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Supplement: Perturbation expansion for non-small Ξ .

This is somewhat of an artistic endeavor rather than a step-by-step plod. In our hands, it went like this.

First, define $\varepsilon = \Xi^{-1}$. Eq. (19) then becomes

$$\varepsilon\omega^{\#\#} - \varepsilon\omega^{\#}\,\omega + \omega[\kappa\varepsilon] + 1 = 0\ . \tag{S1}$$

For large Ξ , $\varepsilon << 1$ and is a small parameter multiplying the highest order derivative; therefore, it seems reasonable to treat (S1) as a singular perturbation problem (Van Dyke 1964; Jordan and Smith 1990). This then requires splitting the ζ-interval (0,1) into three segments: (i) denoted by a subscript ℓ for a source-like leaf, an "inner" region near $\zeta = 0$ where rapid variation of ω seems likely[†]; (ii) denoted by a subscript $\mathfrak{r}$ for a sink-like root, a second "inner" region near $\zeta = 1$ where rapid variation of ω seems likely; (iii) denoted by a subscript $\boldsymbol{a}$ for the German 'äusser', an "outer" region of transport phloem where ω varies only slowly. Boundary conditions are applied to the inner solutions and the complete solution achieved by "matching" the inner solutions to the outer solution at points far from the boundaries.

For the outer solution, let

$$\omega_{\boldsymbol{a}} = \omega_{\boldsymbol{a}0} + \varepsilon\omega_{\boldsymbol{a}1} + \varepsilon^2\omega_{\boldsymbol{a}2} + \ldots \tag{S2}$$

[†] The "inner" label derives from fluid mechanics where it denotes a region near a boundary where rapid variation of a dependent variable is to be expected. "Outer" then denotes a region far from the boundaries where much less rapid variation is anticipated.

Then observe from Fig. 2 that, for large values of the forcing parameter Ξ , $\kappa\varepsilon \propto \Xi^{-1/3}$, which very crudely can be considered to be of order zero in ε . Plug (S2) into (S1) and, approximating $\kappa\varepsilon = 2(\varepsilon^0)$, group terms in ε^0 , ε^1 , ε^2 , … , ε^n , … to obtain:

Order 0: $$0 = \omega_{a0}[\kappa\varepsilon] + 1 \;\Rightarrow\; \omega_{a0} = -\frac{1}{\kappa\varepsilon} \;; \tag{S3}$$

Order 1: $$0 = \omega_{a0}{}^{\#\#} - \omega_{a0}\,\omega_{a0}{}^{\#} + \omega_{a1}[\kappa\varepsilon] \;\Rightarrow\; \omega_{a1} = 0 \;; \tag{S4}$$

Order 2: $$0 = \omega_{a1}{}^{\#\#} - (\omega_{a0}\,\omega_{a1}{}^{\#} + \omega_{a1}\,\omega_{a0}{}^{\#}) + \omega_{a2}[\kappa\varepsilon] \;\Rightarrow\; \omega_{a2} = 0 \;; \tag{S4}$$

and, by induction,

Order n: $$0 = \omega_{an} \;,\; n \,\hat{}\, 3 \;. \tag{S5}$$

This perturbation is unusual in that the term of order zero is a constant and the higher order terms are approximately zero:

$$\omega_a = \omega_{a0} = -\frac{1}{\kappa\varepsilon} = -\frac{\Xi}{\kappa} \;. \tag{S6}$$

Using the values of ω_{max} derived by numerical integration as a proxy for ω_{a0} , Fig. 6 demonstrates the accuracy of Eq. (S6) for large values of Ξ .

To derive an inner expansion near $\zeta = 0$, define a “stretched” coordinate $\acute{s}$ defined by $\acute{s} = \zeta\varepsilon^{-1/2}$ and transform Eq. (S1) to

$$\frac{d^2\omega_\ell}{d\acute{s}^2} - \varepsilon^{1/2}\omega_\ell\frac{d\omega_\ell}{d\acute{s}} + \omega_\ell[\kappa\varepsilon] + 1 = 0 \;, \tag{S7}$$

where the inner approximation ω_ℓ is expressed as

$$\omega_\ell = (\varepsilon^{1/2})^0\omega_{\ell 0} + (\varepsilon^{1/2})^1\omega_{\ell 1} + (\varepsilon^{1/2})^2\omega_{\ell 2} + 2((\varepsilon^{1/2})^3) \;. \tag{S8}$$

The relevant boundary conditions are

$$\omega_\ell(0) = 0 \tag{S9}$$

and[‡]

$$\lim_{ś\to\infty} \omega_\ell(ś) = \omega_{\boldsymbol{a}} = \omega_{\boldsymbol{a}0} = -\frac{1}{\kappa\varepsilon} \ . \tag{S10}$$

