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Introduction to the Views of Distant Objects in Kerr–Newman Spacetimes

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Abstract

Gravitational effects cause the visual appearance of the external universe to be distorted to observers near a Kerr–Newman black hole. This paper introduces the problem and presents some computer-generated pictures which show how the 'sky' appears to an observer near a Kerr black hole.

1. Introduction

The lensing effects of gravitation upon the images of objects has received increased attention in recent years. Most of the work has been devoted to the effects as seen by an observer at a great distance from the lensing influence. The detailed effects seen by a nearby observer are a less urgent topic but can nonetheless reveal interesting and significant aspects of the gravitational field.

This paper provides an introduction to the lensing effects seen by an observer in a stationary, asymptotically flat, black-hole spacetime. Cunningham (1975) investigated the problem in the case of a Schwarzschild spacetime. In such a spacetime, one can normalise by a factor of M (the mass parameter of the black hole) and thereby reduce the investigation to that of a single spacetime. Metzenthen (1990) did a similar investigation for a charged non-rotating black hole spacetime (the Reissner–Nordström spacetimes). In this case there is an essential one-parameter (the charge/mass ratio) family of spacetimes to be investigated. The Reissner–Nordström spacetime includes the Schwarzschild spacetime as a limiting case provided that the limit is taken in the appropriate manner (the difficulty being that the two spacetimes have a different topology).

The most general class of a stationary, asymptotically flat, electro-vac black-hole spacetime is the Kerr–Newman family of spacetimes. These are believed to describe the gravitational field of a rotating charged black hole. There is essentially a two parameter (charge/mass and rotation/mass) family to be considered in the Kerr–Newman spacetimes, although the charge is less effective than the rotation in providing interesting features.

The Reissner–Nordström family (which is a limiting case of the Kerr–Newman family, provided that the appropriate limit is taken) has a spherical symmetry which simplifies the required analysis. Any geodesic will be confined entirely to some 3-dimensional hypersurface, thus reducing the complexity of the required

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analysis. Using appropriate coordinates, the projection of a geodesic onto a three-spacelike-dimension view will lie entirely within a plane. This symmetry is lacking in the Kerr–Newman family and considerably complicates the analysis; with the exception of a few special cases, geodesics are not confined to simply defined lower-dimensional surfaces.

2. Geodesics in Kerr–Newman Spacetimes

We make the usual assumptions: principally that the observer has no influence upon the spacetime or the propagation of light, and that light is described by photons which follow null geodesics.

(2a) The Metric

The Kerr–Newman metric in Boyer–Lindquist coordinates (t, r, θ, ϕ) is

$$g_{\mu\nu} = \begin{bmatrix} 1-(2Mr-e^2/\Sigma) & 0 & 0 & a(2Mr-e^2)\sin^2\theta/\Sigma \\ 0 & -\Sigma/\Delta & 0 & 0 \\ 0 & 0 & -\Sigma & 0 \\ a(2Mr-e^2)\sin^2\theta/\Sigma & 0 & 0 & -((r^2+a^2)^2-a^2\Delta\sin^2\theta)\sin^2\theta/\Sigma \end{bmatrix}$$

where M is the mass parameter, e is the charge parameter, a is the rotation parameter, and

$$\Sigma = r^2 + a^2\cos^2\theta\,, \tag{1}$$

$$\Delta = r^2 + a^2 - (2Mr - e^2)\,. \tag{2}$$

(2b) The Geodesic Equations

The classical method of solving the geodesic equations in a Kerr–Newman spacetime is to solve the Hamilton-Jacobi equation (Carter 1968). Alternatively, we may use proposition C.3.1 of Wald (1984): *If ξ^a is a Killing vector field and γ^b is the tangent vector of a geodesic then $\xi^a\gamma_a$ is constant along the geodesic.* $\partial/\partial t$ and $\partial/\partial\phi$ are obvious Killing vectors. Therefore any geodesic with tangent γ^μ will have

