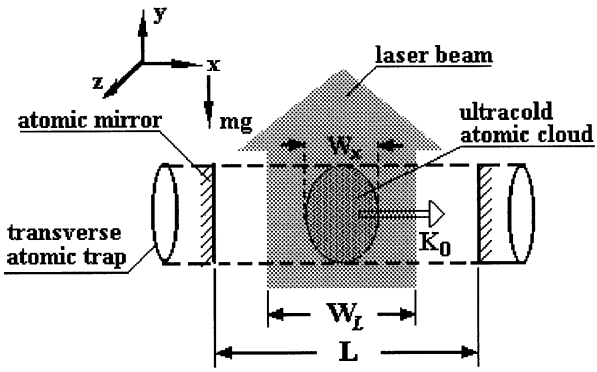

Fig. 6. Schematic diagram for a nonlinear atomic cavity.

$$i\hbar\frac{\partial\phi}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_L(x) + V_0\left(1 - i\frac{\gamma}{2\delta}\right)(1 - \kappa|\phi(x)|^2)\right]\phi, \qquad (4\cdot4\cdot1)$$

where we have defined the parameter $V_0 = \hbar\Omega_0^2\,\delta/(4\delta^2+\gamma^2)$ and a characteristic length

$$\kappa = \gamma\lambda_L^2(\delta\lambda_L\,\beta/\pi^{\frac{5}{2}}W_T + 3\gamma/2^{\frac{3}{2}})/[W_T(4\delta^2+\gamma^2)\pi^{\frac{3}{2}}],$$

which is due to the nonlinearity caused by the light-induced dipole–dipole interaction and the photon absorption. The coefficient β is determined by a spatial integral of the dipole–dipole potential (Zhang *et al.* 1995). In equation $(4\cdot4\cdot1)$, $V_L(x)$ is introduced to describe the effect of the atomic mirrors at the ends of the atomic cavity. In addition, the spontaneous emission decay is included in $(4\cdot4\cdot1)$ to describe the loss of atoms from the cavity into other incoherent channels due to inelastic scattering during the spontaneous emission. Such an inelastic scattering process has been neglected in the previous sections where we assumed a large laser detuning and a short period of interaction of atoms with the laser beam. However, in the present case we include the loss due to spontaneous emission for general consideration. In terms of $(4\cdot4\cdot1)$, we clearly see that the real part of the nonlinear potential is analogous to the Kerr-type nonlinearity of conventional nonlinear optics, and the imaginary part of the nonlinear potential is similar to the self-saturation absorption of conventional nonlinear optics. Hence we conclude that the laser beam acts as a nonlinear absorption 'crystal' for the cold atomic wave when the effects of spontaneous emission are included. In view of this discussion, it is clear that the proposed nonlinear atomic cavity is analogous to a conventional optical cavity with a saturable absorber (Yariv 1967). Here the laser beam plays the role of the saturation absorption cell for the atoms. We numerically simulate $(4\cdot4\cdot1)$ for the initially loaded macroscopic atomic wavefunction

$$\Phi(x,0) = \left(\frac{N}{\sqrt{\pi}\,W_x}\right)^{\frac{1}{2}} \exp\left(-\frac{(x-L/2)^2}{2W_x^2} + \mathrm{i}K_0\,x\right), \qquad (4\cdot4\cdot2)$$

where W_x is the initial width of the atomic wave packet in the longitudinal direction and L is the cavity length.

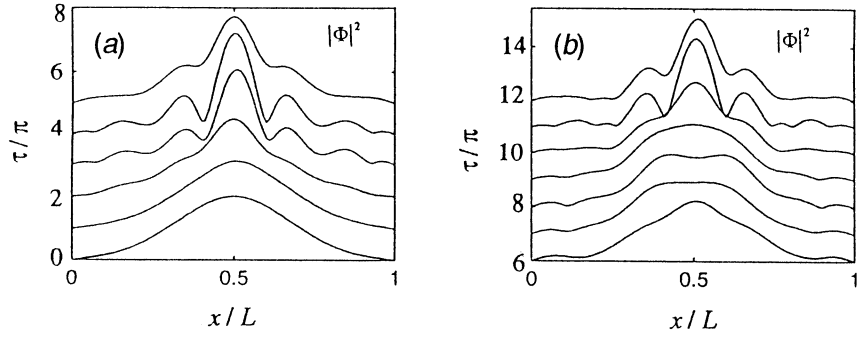

Fig. 7. Self-phase modulation of a coherent atomic wave packet in a nonlinear atomic cavity. The spatial density distributions plotted correspond to a sequence of times varying (*a*) from $\tau \equiv \hbar K_0 \pi t/mL = 0$ to 5π, and (*b*) from $\tau = 6\pi$ to $\tau = 12\pi$. The other parameters chosen for the simulation are $W_x = L/4\sqrt{2}, \delta = 1000\gamma$ and $\kappa(N-1)/\sqrt{\pi}\,W_x = 0{\cdot}3$.