Next, substitute (S8) into (S7), group terms by order of $(\varepsilon^{½})$ while presuming that $[\kappa\varepsilon]$ can be treated as of order zero for grouping purposes, and obtain:

$$\text{Order 0:} \qquad \frac{d^2\omega_{\ell 0}}{dś^2} - [-\kappa\varepsilon]\omega_{\ell 0} = -1 \ . \tag{S11}$$

$$\text{Order 1:} \qquad \frac{d^2\omega_{\ell 1}}{dś^2} - [-\kappa\varepsilon]\omega_{\ell 1} = \omega_{\ell 0}\frac{d\omega_{\ell 0}}{dś} \ . \tag{S12}$$

$$\text{Order 2:} \qquad \frac{d^2\omega_{\ell 2}}{dś^2} - [-\kappa\varepsilon]\omega_{\ell 2} = \omega_{\ell 0}\frac{d\omega_{\ell 1}}{dś} + \omega_{\ell 1}\frac{d\omega_{\ell 0}}{dś} \ . \tag{S13}$$

The solution of Eq. (S11) subject to the boundary conditions (S9) and (S10) can be found by elementary means to be

$$\omega_{\ell 0}(ś) = \omega_{\boldsymbol{a}}[1 - e^{-ś\text{þ}}] = \text{þ}^{-2}[1 - e^{-ś\text{þ}}] \ , \tag{S14}$$

where $\text{þ}^2 = [-\kappa\varepsilon]$ and $\text{þ} = \sqrt{-\kappa\varepsilon}$; it is completely determined except for the value of κ . Subject to the boundary conditions $\omega_{\ell 1}(0) = 0$ and $\lim_{ś\to\infty}\omega_{\ell 1}(ś) = 0$, (S12) can also be solved by elementary means as

$$\omega_{\ell 1}(ś) = \text{þ}^{-5}[⅓e^{-ś\text{þ}} - ⅓e^{-2ś\text{þ}} - ½ś\text{þ}e^{-ś\text{þ}}] \ , \tag{S15}$$

‡ More exactly, the condition (C10) should be that $\omega_\ell(ś)$ approach $\omega_{\boldsymbol{a}}$ closely as $ś$ becomes very large. In practice, it is usually applied as stated.

although the algebra is slightly more complicated. Subject to the boundary conditions $\omega_{\ell 2}(0) = 0$ and $\lim\limits_{\acute{s}\to\infty} \omega_{\ell 2}(\acute{s}) = 0$, (S13) can likewise be solved by elementary means as

$$\omega_{\ell 2}(\acute{s}) = \text{þ}^{-8}[-\,{}^{1}/_{24}e^{-\acute{s}\text{þ}} + {}^{1}/_{6}e^{-2\acute{s}\text{þ}} - {}^{1}/_{8}e^{-3\acute{s}\text{þ}} + {}^{7}/_{24}\acute{s}\text{þ}e^{-\acute{s}\text{þ}} - {}^{1}/_{3}\acute{s}\text{þ}e^{-2\acute{s}\text{þ}} - {}^{1}/_{8}(\acute{s}\text{þ})^{2}e^{-\acute{s}\text{þ}}], \quad \text{(S16)}$$

although the algebra is rather more complicated.

To derive an inner expansion near $\zeta = 1$, it is useful to employ the variable $\xi = 1 - \zeta$ and to define a new “stretched” coordinate $\text{ş} = \xi\varepsilon^{-½}$. This yields the differential equation

$$\frac{d^2\omega_\ell}{d\text{ş}^2} + \varepsilon^{½}\omega_\ell\frac{d\omega_\ell}{d\text{ş}} + \omega_\ell[\kappa\varepsilon] + 1 = 0 , \qquad \text{(S17)}$$

where the inner approximation ω_ℓ can be expanded as

$$\omega_\ell = (\varepsilon^{½})^0\omega_{\ell 0} + (\varepsilon^{½})^1\omega_{\ell 1} + (\varepsilon^{½})^2\omega_{\ell 2} + 2((\varepsilon^{½})^3) \ . \qquad \text{(S18)}$$

The relevant boundary conditions are

$$\omega_\ell(0) = 0 \qquad \text{(S19)}$$

and

$$\lim_{\text{ş}\to\infty} \omega_\ell(\text{ş}) = \omega_{\boldsymbol{a}} = \omega_{\boldsymbol{a}0} = -\frac{1}{\kappa\varepsilon} = \text{þ}^{-2} . \qquad \text{(S20)}$$

At this boundary, however, we have the additional condition (20c) which must *eventually* be met, but not just yet:

$$\frac{d\omega_\ell}{d\text{ş}} = -\kappa\varepsilon^{½} = \text{þ}^2\,\varepsilon^{-½} , \ \ \text{ş} = 0 \ . \qquad \text{(S21)}$$