$$\gamma_\mu = \{E, \gamma_r, \gamma_\theta, -L_z\}\,, \tag{3}$$

where E and L_z are constants. Two other constants of the motion can be found by using a similar property of Killing tensor fields, and by scaling the affine parameter. This leads to

$$\gamma_r = \pm E\frac{\sqrt{\tilde{R}}}{\Delta}\,, \tag{4}$$

$$\gamma_\theta = \pm E\sqrt{\tilde{\Theta}}\,, \tag{5}$$

where

$$\tilde{\Theta} = \tilde{\kappa} - (a\sin\theta - \tilde{L}_z/\sin\theta)^2 - a^2\tilde{\mu}^2\cos^2\theta\,, \tag{6}$$

$$\tilde{R} = \left(r^2 + a^2 - a\tilde{L}_z\right)^2 - \Delta(\tilde{\mu}^2 r^2 + \tilde{\kappa})\,, \tag{7}$$

and where $\tilde{\kappa}$ and $\tilde{\mu}$ are the two extra constants of the motion. A null geodesic has $\tilde{\mu} = 0$.

Equations (3) to (5) can be manipulated to give

$$\pm\int\frac{\mathrm{d}\theta}{\sqrt{\tilde{\Theta}}} = \pm\int\frac{\mathrm{d}r}{\sqrt{\tilde{R}}}\,, \tag{8}$$

$$\phi = \mp a\int\frac{(2Mr - e^2) - a\tilde{L}_z}{\Delta\sqrt{\tilde{R}}}\mathrm{d}r \mp \tilde{L}_z\int\frac{\mathrm{d}\theta}{\sin^2\theta\sqrt{\tilde{\Theta}}}\,. \tag{9}$$

With the exception of degenerate cases, these are elliptic integrals.

(2c) The Roots of the Quartics

The roots of the polynomials $\tilde{R}(r)$ and $\tilde{\Theta}(\theta)$ are a key to the behaviour of the integrals:

- 'Bounce'. Near a simple root (multiplicity $= 1$), the behaviour of the geodesic is approximately parabolic (e.g. near a simple root of $\tilde{R}(r)$, r as a function of θ approximates a parabola).
- 'Spiral'. In the neighbourhood of a root with multiplicity two, geodesics exhibit an approximately exponential behaviour (e.g. near a simple root of $\tilde{R}(r)$, r as a function of θ approximately decays exponentially towards or grows exponentially away from the root).
- Higher multiplicity roots occur only for exceptional geodesics.

The quartic $\Theta(\theta)$ did not arise in the earlier investigations, but it has a simple behaviour, the two typical cases being shown in Fig. 1. Here $\eta = \tilde{\kappa} - (a - \tilde{L}_z)^2$ is found to be a useful combination of parameters.

The $\tilde{R}(r)$ quartic has a more complicated behaviour than the corresponding quartic of the earlier studies (Metzenthen 1990), but the behaviour of the roots above the event horizon (at $r_+ = M + \sqrt{M^2 - (a^2 + e^2)}$) is essentially the same.

(2d) A Geodesic Example

The integrals in equations (8) and (9) may be evaluated using existing algorithms (Bulirsch 1965*a*, 1965*b*, 1969) for elliptic integrals after applying the appropriate transformations. Fig. 2 shows an example of a computed geodesic.

As the spiralling behaviour (in r) shows, this geodesic is close to having a double root of $\tilde{R}(r)$. This geodesic demonstrates a behaviour which is visually much more interesting than those possible for similar cases in the spherically symmetric spacetimes.