When the laser field is turned on the motion of the atomic cloud is affected by its interaction with the laser. In terms of (4·4·1) this interaction leads to a loss and a nonlinearity for the intracavity atomic wave. The magnitude of the loss and the nonlinearity of atoms can be controlled by choosing both the laser and the atomic parameters appropriately. In the example shown here, we choose a laser detuning large enough to reduce the random scattering of atoms into other incoherent channels due to spontaneous emission and permit a negligible loss of ground-state atoms in the cavity. In such an off-resonance regime, the dipole–dipole interaction is weakened as well. However, one can still obtain a considerable nonlinearity by choosing the appropriate peak density and transverse width of the atomic wave packet. In this case, the laser field just acts as a 'nonlinear crystal' with a Kerr-type nonlinearity for the intracavity atomic wave. Such an atomic nonlinearity results in the self-phase modulation of the intracavity atomic wave. In Fig. 7 we show the time evolution of the atomic density in the cavity due to the self-phase modulation. Such a self-phase modulation of atomic waves is very similar to that of a coherent optical pulse in a lossless nonlinear optical cavity. On the other hand, since some loss of ground-state atoms to other incoherent channels always occurs due to spontaneous emission, we finally simulate the intracavity atomic dynamics including the effects of spontaneous emission. The results are plotted in Fig. 8, which displays the compression of the spatial distribution of the atomic wave packet. This compression is induced by the combined effects of self-phase modulation and the intracavity atomic loss. When spontaneous emission is not negligible, the atoms in the cavity are partially scattered into incoherent channels by random photon recoil. The incoherently scattered atoms account for the atomic cavity losses. As we have seen in (4·4·1),

for a coherent atomic wave packet the atomic cavity losses depend on the spatial distribution of the atomic density in this cavity. In terms of (4·4·1), the total loss of the atomic cavity has the form $\Gamma(x) \approx V_0\,\gamma/2\delta\hbar[1-\kappa|\phi(x)|^2]$. Fig. 9 shows the spatial dependence of the total cavity losses at different times. We see that the edge of the atomic wave packet 'feels' a larger loss rate than the centre of the wave packet. As a result, a dip appears in the spatially dependent cavity loss rate. The shape of the dip changes with time, due to the self-modulation of the atomic density profile. This spatial dip in the loss rate modifies the atomic wave packet and narrows its spatial distribution at the centre of the atomic cavity. This is similar to passive mode-locking in a conventional optical cavity with a nonlinear absorber (Yariv 1967), which can be used to compress a coherent light pulse to generate an ultrashort pulse. The only difference is that the compression of optical pulses is in the time domain whereas the compression of the coherent atomic wave packet occurs in the spatial domain.

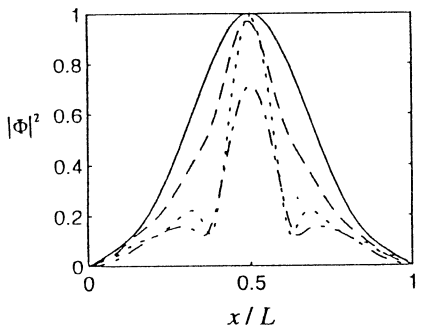

Fig. 8. Self-induced compression of a coherent atomic wave packet in a nonlinear atomic cavity due to the presence of spontaneous emission. The laser and atomic parameters chosen for the simulations are $W_x = L/4\sqrt{2}$, $\delta = 10\gamma$ and $\kappa(N-1)/\sqrt{\pi}W_x = 0{\cdot}3$. The various times are solid curve ($\tau = 0$), dashed curve ($\tau = \pi$), dotted curve ($\tau = 2\pi$) and dash–dot curve ($\tau = 3\pi$).

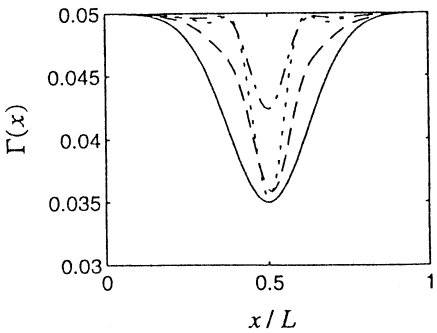

Fig. 9. Spatial dependence of the total loss rate of the nonlinear atomic cavity. The parameters and curves are the same as in Fig. 8.

5. Summary

In this paper we have systematically introduced theoretical methods to study the interaction of ultracold atoms with a laser field in the frame of a vector quantum field theory. Both the atoms and the laser field are treated as quantum fields. A stochastic nonlinear Schrödinger equation for ultracold atoms and a quantum propagation equation for laser photons are derived. These equations form the basis on which to study many-body quantum statistics and atomic nonlinearity in atom optics. As a straightforward application of the theory, we construct a general formalism of nonlinear atomic optics for a coherent atomic

beam. Applying the formalism to the propagation of a coherent atomic wave packet in a travelling-wave laser beam, we find that the travelling-wave laser beam acts as a nonlinear atomic waveguide which allows the propagation of atomic solitons. For a standing-wave laser beam, we study the diffraction of a coherent atomic beam in the Bragg resonance regime. We find that the photon exchanges between ultracold atoms in the coherent beam induce an effective atomic nonlinearity for the atomic waves which is analogous to an optical Kerr-type nonlinearity for coherent light waves. Such an atomic nonlinearity can result in self-phase modulation and cross-phase modulation of the diffracted atomic waves. Finally, we have discussed a nonlinear cavity for ultracold atoms which is analogous to the nonlinear optical cavity in coherent optics. In these examples we see that in the ultracold regime, the interaction of atoms with photons exhibits many novel phenomena which cannot be explained by the single-atom theory. With the recent realisation of Bose–Einstein condensation of rubidium and lithium atoms, the quantum statistics of ultracold atoms is expected to play an important role in the study of atomic physics and quantum optics. The vector quantum field theory introduced here will be the most useful tool with which to study such cold atomic samples.

Acknowledgments

I am very grateful to Dr Barry Sanders and Professors D. F. Walls, Pierre Meystre and E. M. Wright for their many helpful discussions during our collaboration. I would also like to thank Dr Sigurd Dyrting for his stimulating suggestions which greatly improved the manuscript. This work is supported by Australian Research Council and Macquarie University Research grants.

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Manuscript received 29 September 1995, accepted 13 February 1996