First, substitute (S18) into (S17) to get (in analogy to the development for the boundary at $\zeta = 0$):

$$\text{Order 0:} \qquad \frac{d^2\omega_{\iota 0}}{d\text{ş}^2} - \text{þ}^2\omega_{\iota 0} = -1\ . \tag{S22}$$

$$\text{Order 1:} \qquad \frac{d^2\omega_{\iota 1}}{d\text{ş}^2} - \text{þ}^2\omega_{\iota 1} = -\,\omega_{\iota 0}\frac{d\omega_{\iota 0}}{d\text{ş}}\ . \tag{S23}$$

$$\text{Order 2:} \qquad \frac{d^2\omega_{\iota 2}}{d\text{ş}^2} - \text{þ}^2\omega_{\iota 2} = -\,\omega_{\iota 0}\frac{d\omega_{\iota 1}}{d\text{ş}} - \omega_{\iota 1}\frac{d\omega_{\iota 0}}{d\text{ş}}\ . \tag{S24}$$

The solution of Eq. (S22) subject to the boundary conditions (S19) and (S20) can be found by elementary means to be

$$\omega_{\iota 0}(\text{ş}) = \omega_{\alpha}\,[1 - e^{-\text{şþ}}] = \text{þ}^{-2}[1 - e^{-\,\text{şþ}}]\ ; \tag{S25}$$

it is completely determined except for the value of κ. Subject to the boundary conditions $\omega_{\iota 1}(0) = 0$ and $\lim_{\text{ş}\to\infty} \omega_{\iota 1}(\text{ş}) = 0$, (S23) can also be solved by elementary means as

$$\omega_{\iota 1}(\text{ş}) = -\text{þ}^{-5}[\tfrac{1}{3}e^{-\text{şþ}} - \tfrac{1}{3}e^{-2\text{şþ}} - \tfrac{1}{2}\,\text{şþ}e^{-\text{şþ}}]\ , \tag{S26}$$

although the algebra is slightly more complicated. Subject to the boundary conditions $\omega_{\iota 2}(0) = 0$ and $\lim_{\text{ś}\to\infty} \omega_{\iota 2}(\text{ś}) = 0$, (S24) can likewise be solved by elementary means as

$$\omega_{\iota 2}(\text{ş}) = \text{þ}^{-8}[-\,{}^{1}/_{24}e^{-\text{şþ}} + {}^{1}/_{6}e^{-2\text{şþ}} - {}^{1}/_{8}e^{-3\text{şþ}} + {}^{7}/_{24}\text{şþ}e^{-\text{şþ}} - {}^{1}/_{3}\text{şþ}e^{-2\text{şþ}} - {}^{1}/_{8}(\text{şþ})^2e^{-\text{şþ}}]\ . \tag{S27}$$

The final step is the determination of κ by application of the boundary condition (S21) at $\text{ş} = 0$. Note first that:

$$\frac{d\omega_{\iota 0}}{d\text{ş}}\Big]_{\text{ş}=0} = \text{þ}^{-1}\ ; \tag{S28a}$$

$$\frac{d\omega_{\iota 1}}{d\text{ş}}\Big]_{\text{ş}=0} = \text{þ}^{-4}/6\ ; \tag{S28b}$$

$$\frac{d\omega_{\iota 2}}{d\text{ş}}\Big]_{\text{ş}=0} = \text{þ}^{-7}/24\ . \tag{S28c}$$

Order 0: $(\varepsilon^{1/2})^0[\text{þ}^{-1}] = \text{þ}^2\,\varepsilon^{-1/2} \;\Rightarrow\; \kappa_0 = -\,\Xi^{2/3}\,;$ (S29a)

Order 1: $(\varepsilon^{1/2})^0[\text{þ}^{-1}] + (\varepsilon^{1/2})^1[(^1/_6)\text{þ}^{-4}] = \text{þ}^2\,\varepsilon^{-1/2} \;\Rightarrow$

$$\kappa_1 = -\,\Xi^{2/3}\,[\tfrac{1}{2}(1 + \sqrt{^5/_3}\,)]^{2/3}\;\; 8 -\Xi^{2/3}\,[1.0948]\,; \quad \text{(S29b)}$$

Order 2: $(\varepsilon^{1/2})^0[\text{þ}^{-1}] + (\varepsilon^{1/2})^1[(^1/_6)\text{þ}^{-4}] + (\varepsilon^{1/2})^2[(^1/_{24})\text{þ}^{-7}] = \text{þ}^2\,\varepsilon^{-1/2} \;\Rightarrow$

$$\kappa_2 \; 8 -\Xi^{2/3}\,[1.1120] \; 8 - {}^{19}/_{17}\,\Xi^{2/3}\,. \quad \text{(S29c)}$$