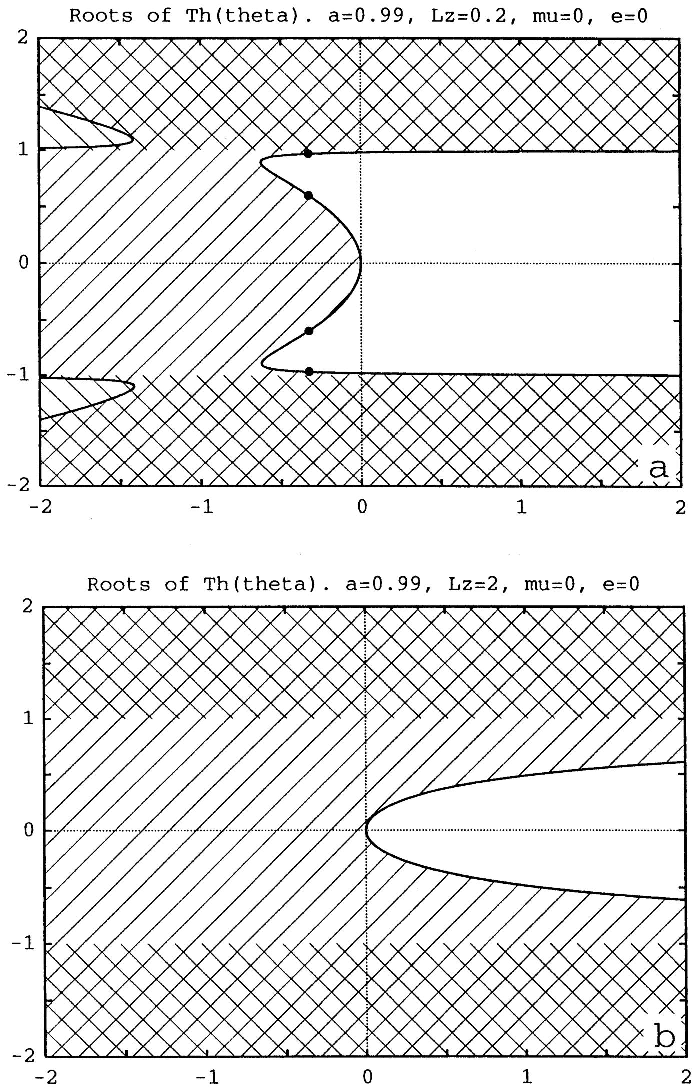


Fig. 1. Roots of $\tilde{\Theta}(\cos\theta)$ versus η for null geodesics. Both diagrams are for a Kerr spacetime with $a = 0{\cdot}99$. For illustration, the black dots show the four real roots at $\eta = -\frac{1}{3}$. The hatching shows where $\tilde{\Theta}$ is negative; a null geodesic with the given $\tilde{L}_z$ cannot have a (η, cos θ) in these regions (see equation 5).

3. Views of Distant Objects

The views seen by an observer can be computed by establishing a suitable arrangement of distant objects and then tracing null geodesics from the distant

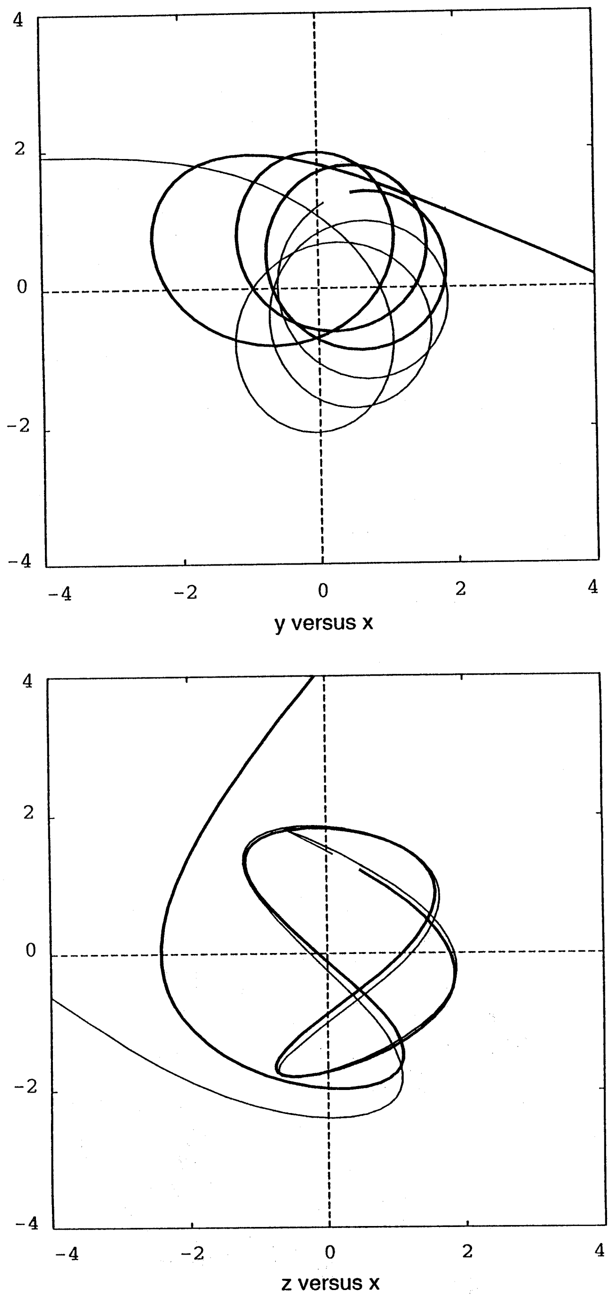


Fig. 2. Geodesic, $L_z = 1 \cdot 27237$, $\eta = 13 \cdot 9588$, where the ingoing portion (thicker line) passes through $r_0 = 10$, $\nu_0 = 0 \cdot 9$ for Kerr spacetime ($a = 0 \cdot 99$).

objects to the observer. The view is then determined by the motion of the observer. Some views which would be seen by a static observer, i.e. one at fixed r, θ, ϕ, are shown in Figs 3, 4 and 5. Each of these diagrams shows two views, a fore view where the centre of the diagram is in a direction towards $r = 0$, and

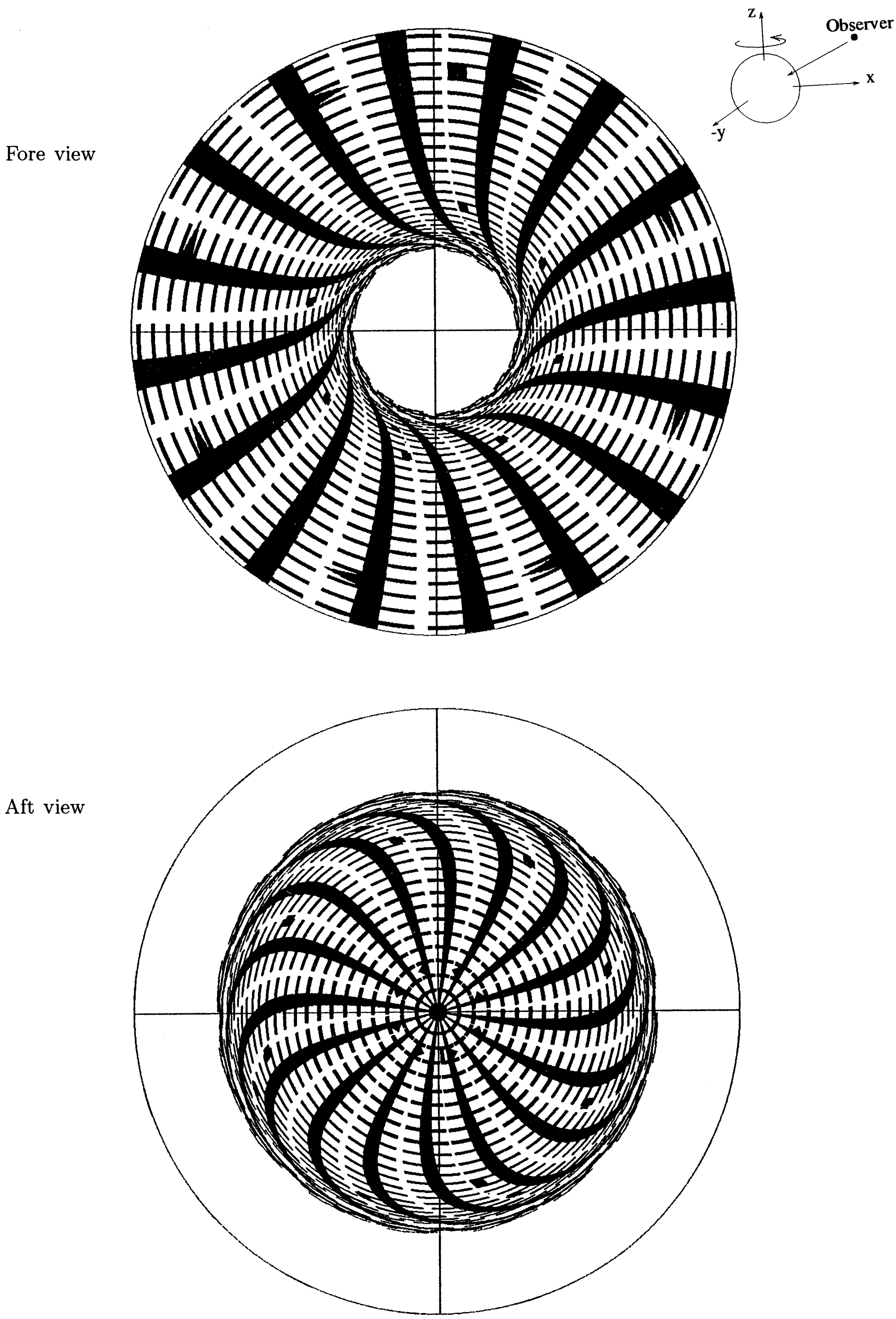
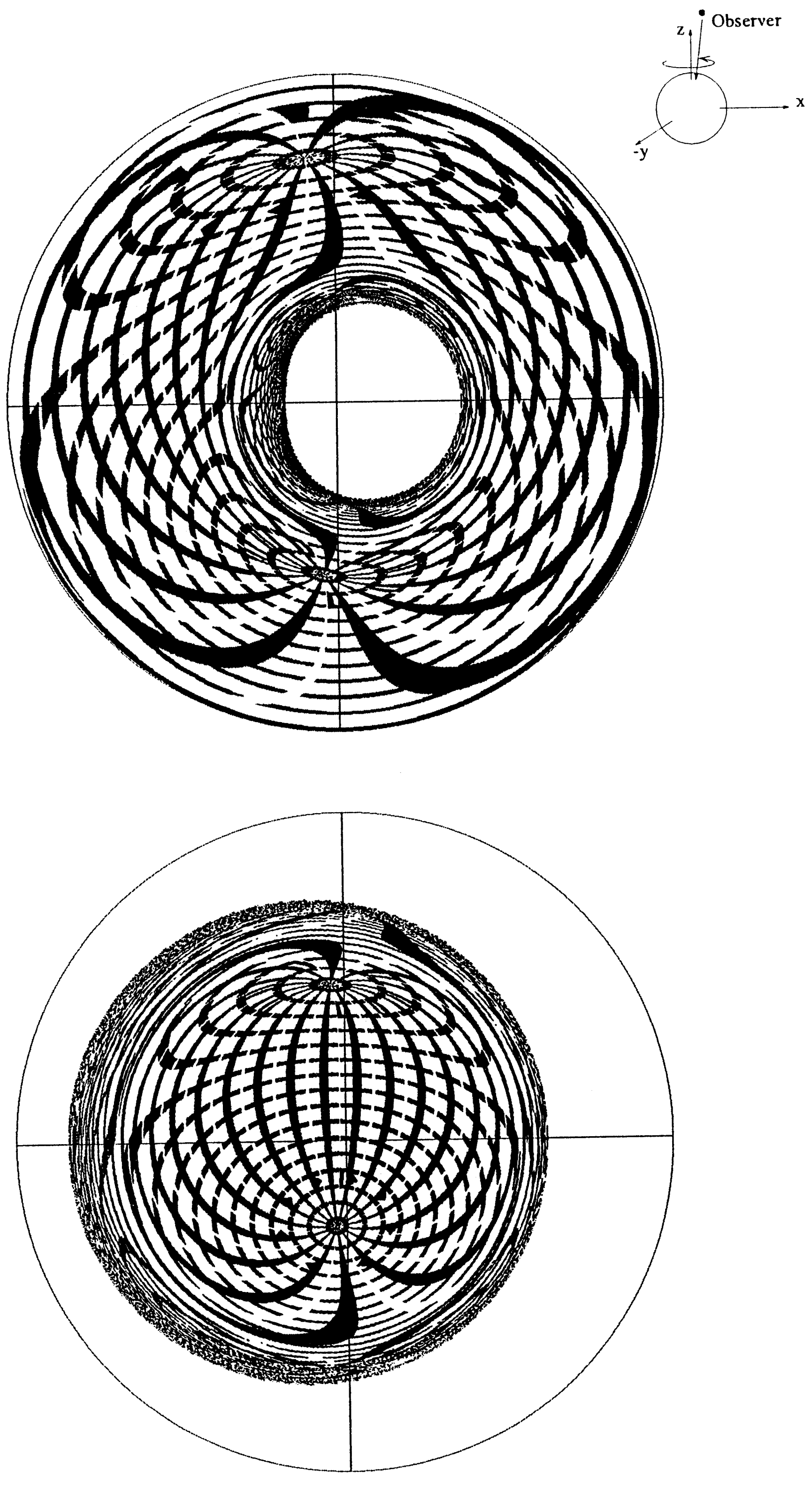


Fig. 3. The view seen by a static observer in a Kerr spacetime at $r_0 = 5$, $\theta_0 = 0°$. The spacetime has $M = 1$, $a = 0 \cdot 99$, $e = 0$. The small diagram at the upper right gives an indication of the position of the observer in the spacetime.



Fore view

Aft view

Fig. 4. The sky for a static observer at $r_0 = 5$, $\theta_0 = 60°$ in a Kerr spacetime with $M = 1$, $a = 0 \cdot 99$.

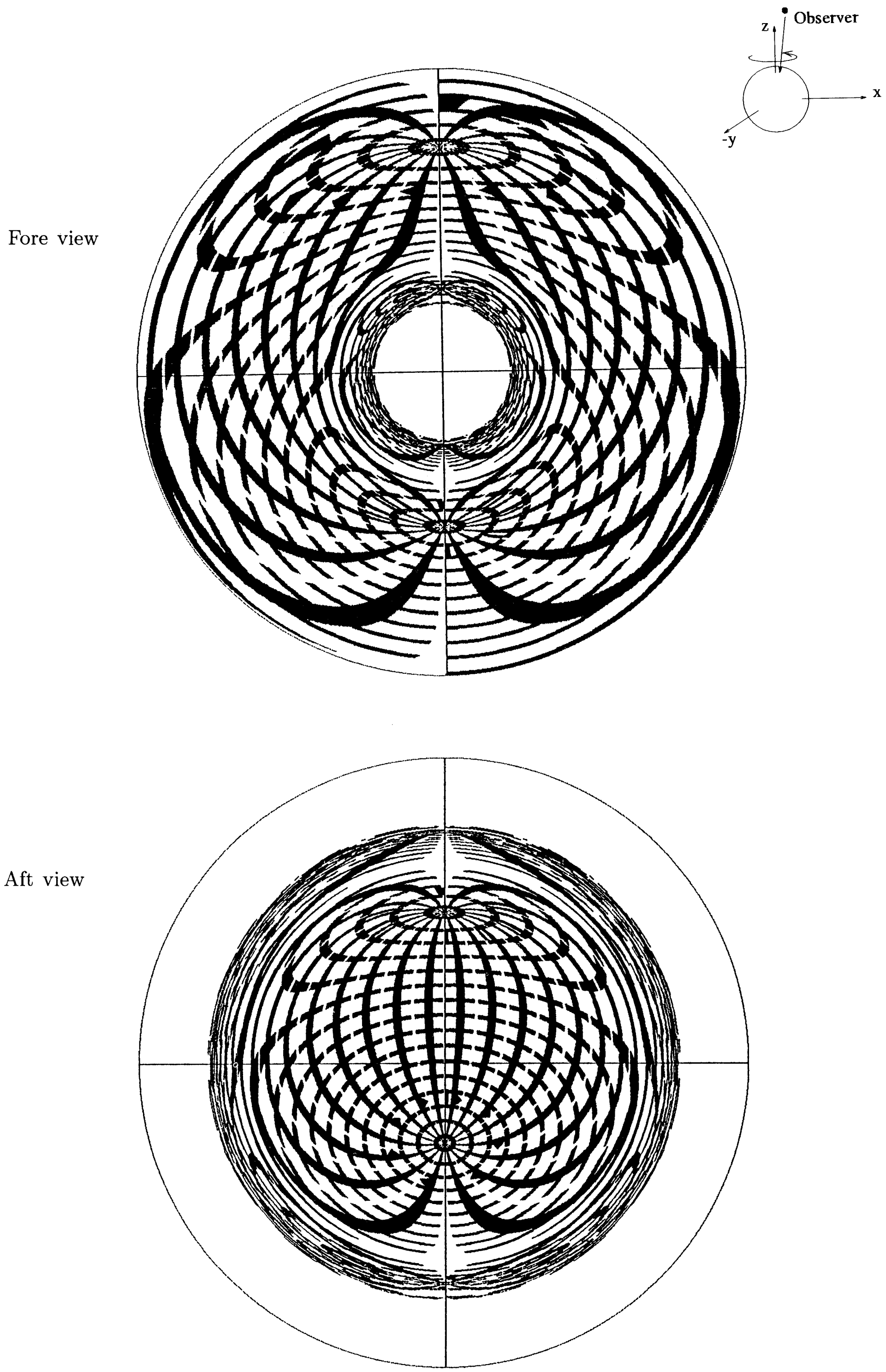


Fig. 5. A static observer's sky in a Reissner–Nordström spacetime for $r_0 = 5$, $\theta_0 = 60^\circ$ with $M = 1$, $e = 0 \cdot 99$.

an aft view where the centre is in the opposite direction. Each view shows the whole sky as seen by the observer, thus the periphery of each view is the same point as the centre of the other view. Consequently the projections have large distortions of shape away from the centres of the views.

The various features on the distant sky are designed to aid the interpretation of the diagrams. Apart from the broad lines between the poles and around the poles, there are arrow heads and blocks. Fig. 3 shows on-axis views. Figs 4 and 5 show off-axis views, from similar positions, in a Kerr and a Reissner–Nordström spacetime. Some of the main effects of the rotation in the Kerr spacetime are apparent by comparing these two diagrams.

The results presented here were obtained by direct evaluation of the integrals rather than finite-difference techniques or similar. This was believed to be computationally more efficient for this investigation. This appears to be supported by the fact that the views can be computed on a personal computer in about five minutes, a time which is believed to be comparable with that taken on a super-computer when finite-difference techniques are used.

4. Conclusion

This paper has presented an introduction to the views seen by observers in Kerr–Newman spacetimes. A more complete treatment appears in a PhD thesis (Metzenthen 1995). Topics such as the views seen by non-static observers, caustics, and redshift are dealt with there. Some of those results will be submitted for publication.

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